International Electronic Journal of Algebra
Volume 31 (2022) 13-37
DOI: 10.24330/ieja.1058385

ON LATTICES ASSOCIATED TO RINGS WITH RESPECT TO A PREADICAL

Erwin Cerda-León and Hugo Rincón-Mejía

Received: 1 August 2020; Revised: 14 May 2021; Accepted: 21 May 2021
Communicated by Christian Lomp

Abstract. We introduce some new lattices of classes of modules with respect to appropriate preradicals. We introduce some concepts associated with these lattices, such as the σ-semiartinian rings, the σ-retractable modules, the σ-V-rings, the σ-max rings. We continue to study σ-torsion theories, σ-open classes, σ-stable classes. We prove some theorems that extend some known results. Our results fall into well known situations when the preradical σ is chosen as the identity preradical.

Mathematics Subject Classification (2020): 16W30
Keywords: Preradical, lattice of module class, hereditary torsion class, boolean lattice, V-ring, max-ring

1. Introduction

Lattices and big lattices of module classes has been studied to obtain information about the underlying ring R and about its associated module category. For example, the big lattice of preradicals and some associated lattices of special kinds of preradicals have provided a wealth of information about the rings and their module categories.

Similar considerations can be made about module classes lattices defined by closure properties. Some examples of these lattices are: the lattice of the natural classes, the lattice of the hereditary torsion classes, the lattice of Serre classes, that of the Wisbauer classes and some others.

In [6], the big lattices of module classes induced by a preradical σ over R-Mod were introduced, for example, the lattices of σ-hereditary classes, of σ-cohereditary classes, of σ-natural classes, and of σ-conatural classes. Note that the σ-open classes lattice and the σ-torsion theories lattice were also introduced in the same paper.

Our objective in this work is to introduce some new lattices of module classes with respect to a preradical σ, to use these lattices to set properties for rings and for

This research was supported by a grant from the PostDoctoral Scholarship Program at the National Autonomous University of Mexico.

their module categories. We introduce the σ-semiartinian rings, the σ-retractable modules, σ-V-rings, and σ-max rings. We extend some well known results in the literature.

2. Preliminaries

2.1. Preradicals and classes of modules. In this section, we present basic results about preradicals on R-Mod and about classes of modules. For more information about preradicals, see [5], [11] and [14]. We refer to [1], [4], and [7], for basic results about hereditary, cohereditary, natural, conatural and open classes.

A preradical on R-mod is an assignment $\sigma : R\text{-Mod} \to R\text{-Mod}$ such that for each $M \in R$-Mod, $\sigma(M) \leq M$ and for each R-morphism $f : M \to N$, $f(\sigma(M)) \leq \sigma(N)$. σ is a subfunctor of the identity functor on R-Mod. R-pr denotes the collection of all preradicals on R-Mod.

In R-pr, we have two binary operations, one of them called the *product*, denoted with "$\cdot$" and the other called the *coproduct* denoted with ":" given by:

$$(\sigma \cdot \tau)(M) = \sigma(\tau(M)),$$

$$(\sigma : \tau)(M) \text{ is defined by } (\sigma : \tau)(M)/\sigma(M) = \tau(M/\sigma(M)).$$

We will write $\sigma\tau$ instead of $\sigma \cdot \tau$. A preradical σ is called *idempotent* if $\sigma\sigma = \sigma$, and σ is called a *radical* if $(\sigma : \sigma) = \sigma$.

Let us recall that σ is a radical if and only if $\sigma(M/\sigma(M)) = 0$, for each $M \in R$-Mod. σ is *t-radical* if and only if $\sigma(M) = \sigma(R)M$. t-radicals are precisely the preradicals preserving epimorphisms. t-radicals are also called *cohereditary* radicals.

A preradical σ is a *left exact* preradical if it is a left exact functor. This is equivalent to the condition that for each submodule N of a module M we have that $\sigma(N) = N \cap \sigma(M)$. That σ is a left exact preradical it is also equivalent to that σ is an idempotent preradical and $\mathbb{T}_\sigma$ is a hereditary class.

We will denote R-id, R-rad, R-lep, R-radid the collections of idempotent preradicals, of radicals, of left exact preradicals, and of idempotent radicals, respectively.

Each preradical σ has associated the class $\mathbb{T}_\sigma = \{M \in R\text{-Mod} \mid \sigma(M) = M\}$. This class is closed under quotients and direct sums and it is called the σ-pretorsion class.

Let us recall that a class of R-modules is a *pretorsion free* class if it is closed under taking submodules and direct products. Each $\sigma \in R$-pr has associated the pretorsion free class $\mathbb{F}_\sigma = \{N \in R\text{-Mod} \mid \sigma(N) = 0\}$.

