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RESEARCH ARTICLE

COMMON QUADRATIC LYAPUNOV FUNCTIONS FOR TWO STABLE MATRICES

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ABSTRACT

In this paper we consider the problem of existence and evaluation of common solutions to Lyapunov equations for switched systems consisting of two real or complex matrices. Conditions for existence and solution algorithm based on the gradient of a matrix function are given. Number of examples are provided.

Keywords: Lyapunov function, Common solution, Hurwitz stability

1. INTRODUCTION

Let $\mathbb{R}^{n\times n}$ ($\mathbb{C}^{n\times n}$) be the set of $n\times n$ real (complex) matrices. A matrix A is called Hurwitz stable if its all eigenvalues lie in the open left-half plane. A necessary and sufficient condition for the matrix $A\in\mathbb{R}^{n\times n}$ ($A\in\mathbb{C}^{n\times n}$) to be Hurwitz stable is that for any given positive definite matrix Q there exists a (unique) positive definite matrix P that satisfies the Lyapunov equation.

Consider the following linear switched system

$$\dot{x} = Ax, \qquad A\in\{A_1, A_2, \dots, A_N\} \tag{1}$$

where $A_i\in\mathbb{R}^{n\times n}$ ($A_i\in\mathbb{C}^{n\times n}$) $(i=1,2,\dots,N)$ are Hurwitz stable.

If there exists a $\mathrm{P} > 0$ such that

$$A_i^T P + PA_i < 0, \quad (A_i^* P + PA_i < 0) \quad (i=1,2,\dots,N) \tag{2}$$

the matrix P is called a common solution to Lyapunov inequalities (2) and the function $V(x) = x^T Px$ ($V(x) = x^* Px$) is called common quadratic Lyapunov function of the switched system (1). A sufficient condition for the switched system (1) to be global uniformly asimptotically stable is that the inequalities (2) have a common solution P (see [1,2]).

The problem of existence and evaluation of common solution to system (2) have been studied in a lot of works (see [1-6] and references therein). Theoretical condition on the existence of a solution to this problem for two real 2×2 dimensional matrices is given in [3]. In the general n-dimensional case, there is no theoretical solution in the literature except for the theoretical results that can be given without resorting to numerical methods for the existence of the solution (see [3]). In [4] for a pair of 2×2 complex matrices a necessary and sufficient condition for the existence of common solution P is given. This result relates to a special class of 4×4 real matrices.

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In this paper we give a necessary and sufficient condition for the existence of a common solution for two $n \times n$ dimensional real matrix. In the second part, we propose a gradient algorithm for common solution of Lyapunov inequalities for two $n \times n$ dimensional complex matrix. Here we establish a formula for the derivative of a matrix functional (Theorem 2).

2. CONDITION FOR COMMON SOLUTION FOR TWO REAL MATRICES

Let A be Hurwitz stable real matrix. For a given positive definite matrix Q, we denote the unique $P > 0$ solution of the equation $A^T P + PA = -Q$ by $P_{A(Q)}$.

Let two $n \times n$ dimensional Hurwitz stable real matrices A_1 and A_2 be given. For $Q_1 > 0$ and $Q_2 > 0$ we define

$$P_1 = P_{A_1(Q_1)} > 0, \qquad P_2 = P_{A_2(Q_2)}$$

and

$$l_{ij} = \lambda_{\max}(A_i^T P_j + P_j A_i), \qquad (i, j = 1,2)$$

where $\lambda_{\max}(C)$ stands for the maximum eigenvalue of C. From stability of A_1 and A_2 it follows that

$$l_{11} < 0 \text{ and } l_{22} < 0. \tag{3}$$

Define 2×2 dimensional matrix

$$L = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}.$$

If $l_{21} < 0$ then $A_2^T P_1 + P_1 A_2 < 0$ therefore P_1 is a common solution for $\{A_1, A_2\}$. If $l_{12} < 0$ then $A_1^T P_2 + P_2 A_1 < 0$ and P_2 is a common solution. Therefore we assume that

$$l_{12} \geq 0 \quad \text{and} \quad l_{21} \geq 0. \tag{4}$$

We investigate a weighted sum P_1 and P_2 for a common solution.

