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Generalized Woodall Numbers: An Investigation of Properties of Woodall and Cullen Numbers via Their Third Order Linear Recurrence Relations

Yüksel Soykan[1*] and Vedat İrge[1]

[1] *Department of Mathematics, Art and Science Faculty, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey*
[*] *Corresponding author*

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Abstract

In this paper, we investigate the generalized Woodall sequences and we deal with, in detail, four special cases, namely, modified Woodall, modified Cullen, Woodall and Cullen sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

1. Introduction

The Woodall numbers $\{R_n\}$, sometimes called Riesel numbers, and also called Cullen numbers of the second kind, are numbers of the form

$$R_n = n \times 2^n - 1.$$

The first few Woodall numbers are:

$$1, 7, 23, 63, 159, 383, 895, 2047, 4607, 10239, 22527, 49151, 106495, 229375, 491519, 1048575, \ldots$$

(sequence A003261 in the OEIS [22]). Woodall numbers were first studied by Allan J. C. Cunningham and H. J. Woodall in [6] in 1917, inspired by James Cullen's earlier study of the similarly-defined Cullen numbers.
The Cullen numbers $\{C_n\}$ are numbers of the form

$$C_n = n \times 2^n + 1.$$

The first few Cullen numbers are:

$$1, 3, 9, 25, 65, 161, 385, 897, 2049, 4609, 10241, 22529, 49153, 106497, 229377, 491521, \ldots$$

(sequence A002064 in the OEIS).
Woodall and Cullen sequences have been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [1,2,6,9,10,11,13,15,16,17,18] and references therein.
Note that $\{R_n\}$ and $\{C_n\}$ hold the following relations:

$$R_n = 4R_{n-1} - 4R_{n-2} - 1,$$
$$C_n = 4C_{n-1} - 4C_{n-2} + 1.$$

Email addresses and ORCID numbers: **ysoykan@beun.edu.tr, 0000-0002-1895-211X (Y. Soykan), vedatirge_1986@hotmail.com, 0000-0003-2878-3505 (V. İrge)**

Note also that the sequences $\{R_n\}$ and $\{C_n\}$ satisfy the following third order linear recurrences:

$$R_n = 5R_{n-1} - 8R_{n-2} + 4R_{n-3}, \quad R_0 = -1, R_1 = 1, R_2 = 7, \tag{1.1}$$
$$C_n = 5C_{n-1} - 8C_{n-2} + 4C_{n-3}, \quad C_0 = 1, C_1 = 3, C_2 = 9. \tag{1.2}$$

The purpose of this article is to generalize and investigate these interesting sequence of numbers (i.e., Woodall, Cullen numbers) via their third order linear recurrence relations (1.1) and (1.2). First, we recall some properties of generalized Tribonacci numbers.
The generalized (r,s,t) sequence (or generalized Tribonacci sequence or generalized 3-step Fibonacci sequence)

$$\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$$

(or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \quad W_0 = a, W_1 = b, W_2 = c, \ \ n \geq 3 \tag{1.3}$$

where W_0, W_1, W_2 are arbitrary complex (or real) numbers and r, s, t are real numbers.
This sequence has been studied by many authors, see for example [3,4,5,7,8,14,19,20,21,24,25,27,28,29].
The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$ when $t \neq 0$. Therefore, recurrence (1.3) holds for all integer n.
As $\{W_n\}$ is a third-order recurrence sequence (difference equation), it's characteristic equation is

$$x^3 - rx^2 - sx - t = 0 \tag{1.4}$$

whose roots are

$$\alpha = \alpha(r,s,t) = \frac{r}{3} + A + B,$$
$$\beta = \beta(r,s,t) = \frac{r}{3} + \omega A + \omega^2 B,$$
$$\gamma = \gamma(r,s,t) = \frac{r}{3} + \omega^2 A + \omega B,$$

where

$$A = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{1/3}, \ B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}\right)^{1/3},$$
$$\Delta = \Delta(r,s,t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \ \ \omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that we have the following identities

$$\alpha + \beta + \gamma = r,$$
$$\alpha\beta + \alpha\gamma + \beta\gamma = -s,$$
$$\alpha\beta\gamma = t.$$

In the case of two distinct roots, i.e., $\alpha = \beta \neq \gamma$, Binet's formula can be given as follows:

Theorem 1.1. *(Two Distinct Roots Case: $\alpha = \beta \neq \gamma$) Binet's formula of generalized Tribonacci numbers is*

$$W_n = (A_1 + A_2 n) \times \alpha^n + A_3 \gamma^n$$

where

$$A_1 = \frac{-W_2 + 2\alpha W_1 - \gamma(2\alpha - \gamma)W_0}{(\alpha - \gamma)^2},$$
$$A_2 = \frac{W_2 - (\alpha + \gamma)W_1 + \alpha\gamma W_0}{\alpha(\alpha - \gamma)},$$
$$A_3 = \frac{W_2 - 2\alpha W_1 + \alpha^2 W_0}{(\alpha - \gamma)^2}.$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

Lemma 1.2. *Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized (r,s,t) sequence (the generalized Tribonacci sequence) $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x + (W_2 - rW_1 - sW_0)x^2}{1 - rx - sx^2 - tx^3}. \tag{1.5}$$

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{1.6}$$

For matrix formulation (1.6), see [12]. In fact, Kalman gave the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ r & s & t \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \end{pmatrix}.$$

Now, we present Simson's formula of generalized Tribonacci numbers.

Theorem 1.3 (Simson's Formula of Generalized Tribonacci Numbers)**.** *For all integers n, we have*

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = t^n \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{vmatrix}. \tag{1.7}$$

Proof. For a proof, see Soykan [23]. □
Next, we consider two special cases of the generalized (r,s,t) sequence $\{W_n\}$ which we call them (r,s,t) and Lucas (r,s,t) sequences. (r,s,t) sequence $\{G_n\}_{n\geq 0}$ and Lucas (r,s,t) sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$G_{n+3} = rG_{n+2} + sG_{n+1} + tG_n, \quad G_0 = 0, G_1 = 1, G_2 = r, \tag{1.8}$$

$$H_{n+3} = rH_{n+2} + sH_{n+1} + tH_n, \quad H_0 = 3, H_1 = r, H_2 = 2s + r^2. \tag{1.9}$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n} = -\frac{s}{t}G_{-(n-1)} - \frac{r}{t}G_{-(n-2)} + \frac{1}{t}G_{-(n-3)},$$

$$H_{-n} = -\frac{s}{t}H_{-(n-1)} - \frac{r}{t}H_{-(n-2)} + \frac{1}{t}H_{-(n-3)}$$

for $n = 1,2,3,...$ respectively. Therefore, recurrences (1.8)-(1.9) hold for all integers n.
In the case of two distinct roots, i.e., $\alpha = \beta \neq \gamma$, for all integers n, Binet's formula of (r,s,t) and Lucas (r,s,t) numbers (using initial conditions in (1.8)-(1.9)) can be expressed as follows:

Theorem 1.4. *(Two Distinct Roots Case: $\alpha = \beta \neq \gamma$) For all integers n, Binet's formula of (r,s,t) and Lucas (r,s,t) numbers are*

$$G_n = \left(\frac{-\gamma}{(\alpha-\gamma)^2} + \frac{1}{(\alpha-\gamma)}n\right) \times \alpha^n + \frac{\gamma}{(\alpha-\gamma)^2}\gamma^n,$$

$$H_n = 2\alpha^n + \gamma^n,$$

respectively.