We say that a module M splits in a preradical σ if $M = \sigma(M) \oplus M'$ for some $M' \leq M$. Notice that in this case, $\sigma(M') = 0$, $\sigma(\sigma(M)) = \sigma(M)$ and $\sigma(M/\sigma(M)) = 0$.

$\sigma \in R$-pr is *stable* (*costable*) if for each injective (projective) module Q, Q splits in σ. This implies that $\sigma(Q)$ is an injective (projective) module. We say that σ *centrally splits* if for each R-module M we have that $M = \sigma(R)M \oplus M'$, with $M' = \{m \in M \mid \sigma(R)m = 0\}$, for further information, see [5], Chapter I.

We say that a two sided ideal I of a ring R is *pure* if $IJ = I \cap J$ for every ideal J of R. For a two sided ideal I we have that I is a pure ideal $\Leftrightarrow$ for each $M \in R$-Mod and $N \leq M$, $IN = N \cap IM \Leftrightarrow R/I$ is a flat module (see [14] Chap. I, §11) $\Leftrightarrow$ for all $a \in I$, $a \in Ia$. Notice that if I is a pure ideal, then the preradical $I \cdot -$ is exact.

Remark 2.1. ([14], page 157) Take $\sigma \in R$-pr. The following conditions are equivalent:

(1) σ is a t-radical.
(2) σ preserves epimorphisms.
(3) σ is a radical and $\mathbb{F}_\sigma$ is closed under quotient modules.

If σ is an exact preradical, then σ is a t-radical and $\sigma(R)$ is a pure ideal because for each $M \in R$-Mod and $N \leq M$ we have that $\sigma(R)N = \sigma(N) = N \cap \sigma(M) = N \cap \sigma(R)M$.
If $I \leq R$ is a pure ideal, then I defines an exact preradical σ by $\sigma(M) = IM$.

2.2. Classes of modules. A lattice L is *bounded* if it has a smallest element (usually denoted by $\mathbf{0}$) and a largest element (usually denoted by $\mathbf{1}$). In a lattice L with $\mathbf{0}$, an element a^* is a *pseudocomplement* of $a \in L$, if $a \wedge a^* = 0$ and a^* is maximal in L with respect to this property. We say that a^* is a *strong pseudocomplement* of a if it is the largest element in L with respect to $a \wedge a^* = 0$.

We will denote $Skel(L) = \{a^* \mid a \in L\}$ and we will call it the *skeleton of* L. In a bounded lattice L, we will say that $a^* \in L$ is a complement of $a \in L$ if $a^* \wedge a = 0$ and $a^* \vee a = 1$.

If L is a proper class instead of a set, we will say that L is a *big lattice*.

A class of left R-modules is called an abstract class if it is closed under taking isomorphic copies of its members. We consider some closure properties of a class of modules, like being closed under submodules, quotients, extensions, direct sums, injective hulls, products or projective covers, we will use the symbols $\leq$, $\twoheadrightarrow$, ext, $\oplus$, E, $\prod$, P respectively, to abbreviate. If A denotes a set of these closure properties, we denote $\mathscr{L}_A$ the proper class of classes of modules closed under each closure property in A. So, $\mathscr{L}_{\{\leq\}}$ denotes the proper class of hereditary classes in R-mod, $\mathscr{L}_{\{\leq,\oplus,E\}}$ denotes the class of natural classes, and so on.

We should notice that $\mathscr{L}_A$ becomes a complete big with inclusion of classes as the order and with infima given by intersections. We will denote $\xi_A(\mathscr{C})$ the least class in $\mathscr{L}_A$ containing $\mathscr{C}$ and by $\chi_A(\mathscr{C})$ the largest class in $\mathscr{L}_A$ contained in $\mathscr{C}$. Thus $\xi_{\{\leq\}}(\mathscr{C})$ denotes the hereditary closure of $\mathscr{C}$, and $\xi_{\{\twoheadrightarrow\}}(\mathscr{C})$ denotes the homomorphic image closure of $\mathscr{C}$. $\xi_{\{\leq\}}(\mathscr{C})$ will be denoted also $S(\mathscr{C})$ and $\xi_{\{\twoheadrightarrow\}}(\mathscr{C})$ will be denoted also $H(\mathscr{C})$.

The big lattice of torsion theories is denoted by R-TORS (see [14], Chapter VI), and the lattice of hereditary torsion theories is denoted by R-tors (see [9]). Often it will be convenient to identify each torsion theory with its torsion class, that is, R-TORS$= \mathscr{L}_{\{\twoheadrightarrow,\oplus,ext\}}$ and R-tors$= \mathscr{L}_{\{\leq,\twoheadrightarrow,\oplus,ext\}}$. We denote R-jtors $= \{\mathscr{C} \in R\text{-tors} \mid \mathscr{C} \in \mathscr{L}_{\{\prod\}}\}$.