Lemma 1. Let the pair (w_1, w_2) is given where

$$l_{11} w_1 + l_{12} w_2 < 0, \quad l_{21} w_1 + l_{22} w_2 < 0, \quad w_1 > 0, \quad w_2 > 0. \tag{5}$$

Then $P_* = w_1 P_1 + w_2 P_2$ is a common solution for $\{A_1, A_2\}$.

Proof. Using convexity of the function $P \mapsto \lambda_{\max}(A_i^T P + PA_i)$ (see [5, p.34], [6]), we obtain

$$\begin{aligned} \lambda_{\max}\left(A_i^T P_* + P_* A_i\right) &= \lambda_{\max}\left(w_1\left(A_i^T P_1 + P_1 A_i\right) + w_2\left(A_i^T P_2 + P_2 A_i\right)\right) \\ &\leq w_1 \lambda_{\max}\left(A_i^T P_1 + P_1 A_i\right) + w_2\left(A_i^T P_2 + P_2 A_i\right) \\ &= w_1 l_{i1} + w_2 l_{i2} < 0 \quad (i = 1,2). \end{aligned}$$

Therefore the matrix $P_* = w_1 P_1 + w_2 P_2$ is a common solution.

□

By (5) we consider for a positive solution of the system

$$\begin{cases} l_{11}x + l_{12}y < 0 \\ l_{21}x + l_{22}y < 0 \end{cases}. \tag{6}$$

If $l_{12} = 0$ then any pair (x, y), where $x > 0$ is arbitrary and $y > -\frac{l_{21}}{l_{22}}x$ is a feasible pair. If $l_{21} = 0$ then any pair (x, y), where $y > 0$ is arbitrary and $x > -\frac{l_{12}}{l_{11}}y$ is a feasible pair. Therefore we assume that

$$l_{12} > 0 \text{ and } l_{21} > 0. \tag{7}$$

Theorem 1. Assume that (7) is satisfied. Then (6) has positive solution (x, y) if and only if

$$l_{11}l_{22} > l_{12}l_{21}. \tag{8}$$

Proof. From (6) it follows that

$$y < -\frac{l_{11}}{l_{12}}x, \quad y > -\frac{l_{21}}{l_{22}}x, \quad \text{or} \quad y < k_1x, \quad y > k_2x,$$

where $k_1 = -\frac{l_{11}}{l_{12}} > 0$, $k_2 = -\frac{l_{21}}{l_{22}} > 0$. Then (6) has a positive solution pair (x, y) if and only if $k_1 > k_2$ (see Fig. 1). The inequality $k_1 > k_2$ is equivalent to (8).

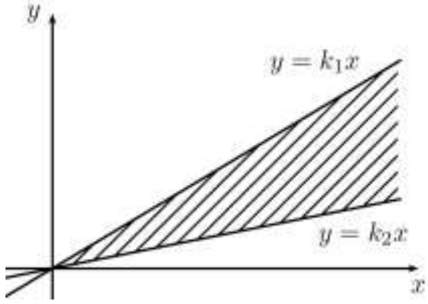

Figure 1.

□

If (8) is satisfied the weighted coefficients

$$w_1^0 = \frac{l_{12} - l_{22}}{l_{11}l_{22} - l_{21}l_{12}}, \quad w_2^0 = \frac{l_{21} - l_{11}}{l_{11}l_{22} - l_{21}l_{12}}$$

satisfy the inequalities

$$w_1^0 > 0, \quad w_2^0 > 0, \quad l_{11}w_1^0 + l_{12}w_2^0 = -1 < 0, \quad l_{21}w_1^0 + l_{22}w_2^0 = -1 < 0$$

and the matrix $P_* = w_1^0P_1 + w_2^0P_2$ is a common solution.