Lemma 1.2 gives the following results as particular examples (generating functions of (r,s,t) and Lucas (r,s,t) numbers).

Corollary 1.5. *Generating functions of (r,s,t) and Lucas (r,s,t) numbers are*

$$\sum_{n=0}^{\infty} G_n x^n = \frac{x}{1 - rx - sx^2 - tx^3},$$

$$\sum_{n=0}^{\infty} H_n x^n = \frac{3 - 2rx - sx^2}{1 - rx - sx^2 - tx^3},$$

respectively.

The following theorem shows that the generalized Tribonacci sequence W_n at negative indices can be expressed by the sequence itself at positive indices.

Theorem 1.6. *For $n \in \mathbb{Z}$, we have*

$$W_{-n} = t^{-n}(W_{2n} - H_n W_n + \frac{1}{2}(H_n^2 - H_{2n})W_0).$$

Proof. For the proof, see Soykan [26, Theorem 2.]. □
Now, we present a basic relation between $\{H_n\}$ and $\{W_n\}$ which can be used to write H_n in terms of W_n.

Lemma 1.7. *The following equality is true:*
$(W_2^3 + (t+rs)W_1^3 + t^2W_0^3 + (r^2-s)W_1^2W_2 - 2rW_1W_2^2 - sW_0W_2^2 + rtW_0^2W_2 + (s^2+rt)W_0W_1^2 + 2stW_0^2W_1 + (rs-3t)W_0W_1W_2)H_n = (3W_2^2 + (r^2-s)W_1^2 + rtW_0^2 - 4rW_1W_2 - 2sW_0W_2 + (rs-3t)W_0W_1)W_{n+2} + (-2rW_2^2 + 3tW_1^2 - 2sW_1W_2 - 3tW_0W_2 + 3rsW_1^2 + 2stW_0^2 + 2r^2W_1W_2 + 2s^2W_0W_1 + rsW_0W_2 + 2rtW_0W_1)W_{n+1} + (-sW_2^2 + (s^2+rt)W_1^2 + 3t^2W_0^2 + (rs-3t)W_1W_2 + 2rtW_0W_2 + 4stW_0W_1)W_n.$

Proof. It is given in Soykan [25]. □
Using Theorem 1.6, we have the following corollary, see Soykan [26, Corollary 6].

Corollary 1.8. *For $n \in \mathbb{Z}$, we have*

(a)

$$G_{-n} = \frac{1}{t^{n+1}}((2rt - s^2)G_n^2 + tG_{2n} + sG_{n+2}G_n - (3t + rs)G_{n+1}G_n).$$

(b)

$$H_{-n} = \frac{1}{2t^n}(H_n^2 - H_{2n}).$$

Note that G_{-n} and H_{-n} can be given as follows by using $G_0 = 0$ and $H_0 = 3$ in Theorem 1.6,

$$G_{-n} = t^{-n}(G_{2n} - H_nG_n + \frac{1}{2}(H_n^2 - H_{2n})G_0) = t^{-n}(G_{2n} - H_nG_n),$$
$$H_{-n} = t^{-n}(H_{2n} - H_nH_n + \frac{1}{2}(H_n^2 - H_{2n})H_0) = \frac{1}{2t^n}(H_n^2 - H_{2n}),$$

respectively.

2. Generalized Woodall Sequence

In this paper, we consider the case $r = 5, s = -8, t = 4$. A generalized Woodall sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0,W_1,W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = 5W_{n-1} - 8W_{n-2} + 4W_{n-3} \tag{2.1}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero.
The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-1)} - \frac{5}{4}W_{-(n-2)} + \frac{1}{4}W_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$. Therefore, recurrence (2.1) holds for all integer n.
Theorem 1.1 can be used to obtain Binet formula of generalized Woodall numbers. Binet formula of generalized Woodall numbers can be given as
(two distinct roots case: $\alpha = \beta \neq \gamma$)

$$W_n = (A_1 + A_2n) \times \alpha^n + A_3\gamma^n$$

where

$$A_1 = \frac{-W_2 + 2\alpha W_1 - \gamma(2\alpha - \gamma)W_0}{(\alpha - \gamma)^2},$$
$$A_2 = \frac{W_2 - (\alpha + \gamma)W_1 + \alpha\gamma W_0}{\alpha(\alpha - \gamma)},$$
$$A_3 = \frac{W_2 - 2\alpha W_1 + \alpha^2 W_0}{(\alpha - \gamma)^2}.$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - 5x^2 + 8x - 4 = (x - 2)^2(x - 1) = 0.$$

Moreover

$$\alpha = \beta = 2,$$
$$\gamma = 1.$$

So,

$$W_n = (A_1 + A_2n) \times 2^n + A_3$$

where

$$A_1 = -W_2 + 4W_1 - 3W_0,$$
$$A_2 = \frac{W_2 - 3W_1 + 2W_0}{2},$$
$$A_3 = W_2 - 4W_1 + 4W_0,$$

i.e.,

$$W_n = ((-W_2 + 4W_1 - 3W_0) + \frac{W_2 - 3W_1 + 2W_0}{2} n) \times 2^n + (W_2 - 4W_1 + 4W_0). \tag{2.2}$$

The first few generalized Woodall numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Woodall numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$\frac{1}{4}(8W_0 - 5W_1 + W_2)$
2	W_2	$(11W_0 - 9W_1 + 2W_2)$
3	$4W_0 - 8W_1 + 5W_2$	$\frac{1}{16}(52W_0 - 47W_1 + 11W_2)$
4	$20W_0 - 36W_1 + 17W_2$	$(57W_0 - 54W_1 + 13W_2)$
5	$68W_0 - 116W_1 + 49W_2$	$\frac{1}{64}(240W_0 - 233W_1 + 57W_2)$
6	$196W_0 - 324W_1 + 129W_2$	$(247W_0 - 243W_1 + 60W_2)$
7	$516W_0 - 836W_1 + 321W_2$	$\frac{1}{256}(1004W_0 - 995W_1 + 247W_2)$
8	$1284W_0 - 2052W_1 + 769W_2$	$\frac{1}{256}(1013W_0 - 1008W_1 + 251W_2)$
9	$3076W_0 - 4868W_1 + 1793W_2$	$\frac{1}{1024}(4072W_0 - 4061W_1 + 1013W_2)$
10	$7172W_0 - 11268W_1 + 4097W_2$	$\frac{1}{1024}(4083W_0 - 4077W_1 + 1018W_2)$
11	$16388W_0 - 25604W_1 + 9217W_2$	$\frac{1}{4096}(16356W_0 - 16343W_1 + 4083W_2)$
12	$36868W_0 - 57348W_1 + 20481W_2$	$\frac{1}{4096}(16369W_0 - 16362W_1 + 4089W_2)$
13	$81924W_0 - 126980W_1 + 45057W_2$	$\frac{1}{16384}(65504W_0 - 65489W_1 + 16369W_2)$