For a module class $\mathfrak{a}$, we denote $\xi(\mathfrak{a})$ the least hereditary torsion theory containing $\mathfrak{a}$, and by $\chi(\mathfrak{a})$ the largest hereditary torsion theory such that each one of its modules has no nonzero submodules in $\mathfrak{a}$.

Remark 2.2. $\wp(R\text{-Mod}) := \{\mathscr{C} \mid \mathscr{C} \subseteq R\text{-Mod}\}$. Each $\sigma \in R$-pr define two assignments:

(1) $\sigma^* : \wp(R\text{-Mod}) \to \wp(R\text{-Mod})$, where $\sigma^*(\mathscr{C}) = \{\sigma(M) \mid M \in \mathscr{C}\}$.
(2) $\overleftarrow{\sigma}: \wp(R\text{-Mod}) \to \wp(R\text{-Mod})$, where $\overleftarrow{\sigma}(\mathscr{C}) = \{M \in R\text{-Mod} \mid \sigma(M) \in \mathscr{C}\}$.

Notice that $\sigma^*(\overleftarrow{\sigma}(\sigma^*(\mathscr{C}))) = \sigma^*(\mathscr{C})$ and $\overleftarrow{\sigma}(\sigma^*(\overleftarrow{\sigma}(\mathscr{C}))) = \overleftarrow{\sigma}(\mathscr{C})$ for each $\mathscr{C} \subseteq R$-Mod.

3. σ-(R-tors) and σ-(R-TORS)

Let us take a preradical σ. We will say that a class $\mathscr{C} \subseteq R$-Mod is *σ-hereditary* (*σ-cohereditary*) if it has the following two conditions: $\mathbb{F}_\sigma \subseteq \mathscr{C}$, and for each $M \in \mathscr{C}$ and $N \leq M$ ($M \twoheadrightarrow N$) it happens that $\sigma(N) \in \mathscr{C}$. We denote $\mathscr{L}_{\{\leq_\sigma\}}$ ($\mathscr{L}_{\{\twoheadrightarrow_\sigma\}}$) the collection of all hereditary σ-hereditary (σ-cohereditary) classes. $\mathscr{L}_{\{\leq_\sigma\}}$ is a bounded pseudocomplemented big lattice, whose least member is $\mathbb{F}_\sigma$ and whose largest member is R-Mod, where infima is given by class intersections. If σ is an idempotent preradical then the pseudocomplements are strong and $Skel(\mathscr{L}_{\{\leq_\sigma\}})$ is a boolean lattice. If σ is an idempotent cohereditary preradical, then $\mathscr{L}_{\{\twoheadrightarrow_\sigma\}}$ is a strongly pseudocomplemented big lattice. The big lattice of σ-open classes is denoted by $\mathscr{L}_{\{\leq_\sigma,\twoheadrightarrow_\sigma\}}$. If σ is an exact preradical, then $Skel(\mathscr{L}_{\{\leq_\sigma,\twoheadrightarrow_\sigma\}}) = \{\overleftarrow{\sigma}(\mathscr{C}) \mid \mathscr{C} \in Skel(\mathscr{L}_{\{\leq,\twoheadrightarrow\}})\}$ (see [6]).

Definition 3.1. Let σ be a preradical.

(1) R-(σ-TORS) $:= \mathscr{L}_{\{\twoheadrightarrow_\sigma,\oplus,ext\}}$.

(2) R-(σ-tors) $:= \mathscr{L}_{\{\leq_\sigma, \twoheadrightarrow_\sigma, \oplus, ext\}}$.
(3) $\mathscr{L}_{\{\sigma P\}} = \{\mathscr{C} \subseteq R\text{-Mod} \mid \forall\ M \in \mathscr{C}$, and for each projective cover $g : P(M) \to M$, $\sigma(P(M)) \in \mathscr{C}\}$.
(4) $\mathscr{L}_{\{\sigma E\}} = \{\mathscr{C} \subseteq R\text{-Mod} \mid \forall\ M \in \mathscr{C}, \sigma(E(M)) \in \mathscr{C}\}$.