Example 1. Consider the following Hurwitz stable matrices

$$A_1 = \begin{bmatrix} -5 & 2 & -1 & 2 \\ 2 & -4 & 1 & 2 \\ 3 & -2 & -2 & -3 \\ 5 & -2 & 1 & 7 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & -2 & -3 & 2 \\ -2 & -7 & -3 & 4 \\ 4 & -2 & -5 & 1 \\ -3 & 0 & -1 & -1 \end{bmatrix}$$

and positive definite matrices

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, Q_2 = \begin{bmatrix} 12 & -1 & 2 & -3 \\ -1 & 11 & 1 & 0 \\ 2 & 0 & 11 & 0 \\ -3 & 0 & 2 & 11 \end{bmatrix}.$$

Then

$$P_1 = \begin{bmatrix} 0.251 & 0.081 & 0.03 & 0.09 \\ 0.081 & 0.14 & -0.01 & 0.04 \\ 0.03 & -0.01 & 0.19 & -0.05 \\ 0.09 & 0.04 & -0.05 & 0.13 \end{bmatrix}, P_2 = \begin{bmatrix} 10.68 & -2.72 & -2.07 & -2.51 \\ -2.72 & 1.46 & 0.33 & 0.66 \\ -2.07 & 0.33 & 2.09 & 0.24 \\ -2.51 & 0.66 & 0.24 & 3.36 \end{bmatrix}.$$

The matrix L is calculated as

$$L = \begin{bmatrix} -1 & 2.126 \\ 0.541 & -7.507 \end{bmatrix}.$$

Here $l_{11} < 0$, $l_{22} < 0$, $l_{12} > 0$, $l_{21} > 0$ and

$$l_{11}.l_{22} - l_{12}.l_{21} = 6.356834 > 0$$

and the conditions of Theorem 1 are satisfied.

$$k_1 = -\frac{l_{11}}{l_{12}} = 0.470366, \quad k_2 = -\frac{l_{21}}{l_{22}} = 0.072066$$

and the graphs of $y = 0.470366x$ and $y = 0.072066x$ are shown in Fig. 2.

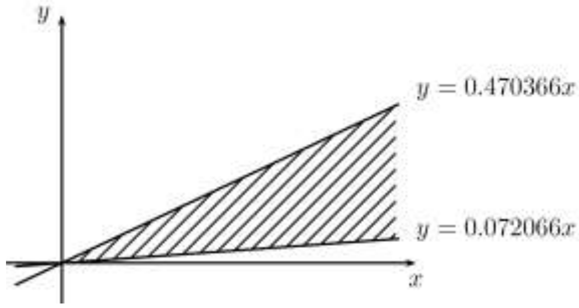

Figure 2.

$w_1^0 = 1.515376$, $w_2^0 = 0.242416$ and by the above the matrix

$$P_* = 1.515376\, P_1 + 0.242416\, P_2$$

is a common solution.

3. AN ALGORITHM FOR TWO COMPLEX MATRICES

In this section we give an algorithm for a common P for given two Hurwitz stable complex matrices. For this purpose we establish the derivative formula for real functionals defined on the set of Hermitian matrices.

Let $\mathcal{H}$ be the set of $n \times n$-dimensional Hermitian matrices. Assume that $f: \mathcal{H} \to \mathbb{R}$ is a real matrix functional.