Now, we define four special cases of the sequence $\{W_n\}$. Modified Woodall sequence $\{G_n\}_{n\geq 0}$, modified Cullen sequence $\{H_n\}_{n\geq 0}$, Woodall sequence $\{R_n\}$ and Cullen sequence $\{C_n\}$ are defined, respectively, by the third-order recurrence relations

$$G_n = 5G_{n-1} - 8G_{n-2} + 4G_{n-3}, \quad G_0 = 0, G_1 = 1, G_2 = 5, \tag{2.3}$$
$$H_n = 5H_{n-1} - 8H_{n-2} + 4H_{n-3}, \quad H_0 = 3, H_1 = 5, H_2 = 9, \tag{2.4}$$
$$R_n = 5R_{n-1} - 8R_{n-2} + 4R_{n-3}, \quad R_0 = -1, R_1 = 1, R_2 = 7, \tag{2.5}$$
$$C_n = 5C_{n-1} - 8C_{n-2} + 4C_{n-3}, \quad C_0 = 1, C_1 = 3, C_2 = 9. \tag{2.6}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$, $\{R_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n} = 2G_{-(n-1)} - \frac{5}{4}G_{-(n-2)} + \frac{1}{4}G_{-(n-3)},$$
$$H_{-n} = 2H_{-(n-1)} - \frac{5}{4}H_{-(n-2)} + \frac{1}{4}H_{-(n-3)},$$
$$R_{-n} = 2R_{-(n-1)} - \frac{5}{4}R_{-(n-2)} + \frac{1}{4}R_{-(n-3)},$$
$$C_{-n} = 2C_{-(n-1)} - \frac{5}{4}C_{-(n-2)} + \frac{1}{4}C_{-(n-3)},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.3)-(2.6) hold for all integer n.

Next, we present the first few values of the modified Woodall, modified Cullen, Woodall and Cullen numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
G_n	0	1	5	17	49	129	321	769	1793	4097	9217	20481	45057	98305
G_{-n}		0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{11}{16}$	$\frac{13}{16}$	$\frac{57}{64}$	$\frac{15}{16}$	$\frac{247}{256}$	$\frac{251}{256}$	$\frac{1013}{1024}$	$\frac{509}{512}$	$\frac{4083}{4096}$	$\frac{4089}{4096}$
H_n	3	5	9	17	33	65	129	257	513	1025	2049	4097	8193	16385
H_{-n}		2	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{9}{8}$	$\frac{17}{16}$	$\frac{33}{32}$	$\frac{65}{64}$	$\frac{129}{128}$	$\frac{257}{256}$	$\frac{513}{512}$	$\frac{1025}{1024}$	$\frac{2049}{2048}$	$\frac{4097}{4096}$
R_n	-1	1	7	23	63	159	383	895	2047	4607	10239	22527	49151	106495
R_{-n}		$-\frac{3}{2}$	$-\frac{3}{2}$	$-\frac{11}{8}$	$-\frac{5}{4}$	$-\frac{37}{32}$	$-\frac{35}{32}$	$-\frac{135}{128}$	$-\frac{33}{32}$	$-\frac{521}{512}$	$-\frac{517}{512}$	$-\frac{2059}{2048}$	$-\frac{1027}{1024}$	$-\frac{8205}{8192}$
C_n	1	3	9	25	65	161	385	897	2049	4609	10241	22529	49153	106497
C_{-n}		$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{27}{32}$	$\frac{29}{32}$	$\frac{121}{128}$	$\frac{31}{32}$	$\frac{503}{512}$	$\frac{507}{512}$	$\frac{2037}{2048}$	$\frac{1021}{1024}$	$\frac{8179}{8192}$

G_n, H_n, R_n and C_n are the sequences A000337, A000051 (and A048578), A003261 and A002064 in [22], respectively. Note that $\{H_n\}$ satisfies the following second order linear recurrence:

$$H_n = 3H_{n-1} - 2H_{n-2}, \; H_0 = 3, H_1 = 5$$

and satisfies the following first order non-linear recurrence:

$$H_n = 2H_{n-1} - 1, \; H_0 = 3.$$

For all integers n, modified Woodall, modified Cullen, Woodall and Cullen numbers (using initial conditions in (2.2)) can be expressed using Binet's formulas as

$$G_n = (n-1)2^n + 1$$
$$H_n = 2^{n+1} + 1$$
$$R_n = n \times 2^n - 1$$
$$C_n = n \times 2^n + 1$$

respectively.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

Lemma 2.1. *Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized Woodall sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 5W_0)x + (W_2 - 5W_1 + 8W_0)x^2}{1 - 5x + 8x^2 - 4x^3}.$$

Proof. Take $r = 5, s = -8, t = 4$ in Lemma 1.2. □

The previous lemma gives the following results as particular examples.

Corollary 2.2. *Generated functions of modified Woodall, modified Cullen, Woodall and Cullen numbers are*

$$\sum_{n=0}^{\infty} G_n x^n = \frac{x}{1 - 5x + 8x^2 - 4x^3},$$
$$\sum_{n=0}^{\infty} H_n x^n = \frac{3 - 10x + 8x^2}{1 - 5x + 8x^2 - 4x^3},$$
$$\sum_{n=0}^{\infty} R_n x^n = \frac{-1 + 6x - 6x^2}{1 - 5x + 8x^2 - 4x^3},$$
$$\sum_{n=0}^{\infty} C_n x^n = \frac{1 - 2x + 2x^2}{1 - 5x + 8x^2 - 4x^3},$$

respectively.

3. Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized Woodall sequence $\{W_n\}_{n\geq 0}$.

Theorem 3.1 (Simson Formula of Generalized Woodall Numbers)**.** *For all integers n, we have*

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = -2^{2n-4}(W_2 - 4W_1 + 4W_0)(W_2 - 3W_1 + 2W_0)^2.$$

Proof. Take $r = 5, s = -8, t = 4$ in Theorem 1.3. □

The previous theorem gives the following results as particular examples.

Corollary 3.2. *For all integers n, Simson formula of modified Woodall, modified Cullen, Woodall and Cullen numbers are given as*

$$\begin{vmatrix} G_{n+2} & G_{n+1} & G_n \\ G_{n+1} & G_n & G_{n-1} \\ G_n & G_{n-1} & G_{n-2} \end{vmatrix} = -2^{2n-2},$$

$$\begin{vmatrix} H_{n+2} & H_{n+1} & H_n \\ H_{n+1} & H_n & H_{n-1} \\ H_n & H_{n-1} & H_{n-2} \end{vmatrix} = 0,$$

$$\begin{vmatrix} R_{n+2} & R_{n+1} & R_n \\ R_{n+1} & R_n & R_{n-1} \\ R_n & R_{n-1} & R_{n-2} \end{vmatrix} = 2^{2n-2},$$

$$\begin{vmatrix} C_{n+2} & C_{n+1} & C_n \\ C_{n+1} & C_n & C_{n-1} \\ C_n & C_{n-1} & C_{n-2} \end{vmatrix} = -2^{2n-2},$$

respectively.

4. Some Identities

In this section, we obtain some identities of generalized Woodall, modified Woodall, modified Cullen, Woodall and Cullen numbers. First, we can give a few basic relations between $\{W_n\}$ and $\{G_n\}$.

Lemma 4.1. *The following equalities are true:*

(a) $16W_n = (52W_0 - 47W_1 + 11W_2)G_{n+4} + (199W_1 - 216W_0 - 47W_2)G_{n+3} + 4(57W_0 - 54W_1 + 13W_2)G_{n+2}$.
(b) $4W_n = (11W_0 - 9W_1 + 2W_2)G_{n+3} + (40W_1 - 47W_0 - 9W_2)G_{n+2} + (52W_0 - 47W_1 + 11W_2)G_{n+1}$.
(c) $4W_n = (8W_0 - 5W_1 + W_2)G_{n+2} + (25W_1 - 36W_0 - 5W_2)G_{n+1} + 4(11W_0 - 9W_1 + 2W_2)G_n$.
(d) $W_n = W_0G_{n+1} + (-5W_0 + W_1)G_n + (8W_0 - 5W_1 + W_2)G_{n-1}$.
(e) $W_n = W_1G_n + (-5W_1 + W_2)G_{n-1} + 4W_0G_{n-2}$.
(f) $4(4W_0 - 4W_1 + W_2)(2W_0 - 3W_1 + W_2)^2G_n$
$= (8W_1^2 - 5W_1W_2 - 4W_0W_1 + W_2^2)W_{n+4} + (-36W_1^2 - 5W_2^2 + 20W_0W_1 - 4W_0W_2 + 25W_1W_2)W_{n+3} + 4(4W_0^2 + 16W_1^2 + 2W_2^2 - 16W_0W_1 + 5W_0W_2 - 11W_1W_2)W_{n+2}$.
(g) $(4W_0 - 4W_1 + W_2)(2W_0 - 3W_1 + W_2)^2G_n$
$= (W_1^2 - W_0W_2)W_{n+3} + (4W_0^2 - 8W_0W_1 + 5W_0W_2 - W_1W_2)W_{n+2} + (8W_1^2 + W_2^2 - 4W_0W_1 - 5W_1W_2)W_{n+1}$.
(h) $(4W_0 - 4W_1 + W_2)(2W_0 - 3W_1 + W_2)^2G_n$
$= (4W_0^2 + 5W_1^2 - 8W_0W_1 - W_1W_2)W_{n+2} + (W_2^2 - 4W_0W_1 + 8W_0W_2 - 5W_1W_2)W_{n+1} + 4(W_1^2 - W_0W_2)W_n$.
(i) $(4W_0 - 4W_1 + W_2)(2W_0 - 3W_1 + W_2)^2G_n$
$= (20W_0^2 + 25W_1^2 + W_2^2 - 44W_0W_1 + 8W_0W_2 - 10W_1W_2)W_{n+1} + 4(-8W_0^2 + 16W_0W_1 - W_2W_0 - 9W_1^2 + 2W_2W_1)W_n + 4(4W_0^2 + 5W_1^2 - 8W_0W_1 - W_1W_2)W_{n-1}$.
(j) $(4W_0 - 4W_1 + W_2)(2W_0 - 3W_1 + W_2)^2G_n$
$= (68W_0^2 + 89W_1^2 + 5W_2^2 - 156W_0W_1 + 36W_0W_2 - 42W_1W_2)W_n + 4(-36W_0^2 + 80W_0W_1 - 16W_0W_2 - 45W_1^2 + 19W_1W_2 - 2W_2^2)W_{n-1} + 4(20W_0^2 + 25W_1^2 + W_2^2 - 44W_0W_1 + 8W_0W_2 - 10W_1W_2)W_{n-2}$.

Proof. Note that all the identities hold for all integers n. We prove (a). To show (a), writing

$$W_n = a \times G_{n+4} + b \times G_{n+3} + c \times G_{n+2}$$

and solving the system of equations

$$W_0 = a \times G_4 + b \times G_3 + c \times G_2$$
$$W_1 = a \times G_5 + b \times G_4 + c \times G_3$$
$$W_2 = a \times G_6 + b \times G_5 + c \times G_4$$

we find that $a = \frac{1}{16}(52W_0 - 47W_1 + 11W_2), b = -\frac{1}{16}(216W_0 - 199W_1 + 47W_2), c = \frac{1}{4}(57W_0 - 54W_1 + 13W_2)$. The other equalities can be proved similarly. □
Note that all the identities in the above Lemma 4.1 can be proved by induction as well.
Next, we present a few basic relations between $\{W_n\}$ and $\{H_n\}$.

Lemma 4.2. *The following equalities are true:*

(a) $2(2W_0 - 3W_1 + W_2)(4W_0 - 4W_1 + W_2)H_n = (8W_0 - 10W_1 + 3W_2)W_{n+4} + (36W_1 - 28W_0 - 11W_2)W_{n+3} + 2(12W_0 - 16W_1 + 5W_2)W_{n+2}$.
(b) $(2W_0 - 3W_1 + W_2)(4W_0 - 4W_1 + W_2)H_n = (6W_0 - 7W_1 + 2W_2)W_{n+3} + (24W_1 - 20W_0 - 7W_2)W_{n+2} + 2(8W_0 - 10W_1 + 3W_2)W_{n+1}$.
(c) $(2W_0 - 3W_1 + W_2)(4W_0 - 4W_1 + W_2)H_n = (10W_0 - 11W_1 + 3W_2)W_{n+2} + 2(18W_1 - 16W_0 - 5W_2)W_{n+1} + 4(6W_0 - 7W_1 + 2W_2)W_n$.
(d) $(2W_0 - 3W_1 + W_2)(4W_0 - 4W_1 + W_2)H_n = (18W_0 - 19W_1 + 5W_2)W_{n+1} + 4(15W_1 - 14W_0 - 4W_2)W_n + 4(10W_0 - 11W_1 + 3W_2)W_{n-1}$.
(e) $(2W_0 - 3W_1 + W_2)(4W_0 - 4W_1 + W_2)H_n = (34W_0 - 35W_1 + 9W_2)W_n + 4(27W_1 - 26W_0 - 7W_2)W_{n-1} + 4(18W_0 - 19W_1 + 5W_2)W_{n-2}$.

Now, we give a few basic relations between $\{W_n\}$ and $\{R_n\}$.