Definition 1. The functional $f(P)$ defined on $\mathcal{H}$ is called differentiable at the point P if there exists a Hermitian matrix $\partial f|_{\mathrm{P}}$ such that

$$f(P+\Delta P)=f(P)+<\partial f|_P,\Delta P> +o(\Delta P),$$

where $\frac{o(\Delta P)}{\|\Delta P\|}\rightarrow 0$ as $\|\Delta P\|\rightarrow 0$, where $\|\cdot\|$ is the Frobenius norm. Here the inner product be defined by $<\partial f|_{\mathrm{P}},\Delta P>=tr(\partial f|_P.\Delta P)$. A Hermitian $n\times n$ matrix P has totally $r=n(n+1)/2$ entries:

$$P=\begin{bmatrix} x_1 & x_2+jy_2 & \cdots & x_n+jy_n \\ x_2-jy_2 & x_{n+1} & \cdots & x_{2n-1}+jy_{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_n-jy_n & x_{2n-1}-jy_{2n-1} & \cdots & x_r \end{bmatrix}.$$

Define real function g by

$$g(x_1,x_2,\dots,x_r,y_2,y_3,y_5,\dots,y_q):=f(P).$$

If the functional is differentiable at P then the function g is differentiable at $(x_1,\dots,x_r,y_2,\dots,y_q)$. We wish calculate the entries of the derivative matrix $\partial f|_P$ in terms of the partial derivative of g.

Theorem 2. If the functional $f(P)$ is differentiable at P, then

$$\partial f|_P=\begin{bmatrix} \frac{\partial g}{\partial x_1} & \frac{1}{2}\left(\frac{\partial g}{\partial x_2}+j\frac{\partial g}{\partial y_2}\right) & \cdots & \frac{1}{2}\left(\frac{\partial g}{\partial x_n}+j\frac{\partial g}{\partial y_n}\right) \\ \frac{1}{2}\left(\frac{\partial g}{\partial x_2}-j\frac{\partial g}{\partial y_2}\right) & \frac{\partial g}{\partial x_{n+1}} & \cdots & \frac{1}{2}\left(\frac{\partial g}{\partial x_{2n-1}}+j\frac{\partial g}{\partial y_{2n-1}}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}\left(\frac{\partial g}{\partial x_n}-j\frac{\partial g}{\partial y_n}\right) & \frac{1}{2}\left(\frac{\partial g}{\partial x_{2n-1}}-j\frac{\partial g}{\partial y_{2n-1}}\right) & \cdots & \frac{\partial g}{\partial x_r} \end{bmatrix}.$$

Proof. To avoid cumbersome expressions we give the proof for the case *n=3*. For an arbitrary n the proof is identical.

Define 3×3 Hermitian matrices

$$P=\begin{bmatrix} x_1 & x_2+jy_2 & x_3+jy_3 \\ x_2-jy_2 & x_4 & x_5+jy_5 \\ x_3-jy_3 & x_5-jy_5 & x_6 \end{bmatrix}$$

and

$$P+\Delta P=\begin{bmatrix} x_1+\Delta x_1 & x_2+\Delta x_2+j(y_2+\Delta y_2) & x_3+\Delta x_3+j(y_3+\Delta y_3) \\ x_2+\Delta x_2-j(y_2+\Delta y_2) & x_4+\Delta x_4 & x_5+\Delta x_5+j(y_5+\Delta y_5) \\ x_3+\Delta x_3-j(y_3+\Delta y_3) & x_5+\Delta x_5-j(y_5+\Delta y_5) & x_6+\Delta x_6 \end{bmatrix}.$$

If $f(P)$ is differentiable at P then the scalar function

$$g(x_1,x_2,x_3,x_4,x_5,x_6,y_2,y_3,y_5)=f(P)$$

is differentiable at the point $u=(x_1,x_2,\dots,x_6,y_2,y_3,y_5)$. The following can be written:

$$\begin{aligned} g(u+\Delta u) &= g(u) + <\partial g(u), \Delta u> + o(\Delta u) \\ &= g(u) + \frac{\partial g}{\partial x_1}\Delta x_1 + \frac{\partial g}{\partial x_2}\Delta x_2 + \cdots + \frac{\partial g}{\partial x_6}\Delta x_6 + \frac{\partial g}{\partial y_2}\Delta y_2 \\ &\quad + \frac{\partial g}{\partial y_3}\Delta y_3 + \frac{\partial g}{\partial y_5}\Delta y_5 + o(\Delta u) \end{aligned}$$

where $\Delta u = (\Delta x_1, \Delta x_2, \dots, \Delta x_6, \Delta y_2, \Delta y_3, \Delta y_5)$, and $\frac{o(\Delta u)}{\| \Delta u \|} \to 0$ as $\| \Delta u \| \to 0$.