Lemma 4.3. *The following equalities are true:*

(a) $8W_n = (42W_1 - 39W_0 - 11W_2)R_{n+4} + (151W_0 - 161W_1 + 42W_2)R_{n+3} + (151W_1 - 144W_0 - 39W_2)R_{n+2}$.
(b) $8W_n = (49W_1 - 44W_0 - 13W_2)R_{n+3} + (168W_0 - 185W_1 + 49W_2)R_{n+2} + 4(42W_1 - 39W_0 - 11W_2)R_{n+1}$.
(c) $2W_n = (15W_1 - 13W_0 - 4W_2)R_{n+2} + (49W_0 - 56W_1 + 15W_2)R_{n+1} + (49W_1 - 44W_0 - 13W_2)R_n$.
(d) $2W_n = (19W_1 - 16W_0 - 5W_2)R_{n+1} + (60W_0 - 71W_1 + 19W_2)R_n + 4(15W_1 - 13W_0 - 4W_2)R_{n-1}$.
(e) $W_n = (12W_1 - 10W_0 - 3W_2)R_n + 2(19W_0 - 23W_1 + 6W_2)R_{n-1} + 2(19W_1 - 16W_0 - 5W_2)R_{n-2}$.
(f) $2(4W_0 - 4W_1 + W_2)(2W_0 - 3W_1 + W_2)^2R_n$
$= (-12W_0^2 + 36W_0W_1 - 13W_0W_2 - 26W_1^2 + 18W_1W_2 - 3W_2^2)W_{n+4} + (52W_0^2 + 108W_1^2 + 12W_2^2 - 152W_0W_1 + 53W_0W_2 - 73W_1W_2)W_{n+3} + (-48W_0^2 + 140W_0W_1 - 48W_0W_2 - 100W_1^2 + 67W_1W_2 - 11W_2^2)W_{n+2}$.
(g) $2(4W_0 - 4W_1 + W_2)(2W_0 - 3W_1 + W_2)^2R_n$
$= (-8W_0^2 + 28W_0W_1 - 12W_0W_2 - 22W_1^2 + 17W_1W_2 - 3W_2^2)W_{n+3} + (48W_0^2 + 108W_1^2 + 13W_2^2 - 148W_0W_1 + 56W_0W_2 - 77W_1W_2)W_{n+2} + 4(-12W_0^2 + 36W_0W_1 - 13W_0W_2 - 26W_1^2 + 18W_1W_2 - 3W_2^2)W_{n+1}$.
(h) $(4W_0 - 4W_1 + W_2)(2W_0 - 3W_1 + W_2)^2R_n$
$= (4W_0^2 - W_1^2 - W_2^2 - 4W_0W_1 - 2W_0W_2 + 4W_1W_2)W_{n+2} + 2(4W_0^2 + 18W_1^2 + 3W_2^2 - 20W_0W_1 + 11W_0W_2 - 16W_1W_2)W_{n+1} + 2(-8W_0^2 + 28W_0W_1 - 12W_0W_2 - 22W_1^2 + 17W_1W_2 - 3W_2^2)W_n$.
(i) $(4W_0 - 4W_1 + W_2)(2W_0 - 3W_1 + W_2)^2R_n$
$= (28W_0^2 + 31W_1^2 + W_2^2 - 60W_0W_1 + 12W_0W_2 - 12W_1W_2)W_{n+1} + 2(-24W_0^2 + 44W_0W_1 - 4W_0W_2 - 18W_1^2 + W_1W_2 + W_2^2)W_n + 4(4W_0^2 - W_1^2 - W_2^2 - 4W_0W_1 - 2W_0W_2 + 4W_1W_2)W_{n-1}$.

(j) $(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2R_n$
$=(92W_0^2+119W_1^2+7W_2^2-212W_0W_1+52W_0W_2-58W_1W_2)W_n+4(-52W_0^2+116W_0W_1-26W_0W_2-63W_1^2+28W_1W_2-3W_2^2)W_{n-1}+4(28W_0^2+31W_1^2+W_2^2-60W_0W_1+12W_0W_2-12W_1W_2)W_{n-2}$.

Next, we present a few basic relations between $\{W_n\}$ and $\{C_n\}$.

Lemma 4.4. *The following equalities are true:*

(a) $8W_n=(25W_0-22W_1+5W_2)C_{n+4}+(95W_1-105W_0-22W_2)C_{n+3}+(112W_0-105W_1+25W_2)C_{n+2}$.
(b) $8W_n=(20W_0-15W_1+3W_2)C_{n+3}+(71W_1-88W_0-15W_2)C_{n+2}+4(25W_0-22W_1+5W_2)C_{n+1}$.
(c) $2W_n=(3W_0-W_1)C_{n+2}+(8W_1-15W_0-W_2)C_{n+1}+(20W_0-15W_1+3W_2)C_n$.
(d) $2W_n=(3W_1-W_2)C_{n+1}+(3W_2-7W_1-4W_0)C_n+4(3W_0-W_1)C_{n-1}$.
(e) $W_n=(4W_1-2W_0-W_2)C_n+2(3W_0-7W_1+2W_2)C_{n-1}+2(3W_1-W_2)C_{n-2}$.
(f) $2(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2C_n$
$=(4W_0^2+10W_1^2+W_2^2-12W_0W_1+3W_0W_2-6W_1W_2)W_{n+4}+(-12W_0^2+40W_0W_1-11W_0W_2-36W_1^2+23W_1W_2-4W_2^2)W_{n+3}+(16W_0^2+44W_1^2+5W_2^2-52W_0W_1+16W_0W_2-29W_1W_2)W_{n+2}$.
(g) $2(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2C_n$
$=(8W_0^2+14W_1^2+W_2^2-20W_0W_1+4W_0W_2-7W_1W_2)W_{n+3}+(-16W_0^2+44W_0W_1-8W_0W_2-36W_1^2+19W_1W_2-3W_2^2)W_{n+2}+4(4W_0^2+10W_1^2+W_2^2-12W_0W_1+3W_0W_2-6W_1W_2)W_{n+1}$.
(h) $(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2C_n$
$=(12W_0^2+17W_1^2+W_2^2-28W_0W_1+6W_0W_2-8W_1W_2)W_{n+2}+2(-12W_0^2+28W_0W_1-5W_0W_2-18W_1^2+8W_1W_2-W_2^2)W_{n+1}+2(8W_0^2+14W_1^2+W_2^2-20W_0W_1+4W_0W_2-7W_1W_2)W_n$.
(i) $(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2C_n$
$=(36W_0^2+49W_1^2+3W_2^2-84W_0W_1+20W_0W_2-24W_1W_2)W_{n+1}+2(-40W_0^2+92W_0W_1-20W_0W_2-54W_1^2+25W_1W_2-3W_2^2)W_n+4(12W_0^2+17W_1^2+W_2^2-28W_0W_1+6W_0W_2-8W_1W_2)W_{n-1}$.
(j) $(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2C_n$
$=(100W_0^2+137W_1^2+9W_2^2-236W_0W_1+60W_0W_2-70W_1W_2)W_n+4(-60W_0^2+140W_0W_1-34W_0W_2-81W_1^2+40W_1W_2-5W_2^2)W_{n-1}+4(36W_0^2+49W_1^2+3W_2^2-84W_0W_1+20W_0W_2-24W_1W_2)W_{n-2}$.

Now, we give a few basic relations between $\{G_n\}$ and $\{H_n\}$.

Lemma 4.5. *The following equalities are true:*

$$\begin{aligned}
4H_n &= 5G_{n+4}-19G_{n+3}+18G_{n+2},\\
2H_n &= 3G_{n+3}-11G_{n+2}+10G_{n+1},\\
H_n &= 2G_{n+2}-7G_{n+1}+6G_n,\\
H_n &= 3G_{n+1}-10G_n+8G_{n-1},\\
H_n &= 5G_n-16G_{n-1}+12G_{n-2}.
\end{aligned}$$

Next, we present a few basic relations between $\{G_n\}$ and $\{R_n\}$.