Let the derivative matrix of f at P be

$$\partial f|_P = \begin{bmatrix} a_1 & a_2 + jb_2 & a_3 + jb_3 \\ a_2 - jb_2 & a_4 & a_5 + jb_5 \\ a_3 - jb_3 & a_5 - jb_5 & a_6 \end{bmatrix}.$$

We aim to express a_i $(i = 1, \dots, 6)$ and b_i $(i = 2,3,5)$ in terms of the partial derivative of g. We have

$$\begin{aligned} tr(\partial f|_P . \Delta P) &= a_1\Delta x_1 + (a_2 + jb_2)(\Delta x_2 - j\Delta y_2) + (a_3 + jb_3)(\Delta x_3 - j\Delta y_3) \\ &\quad + (a_2 - jb_2)(\Delta x_2 + j\Delta y_2) + a_4\Delta x_4 + (a_5 + jb_5)(\Delta x_5 - j\Delta y_5) \\ &\quad + (a_3 - jb_3)(\Delta x_3 + j\Delta y_3) + (a_5 - jb_5)(\Delta x_5 + j\Delta y_5) + a_6\Delta x_6 \\ &= a_1\Delta x_1 + 2a_2\Delta x_2 + 2a_3\Delta x_3 + a_4\Delta x_4 + 2a_5\Delta x_5 + a_6\Delta x_6 + 2b_2\Delta y_2 \\ &\quad + 2b_3\Delta y_3 + 2b_5\Delta y_5 \end{aligned}$$

on the other hand

$$\begin{aligned} <\partial g(u), \Delta u> &= <\left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \dots, \frac{\partial g}{\partial y_5}\right), (\Delta x_1, \Delta x_2, \dots, \Delta y_5)> \\ &= \frac{\partial g}{\partial x_1}\Delta x_1 + \frac{\partial g}{\partial x_2}\Delta x_2 + \frac{\partial g}{\partial x_3}\Delta x_3 + \frac{\partial g}{\partial x_4}\Delta x_4 + \frac{\partial g}{\partial x_5}\Delta x_5 + \frac{\partial g}{\partial x_6}\Delta x_6. \\ &\quad + \frac{\partial g}{\partial y_2}\Delta y_2 + \frac{\partial g}{\partial y_3}\Delta y_3 + \frac{\partial g}{\partial y_5}\Delta y_5 \end{aligned}$$

If the functional f is differentiable at P

$$\mathrm{tr}(\partial f|_P . \Delta P) = <\partial g(u), \Delta u>$$

for all $\Delta u = (\Delta x_1, \Delta x_2, \dots, \Delta x_6, \Delta y_2, \Delta y_3, \Delta y_5)$. Since Δx_i $(i = 1, \dots, 6)$, Δy_i $(i = 2,3,5)$ are arbitrary then

$$a_1 = \frac{\partial g}{\partial x_1}, \quad 2a_2 = \frac{\partial g}{\partial x_2}, \quad 2a_3 = \frac{\partial g}{\partial x_3}, \quad a_4 = \frac{\partial g}{\partial x_4}, \quad 2a_5 = \frac{\partial g}{\partial x_5}, \quad a_6 = \frac{\partial g}{\partial x_6},$$

$$2b_2 = \frac{\partial g}{\partial y_2}, \quad 2b_3 = \frac{\partial g}{\partial y_3}, \quad 2b_5 = \frac{\partial g}{\partial y_5}.$$