Lemma 4.6. *The following equalities are true:*

$$\begin{aligned}
8G_n &= -13R_{n+4}+49R_{n+3}-44R_{n+2},\\
2G_n &= -4R_{n+3}+15R_{n+2}-13R_{n+1},\\
2G_n &= -5R_{n+2}+19R_{n+1}-16R_n,\\
G_n &= -3R_{n+1}+12R_n-10R_{n-1},\\
G_n &= -3R_n+14R_{n-1}-12R_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
8R_n &= -11G_{n+4}+43G_{n+3}-40G_{n+2},\\
2R_n &= -3G_{n+3}+12G_{n+2}-11G_{n+1},\\
2R_n &= -3G_{n+2}+13G_{n+1}-12G_n,\\
R_n &= -G_{n+1}+6G_n-6G_{n-1},\\
R_n &= G_n+2G_{n-1}-4G_{n-2}.
\end{aligned}$$

Now, we give a few basic relations between $\{G_n\}$ and $\{C_n\}$.

Lemma 4.7. *The following equalities are true:*

$$\begin{aligned}
8G_n &= 3C_{n+4}-15C_{n+3}+20C_{n+2},\\
2G_n &= -C_{n+2}+3C_{n+1},\\
G_n &= -C_{n+1}+4C_n-2C_{n-1},\\
G_n &= -C_n+6C_{n-1}-4C_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
8C_n &= 5G_{n+4} - 21G_{n+3} + 24G_{n+2},\\
2C_n &= G_{n+3} - 4G_{n+2} + 5G_{n+1},\\
2C_n &= G_{n+2} - 3G_{n+1} + 4G_n,\\
C_n &= G_{n+1} - 2G_n + 2G_{n-1},\\
C_n &= 3G_n - 6G_{n-1} + 4G_{n-2}.
\end{aligned}$$

Next, we present a few basic relations between $\{H_n\}$ and $\{R_n\}$.

Lemma 4.8. *The following equalities are true:*

$$\begin{aligned}
4H_n &= -3R_{n+4} + 13R_{n+3} - 14R_{n+2},\\
2H_n &= -R_{n+3} + 5R_{n+2} - 6R_{n+1},\\
H_n &= R_{n+1} - 2R_n,\\
H_n &= 3R_n - 8R_{n-1} + 4R_{n-2}.
\end{aligned}$$

Now, we give a few basic relations between $\{H_n\}$ and $\{C_n\}$.

Lemma 4.9. *The following equalities are true:*

$$\begin{aligned}
4H_n &= 5C_{n+4} - 19C_{n+3} + 18C_{n+2},\\
2H_n &= 3C_{n+3} - 11C_{n+2} + 10C_{n+1},\\
H_n &= 2C_{n+2} - 7C_{n+1} + 6C_n,\\
H_n &= 3C_{n+1} - 10C_n + 8C_{n-1},\\
H_n &= 5C_n - 16C_{n-1} + 12C_{n-2}.
\end{aligned}$$

Next, we present a few basic relations between $\{R_n\}$ and $\{C_n\}$.

Lemma 4.10. *The following equalities are true:*

$$\begin{aligned}
4R_n &= -6C_{n+4} + 23C_{n+3} - 21C_{n+2},\\
4R_n &= -7C_{n+3} + 27C_{n+2} - 24C_{n+1},\\
R_n &= -2C_{n+2} + 8C_{n+1} - 7C_n,\\
R_n &= -2C_{n+1} + 9C_n - 8C_{n-1},\\
R_n &= -C_n + 8C_{n-1} - 8C_{n-2},
\end{aligned}$$

and

$$\begin{aligned}
4C_n &= -6R_{n+4} + 23R_{n+3} - 21R_{n+2},\\
4C_n &= -7R_{n+3} + 27R_{n+2} - 24R_{n+1},\\
C_n &= -2R_{n+2} + 8R_{n+1} - 7R_n,\\
C_n &= -2R_{n+1} + 9R_n - 8R_{n-1},\\
C_n &= -R_n + 8R_{n-1} - 8R_{n-2}.
\end{aligned}$$

5. On the Recurrence Properties of Generalized Woodall Sequence

Taking $r = 5, s = -8, t = 4$ in Theorem 1.6, we obtain the following Proposition.

Proposition 5.1. *For $n \in \mathbb{Z}$, generalized Woodall numbers (the case $r = 5, s = -8, t = 4$) have the following identity:*

$$W_{-n} = 4^{-n}(W_{2n} - H_n W_n + \frac{1}{2}(H_n^2 - H_{2n})W_0)$$

where

$$H_n = \frac{((10W_0 - 11W_1 + 3W_2)W_{n+2} - 2(16W_0 - 18W_1 + 5W_2)W_{n+1} + 4(6W_0 - 7W_1 + 2W_2)W_n)}{(2W_0 - 3W_1 + W_2)(4W_0 - 4W_1 + W_2)} \tag{5.1}$$

Note that if we take $r = 5, s = -8, t = 4$ in Lemma 1.7 (or using Lemma 4.2 (c)) we get (5.1).
From the above Proposition 5.1 and Corollary 1.8, we have the following Corollary 5.2 which gives the connection between the special cases of generalized Woodall sequence at the positive index and the negative index: for modified Woodall, modified Cullen, Woodall and Cullen numbers: take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = 5$, take $W_n = H_n$ with $H_0 = 3, H_1 = 5, H_2 = 9$, $W_n = R_n$ with $R_0 = -1, R_1 = 1, R_2 = 7$ and $W_n = C_n$ with $C_0 = 1, C_1 = 3, C_2 = 9$, respectively. Note that in this case $H_n = H_n$.

Corollary 5.2. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a) *Modified Woodall sequence:*

$$G_{-n} = 4^{-n}(-6G_n^2 + G_{2n} - 2G_{n+2}G_n + 7G_{n+1}G_n).$$

(b) *Modified Cullen sequence:*

$$H_{-n} = 2^{-2n-1}\left(H_n^2 - H_{2n}\right).$$

(c) *Woodall sequence:*

$$R_{-n} = 2^{-2n-1}(-R_{n+1}^2 + R_{2n+1} + 2R_{n+1}R_n).$$

(d) *Cullen sequence:*

$$C_{-n} = 2^{-2n-1}(4C_{n+2}^2 + 49C_{n+1}^2 + 24C_n^2 - 2C_{2n+2} + 7C_{2n+1} - 4C_{2n} - 28C_{n+1}C_{n+2} + 20C_nC_{n+2} - 70C_nC_{n+1}).$$

6. Sum Formulas

The following Theorem 6.1 presents some formulas of of generalized Woodall numbers numbers with indices in arithmetic progression.