Therefore gradient of the functional f at P is

$$\partial f|_P = \begin{bmatrix} \frac{\partial g}{\partial x_1} & \frac{1}{2}\left(\frac{\partial g}{\partial x_2} + j\frac{\partial g}{\partial y_2}\right) & \frac{1}{2}\left(\frac{\partial g}{\partial x_3} + j\frac{\partial g}{\partial y_3}\right) \\ \frac{1}{2}\left(\frac{\partial g}{\partial x_2} - j\frac{\partial g}{\partial y_2}\right) & \frac{\partial g}{\partial x_4} & \frac{1}{2}\left(\frac{\partial g}{\partial x_5} + j\frac{\partial g}{\partial y_5}\right) \\ \frac{1}{2}\left(\frac{\partial g}{\partial x_3} - j\frac{\partial g}{\partial y_3}\right) & \frac{1}{2}\left(\frac{\partial g}{\partial x_5} - j\frac{\partial g}{\partial y_5}\right) & \frac{\partial g}{\partial x_6} \end{bmatrix}.$$

□

Let A_1 and A_2 be $n \times n$ dimensional complex Hurwitz stable matrices.

Theorem 3. [6] Let $\varepsilon > 0$ and I be the identity matrix. Then

$$\exists P > 0, \quad \begin{matrix} A_1^* P + P A_1 < 0, \\ A_2^* P + P A_2 < 0, \end{matrix} \quad \Leftrightarrow \quad \exists P > 0, \quad \begin{matrix} A_1^* P + P A_1 + \varepsilon I \leq 0, \\ A_2^* P + P A_2 + \varepsilon I \leq 0. \end{matrix}$$

Let $\varepsilon > 0$ and $Q \geq 0$ be given. Denote by $P_{A_2(Q)} > 0$ the solution of

$$A_2^* P + P A_2 = -Q - \varepsilon I$$

and define the following convex matrix functional:

$$F(Q) = \lambda_{\max}\big(A_1^* P_{A_2(Q)} + P_{A_2(Q)} A_1\big).$$

If $F\big(\tilde{Q}\big) < 0$ at some $\tilde{Q} \geq 0$, then the matrix $P_{A_2(Q)}$ is a common solution.

Consider the minimization of $F(Q)$. For Hermitian matrix H, the projection of H onto the convex cone of nonnegative definite matrices denote by $[H]^+$. For the convergence of the proposed algorithm we impose the following condition (see [6]).

Condition 1. For $t > 0$ there exists $Q^* \geq 0$ such that the closed ball of radius t and centered at Q^* is contained in the set $\{Q \geq 0 : F(Q) < 0\}$.

Consider the following convex problem

$$\begin{cases} F(Q) & \rightarrow \quad \min \\ Q \geq 0. & \end{cases}$$

Using Theorem 2 we suggest the following solution algorithm.

Algorithm 1.

1. Choose $Q^0 \geq 0$.
   If $F(Q^0) < 0$ then stop. Otherwise continue.
2. For $k = 0,1,2, \ldots$ define
   $$Q^{k+1} = \left[Q^k - \mu_k \partial F(Q)|_{Q=Q^k}\right]^+$$
   where
   $$\mu_k := \frac{\alpha F\big(Q^k\big) + t\big\|\partial F(Q)|_{Q=Q^k}\big\|}{\big\|\partial F(Q)|_{Q=Q^k}\big\|^2},$$
   $0 \leq \alpha \leq 2$ and $t > 0$ is defined from Condition 1.
3. If $F\big(Q^k\big) < 0$ for some k then stop. The corresponding $P_{A_2}\big(Q^k\big) < 0$ is a common solution.