Theorem 6.1. *For all integers m and j, we have the following sum formula:*

$$\sum_{k=0}^{n} W_{mk+j} = \frac{1}{2\left(2^m-1\right)^2}(\Gamma_1 + \Gamma_2 + \Gamma_3)$$

where

$$\begin{aligned}
\Gamma_1 &= ((j+mn-2)2^{mn+2m+j} - (j+m+mn-2)2^{mn+m+j} + (m-j+2)2^{m+j} + (j-2)2^j + 2(n+1)(2^m-1)^2)W_2,\\
\Gamma_2 &= (-(3j+3mn-8)2^{mn+2m+j} + (3j+3m+3mn-8)2^{mn+m+j} + (3j-3m-8)2^{m+j} - (3j-8)2^j - 8(n+1)(2^m-1)^2)W_1,\\
\Gamma_3 &= 2((j+mn-3)2^{mn+2m+j} - (j+m+mn-3)2^{mn+m+j} + (m-j+3)2^{m+j} + (j-3)2^j + 4(n+1)(2^m-1)^2)W_0.
\end{aligned}$$

Proof. Use the Binet's formula of generalized Woodall numbers, i.e.,

$$W_n = ((-W_2 + 4W_1 - 3W_0) + \frac{W_2 - 3W_1 + 2W_0}{2}n) \times 2^n + (W_2 - 4W_1 + 4W_0). \quad \square$$

The following Proposition 6.2 presents some formulas of generalized Woodall numbers numbers with positive subscripts.

Proposition 6.2. *For $n \geq 0$, we have the following formulas:*

(a) $\sum_{k=0}^{n} W_k = ((n-3)2^n + n + 3)W_2 - ((3n-11)2^n + 4n + 11)W_1 + ((n-4)2^{n+1} + 4n + 9)W_0.$
(b) $\sum_{k=0}^{n} W_{2k} = \frac{1}{9}(((3n-4)2^{2n+2} + 9n + 16)W_2 - 12((3n-5)2^{2n} + 3n + 5)W_1 + ((6n-11)2^{2n+2} + 36n + 53)W_0).$
(c) $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{9}(((6n-5)2^{2n+2} + 9n + 20)W_2 - 3((6n-7)2^{2n+2} + 12n + 25)W_1 + 4((3n-4)2^{2n+2} + 9n + 16)W_0).$

Proof. Take $m = 1, j = 0$; $m = 2, j = 0$ and $m = 2, j = 1$, respectively, in Theorem 6.1. $\square$
From Theorem 6.1, we have the following Corollary.

Corollary 6.3. *For all integers m and j, we have the following sum formulas:*

(a) $\sum_{k=0}^{n} G_{mk+j} = \frac{1}{(2^m-1)^2}((j+mn-1)2^{mn+2m+j} - (j+m+mn-1)2^{mn+m+j} + (n+1)2^{2m} - (n+1)2^{m+1} - (j-m-1)2^{m+j}$
$+(j-1)2^j + n + 1).$
(b) $\sum_{k=0}^{n} H_{mk+j} = \frac{1}{(2^m-1)}(2^{mn+m+j+1} + (n+1)2^m - 2^{j+1} - n - 1).$
(c) $\sum_{k=0}^{n} R_{mk+j} = \frac{1}{(2^m-1)^2}((j+mn)2^{mn+2m+j} - (j+m+mn)2^{mn+m+j} - (n+1)2^{2m} + (n+1)2^{m+1} + (m-j)2^{m+j} + 2^j j - n - 1).$
(d) $\sum_{k=0}^{n} C_{mk+j} = \frac{1}{(2^m-1)^2}((j+mn)2^{mn+2m+j} - (j+m+mn)2^{mn+m+j} + (n+1)2^{2m} - (n+1)2^{m+1} + (m-j)2^{m+j} + 2^j j + n + 1).$

From the last Proposition 6.2 (or using Corollary 6.3), we have the following Corollary 6.4 which gives sum formulas of modified Woodall numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = 5$).

Corollary 6.4. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} G_k = (n-2)2^{n+1} + n + 4.$
(b) $\sum_{k=0}^{n} G_{2k} = \frac{1}{9}((6n-5)2^{2n+2} + 9n + 20).$
(c) $\sum_{k=0}^{n} G_{2k+1} = \frac{1}{9}((3n-1)2^{2n+4} + 9n + 25).$

Taking $W_n = H_n$ with $H_0 = 3, H_1 = 5, H_2 = 9$ in the last Proposition 6.2 (or using Corollary 6.3), we have the following Corollary 6.5 which presents sum formulas of modified Cullen numbers.

Corollary 6.5. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} H_k = 2^{n+2} + n - 1.$
(b) $\sum_{k=0}^{n} H_{2k} = \frac{1}{3}(2^{2n+3} + 3n + 1).$
(c) $\sum_{k=0}^{n} H_{2k+1} = \frac{1}{3}(2^{2n+4} + 3n - 1).$

From the last Proposition 6.2 (or using Corollary 6.3), we have the following Corollary 6.6 which gives sum formulas of Woodall numbers (take $W_n = R_n$ with $R_0 = -1, R_1 = 1, R_2 = 7$).

Corollary 6.6. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} R_k = (n-1)(2^{n+1}-1)$.
(b) $\sum_{k=0}^{n} R_{2k} = \frac{1}{9}((3n-1)2^{2n+3}-9n-1)$.
(c) $\sum_{k=0}^{n} R_{2k+1} = \frac{1}{9}((6n+1)2^{2n+3}-9n+1)$.

Taking $W_n = C_n$ with $C_0 = 1, C_1 = 3, C_2 = 9$ in the last Proposition 6.2, we have the following Corollary 6.7 which presents sum formulas of Cullen numbers.

Corollary 6.7. *For $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} C_k = (n-1)2^{n+1}+n+3$.
(b) $\sum_{k=0}^{n} C_{2k} = \frac{1}{9}((3n-1)2^{2n+3}+9n+17)$.
(c) $\sum_{k=0}^{n} C_{2k+1} = \frac{1}{9}((6n+1)2^{2n+3}+9n+19)$.

7. Matrices Related With Generalized Woodall numbers

We define the square matrix A of order 3 as:

$$A = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 4$. From (2.1) we have

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} W_{n+1} \\ W_n \\ W_{n-1} \end{pmatrix} \tag{7.1}$$

and from (1.6) (or using (7.1) and induction) we have

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}.$$

If we take $W = G$ in (7.1) we have

$$\begin{pmatrix} G_{n+2} \\ G_{n+1} \\ G_n \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} G_{n+1} \\ G_n \\ G_{n-1} \end{pmatrix}.$$

We also define

$$B_n = \begin{pmatrix} G_{n+1} & -8G_n+4G_{n-1} & 4G_n \\ G_n & -8G_{n-1}+4G_{n-2} & 4G_{n-1} \\ G_{n-1} & -8G_{n-2}+4G_{n-3} & 4G_{n-2} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} W_{n+1} & -8W_n+4W_{n-1} & 4W_n \\ W_n & -8W_{n-1}+4W_{n-2} & 4W_{n-1} \\ W_{n-1} & -8W_{n-2}+4W_{n-3} & 4W_{n-2} \end{pmatrix}$$

Theorem 7.1. *For all integer $m, n \geq 0$, we have*

(a) $B_n = A^n$
(b) $C_1 A^n = A^n C_1$
(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof. Take $r = 5, s = -8, t = 4$ in Soykan [25, Theorem 5.1.]. □
Some properties of matrix A^n can be given as

$$A^n = 5A^{n-1} - 8A^{n-2} + 4A^{n-3}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = 4^n$$

for all integer m and n.