Example 2. This example taken from [4]. Consider the following complex Hurwitz matrices

$$A_1 = \begin{bmatrix} -2 - j & 1 + 2j \\ -1.5 - j & -1.1 + j \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 - j & 3 + 4j \\ -2.5 - j & -2.5 + j \end{bmatrix}.$$

Let $Q^0 = I$ and $\varepsilon = 0.001$. We have

$$P_{A_2(Q^0)} = \begin{bmatrix} 6.673 & -2.558 + 6.450j \\ -2.558 - 6.450j & 7.451 \end{bmatrix}$$

and $F(Q^0) = 10.729 > 0$. By Theorem 2,

$$\partial F(Q)|_{Q=Q^0} = \begin{bmatrix} 8.323 & -0.572 + 4.924j \\ -0.572 - 4.924j & 2.394 \end{bmatrix}.$$

Therefore,

$$Q^1 = \left[Q^0 - \mu_0 \partial F(Q)|_{Q=Q^0}\right]^+ = \begin{bmatrix} 0.270 & 0.055 - 0.472j \\ 0.055 + 0.472j & 0.839 \end{bmatrix}$$

where $\alpha = 2$, $t = 1$ and $\mu_0 = 0.262$.

Algorithm 1 after 5 steps gives

$$Q^5 = \begin{bmatrix} 0.318 & 0.044 - 0.564j \\ 0.044 + 0.564 & 1.006 \end{bmatrix}, \quad F(Q^5) = -\, 0.085 < 0$$

and the matrix

$$P_{A_2(Q^5)} = \begin{bmatrix} 0.0582 & -0.001 - 0.046j \\ -0.001 + 0.046j & 0.126 \end{bmatrix}$$

is a common solution for A_1 and A_2.

Example 3. Consider the following complex Hurwitz matrices

$$A_1 = \begin{bmatrix} -3 & j & 1-j \\ 1+j & -3-3j & -2+j \\ 2-j & -3-2j & -4-j \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.6+j & -2+3j & 1+2j \\ 3+j & -0.6-3j & 2+j \\ -2-j & -5-2j & -1.6-j \end{bmatrix}.$$

Let $Q^0 = I$ and $\varepsilon = 0.001$. We have

$$P_{A_2(Q^0)} = \begin{bmatrix} 2.068 & -0.403 - 1.239j & 0.260 + 1.231j \\ -0.403 + 1.239j & 1.999 & -0.707 + 0.037j \\ 0.260 - 1.231j & -0.707 - 0.037j & 1.153 \end{bmatrix}$$

and $F(Q^0) = 1.8808 > 0$. By Theorem 1,

$$\partial F(Q)|_{Q=Q^0} = \begin{bmatrix} 0.403 & -0.171 + 0.236j & -0.816 + 0.277j \\ -0.171 - 0.236j & 0.368 & 0.094 + 0.805j \\ -0.816 - 0.277j & 0.094 - 0.805j & 1.106 \end{bmatrix}.$$

Therefore,

$$\begin{aligned} Q^1 &= \left[Q^0 - \mu_0 \partial F(Q)|_{Q=Q^0}\right]^+ \\ &= \begin{bmatrix} 0.834 & 0.217 + 0.031j & 0.473 - 0.230j \\ 0.217 - 0.031j & 0.836 & 0.011 - 0.499j \\ 0.473 + 0.230j & 0.011 + 0.499j & 0.574 \end{bmatrix} \end{aligned}$$

where $\alpha = 2$, $t = 1$ and $\mu_0 = 1.308$.

Algorithm 1 after 3 steps gives

$$Q^3 = \begin{bmatrix} 0.813 & 0.277+0.060j & 0.508-0.428j \\ 0.277-0.060j & 0.875 & 0.188-0.652j \\ 0.508+0.428j & 0.188+0.652j & 0.828 \end{bmatrix}, \quad F(Q^3) = -0.08 < 0$$

and the matrix

$$P_{A_2(Q^3)} = \begin{bmatrix} 0.558 & 0.012-0.135j & 0.179+0.112j \\ 0.012+0.135j & 0.643 & -0.109-0.085j \\ 0.179-0.112j & -0.109+0.085j & 0.322 \end{bmatrix}$$

is a common solution for A_1 and A_2.

CONFLICT OF INTEREST

The author stated that there are no conflicts of interest regarding the publication of this article.

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