Corollary 7.2. *For all integers n, we have the following formulas for the modified Woodall, Woodall and Cullen numbers.*

(a) *Modified Woodall Numbers.*

$$A^n=\begin{pmatrix}5&-8&4\\1&0&0\\0&1&0\end{pmatrix}^n=\begin{pmatrix}G_{n+1}&-8G_n+4G_{n-1}&4G_n\\G_n&-8G_{n-1}+4G_{n-2}&4G_{n-1}\\G_{n-1}&-8G_{n-2}+4G_{n-3}&4G_{n-2}\end{pmatrix}$$
$$=\begin{pmatrix}2^{n+1}n+1&4\times 2^n-6\times 2^n n-4&4\times 2^n n-4\times 2^n+4\\2^n n-2^n+1&5\times 2^n-3\times 2^n n-4&2\times 2^n n-4\times 2^n+4\\\frac{1}{2}2^n n-2^n+1&2^{n+2}-\frac{3}{2}2^n n-4&2^n n-3\times 2^n+4\end{pmatrix}.$$

(b) *Woodall Numbers.*

$$A^n=\frac{1}{2}\begin{pmatrix}-5R_{n+3}+19R_{n+2}-16R_{n+1}&24R_{n+2}-92R_{n+1}+76R_n&4(-5R_{n+2}+19R_{n+1}-16R_n)\\-5R_{n+2}+19R_{n+1}-16R_n&24R_{n+1}-92R_n+76R_{n-1}&4(-5R_{n+1}+19R_n-16R_{n-1})\\-5R_{n+1}+19R_n-16R_{n-1}&24R_n-92R_{n-1}+76R_{n-2}&4(-5R_n+19R_{n-1}-16R_{n-2})\end{pmatrix}.$$

(c) *Cullen Numbers.*

$$A^n=\begin{pmatrix}-C_{n+2}+4C_{n+1}-2C_n&6C_{n+1}-26C_n+16C_{n-1}&4(-C_{n+1}+4C_n-2C_{n-1})\\-C_{n+1}+4C_n-2C_{n-1}&6C_n-26C_{n-1}+16C_{n-2}&4(-C_n+4C_{n-1}-2C_{n-2})\\-C_n+4C_{n-1}-2C_{n-2}&6C_{n-1}-26C_{n-2}+16C_{n-3}&4(-C_{n-1}+4C_{n-2}-2C_{n-3})\end{pmatrix}.$$

Proof.

(a) It is given in Theorem 7.1 (a).
(b) Note that, from Lemma 4.6, we have

$$2G_n=-5R_{n+2}+19R_{n+1}-16R_n.$$

Using the last equation and (a), we get required result.
(c) Note that, from Lemma 4.7, we have

$$G_n=-C_{n+1}+4C_n-2C_{n-1}.$$

Using the last equation and (a), we get required result. □

Theorem 7.3. *For all integers m,n, we have*

$$\begin{aligned}W_{n+m}&=W_nG_{m+1}+W_{n-1}(-8G_m+4G_{m-1})+4W_{n-2}G_m\\&=W_nG_{m+1}+(-8W_{n-1}+4W_{n-2})G_m+4W_{n-1}G_{m-1}\end{aligned}\tag{7.2}$$

Proof. Take $r=5,s=-8,t=4$ in Soykan [25, Theorem 5.2.]. □
By Lemma 4.1, we know that

$$\begin{aligned}(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2G_m&=(4W_0^2+5W_1^2-8W_0W_1-W_1W_2)W_{m+2}\\&+(W_2^2-4W_0W_1+8W_0W_2-5W_1W_2)W_{m+1}+4(W_1^2-W_0W_2)W_m.\end{aligned}$$

so (7.2) can be written in the following form
$(4W_0-4W_1+W_2)(2W_0-3W_1+W_2)^2W_{n+m}=W_n((4W_0^2+5W_1^2-8W_0W_1-W_1W_2)W_{m+3}+(W_2^2-4W_0W_1+8W_0W_2-5W_1W_2)W_{m+2}+4(W_1^2-W_0W_2)W_{m+1})+(-8W_{n-1}+4W_{n-2})((4W_0^2+5W_1^2-8W_0W_1-W_1W_2)W_{m+2}$
$+(W_2^2-4W_0W_1+8W_0W_2-5W_1W_2)W_{m+1}+4(W_1^2-W_0W_2)W_m)+4W_{n-1}((4W_0^2+5W_1^2-8W_0W_1-W_1W_2)W_{m+1}+(W_2^2-4W_0W_1+8W_0W_2-5W_1W_2)W_m+4(W_1^2-W_0W_2)W_{m-1})$.

Corollary 7.4. *For all integers m,n, we have*

$$\begin{aligned}G_{n+m}&=G_nG_{m+1}+G_{n-1}(-8G_m+4G_{m-1})+4G_{n-2}G_m,\\H_{n+m}&=H_nG_{m+1}+H_{n-1}(-8G_m+4G_{m-1})+4H_{n-2}G_m,\\R_{n+m}&=R_nG_{m+1}+R_{n-1}(-8G_m+4G_{m-1})+4R_{n-2}G_m,\\C_{n+m}&=C_nG_{m+1}+C_{n-1}(-8G_m+4G_{m-1})+4C_{n-2}G_m,\end{aligned}$$

and

$$\begin{aligned}2R_{m+n}&=-5R_nR_{m+3}+(19R_n+40R_{n-1}-20R_{n-2})R_{m+2}\\&\quad+4(-4R_n-43R_{n-1}+19R_{n-2})R_{m+1}+4(51R_{n-1}-16R_{n-2})R_m-64R_{n-1}R_{m-1},\\2C_{m+n}&=-C_nC_{m+3}+(3C_n+8C_{n-1}-4C_{n-2})C_{m+2}\\&\quad+4(-7C_{n-1}+3C_{n-2})C_{m+1}+12C_{n-1}C_m.\end{aligned}$$

Taking $m = n$ in the last Corollary we obtain the following identities:

$$
\begin{aligned}
G_{2n} &= 4G_{n-1}^2 + (G_{n+1} - 8G_{n-1} + 4G_{n-2})G_n, \\
H_{2n} &= H_nG_{n+1} - 4\left(2H_{n-1} - H_{n-2}\right)G_n + 4H_{n-1}G_{n-1}, \\
R_{2n} &= R_nG_{n+1} - 4\left(2R_{n-1} - R_{n-2}\right)G_n + 4R_{n-1}G_{n-1}, \\
C_{2n} &= C_nG_{n+1} - 4\left(2C_{n-1} - C_{n-2}\right)G_n + 4C_{n-1}G_{n-1},
\end{aligned}
$$

and

$$
\begin{aligned}
2R_{2n} &= -5R_nR_{n+3} + (19R_n + 40R_{n-1} - 20R_{n-2})R_{n+2} + 4(-4R_n - 43R_{n-1} + 19R_{n-2})R_{n+1} \\
&\quad + 4(51R_{n-1} - 16R_{n-2})R_n - 64R_{n-1}R_{n-1}, \\
2C_{2n} &= -C_nC_{n+3} + (3C_n + 8C_{n-1} - 4C_{n-2})C_{n+2} + 4(-7C_{n-1} + 3C_{n-2})C_{n+1} + 12C_{n-1}C_n.
\end{aligned}
$$

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