Remark 3.2. If $\mathscr{C} \subseteq R$-Mod, then

$$\xi_{\{\leq\}}(\mathscr{C}) = \{N \in R\text{-Mod} \mid \exists\ N \rightarrowtail M,\ M \in \mathscr{C}\},$$

$$\xi_{\{\twoheadrightarrow\}}(\mathscr{C}) = \{L \in R\text{-Mod} \mid \exists\ M \twoheadrightarrow L,\ M \in \mathscr{C}\}.$$

Lemma 3.3. *Let σ be a radical and $\mathscr{C} \in \mathscr{L}_{\{ext\}}$ with $\mathscr{C} \supseteqq \mathbb{F}_\sigma$, then $\overleftarrow{\sigma}(\mathscr{C}) \subseteqq \mathscr{C}$.*

Proof. Take σ and $\mathscr{C}$ as in the statement. If $A \in \overleftarrow{\sigma}(\mathscr{C})$ then we have the exact sequence

$$0 \to \sigma(A) \to A \to A/\sigma(A) \to 0$$

with $\sigma(A) \in \mathscr{C}$ and with $A/\sigma(A) \in \mathbb{F}_\sigma \subseteq \mathscr{C}$. Thus $A \in \mathscr{C}$. □

Lemma 3.4. *If $\mathscr{C} \in \mathscr{L}_{\{\leq\}}$, then $\mathscr{C} \subseteq \overleftarrow{\sigma}(\mathscr{C})$.*

Proof. Let $\mathscr{C} \in \mathscr{L}_{\{\leq\}}$ and $M \in \mathscr{C}$, if $N \leq M$ then $N \in \mathscr{C}$, thus $\sigma(M) \in \mathscr{C}$ so that $M \in \overleftarrow{\sigma}(\mathscr{C})$. □

From the two previous lemmas it follows the following remark.

Remark 3.5. If $\mathscr{C} \in \mathscr{L}_{\{\leq, ext\}}$ and σ is a radical with $\mathbb{F}_\sigma \subseteq \mathscr{C}$, then $\mathscr{C} = \overleftarrow{\sigma}(\mathscr{C})$.

Theorem 3.6. *Let $\sigma \in R$-pr:*
(1) $\mathscr{C} \in \mathscr{L}_{\{\leq_\sigma\}} \Leftrightarrow \xi_{\leq}(\mathscr{C}) \subseteq \overleftarrow{\sigma}(\mathscr{C})$,
(2) $\mathscr{C} \in \mathscr{L}_{\{\twoheadrightarrow_\sigma\}} \Leftrightarrow \xi_{\twoheadrightarrow}(\mathscr{C}) \subseteq \overleftarrow{\sigma}(\mathscr{C})$.

Proof. (1) ($\Rightarrow$): Suppose that $\mathscr{C} \in \mathscr{L}_{\{\leq_\sigma\}}$ and $N \in \xi_{\leq}(\mathscr{C})$. Then there is a monomorphism $N \rightarrowtail M$ with $M \in \mathscr{C}$. Thus $\sigma(N) \in \mathscr{C}$, and $N \in \overleftarrow{\sigma}(\mathscr{C})$.

($\Leftarrow$): Suppose that $N \leq M$ with $M \in \mathscr{C}$, then $N \in \xi_{\leq}(\mathscr{C}) \subseteq \overleftarrow{\sigma}(\mathscr{C})$. Hence $\sigma(N) \in \mathscr{C}$. As it is clear that $\mathbb{F}_\sigma \subseteq \mathscr{C}$, we have that $\mathscr{C} \in \mathscr{L}_{\{\leq_\sigma\}}$.

(2) ($\Rightarrow$): Let us take $\mathscr{C} \in \mathscr{L}_{\{\twoheadrightarrow_\sigma\}}$ and $L \in \xi_{\twoheadrightarrow}(\mathscr{C})$, there is an epimorphism $M \twoheadrightarrow L$ with $M \in \mathscr{C}$. Then $\sigma(L) \in \mathscr{C}$, hence $L \in \overleftarrow{\sigma}(\mathscr{C})$.

($\Leftarrow$): Let us assume $\xi_{\twoheadrightarrow}(\mathscr{C}) \subseteq \overleftarrow{\sigma}(\mathscr{C})$ and take an epimorphism $M \twoheadrightarrow L$ with $M \in \mathscr{C}$. Then $L \in \overleftarrow{\sigma}(\mathscr{C})$, thus $\sigma(L) \in \mathscr{C}$. Also, it is clear that $\mathbb{F}_\sigma \subseteq \mathscr{C}$, thus we have that $\mathscr{C} \in \mathscr{L}_{\{\twoheadrightarrow_\sigma\}}$. □

Theorem 3.7. *If σ is a radical then $\mathscr{L}_{\{\leq_\sigma, ext\}} = \{\mathscr{C} \in \mathscr{L}_{\{\leq, ext\}} \mid \mathscr{C} \supseteq \mathbb{F}_\sigma\}$.*

Proof. ($\supseteq$): If $\mathscr{C} \in \mathscr{L}_{\{\leq, ext\}}$ and $\mathscr{C} \supseteq \mathbb{F}_\sigma$, then $\mathscr{C} = \overleftarrow{\sigma}(\mathscr{C})$, by Remark 3.5. Thus $\xi_{\{\leq\}}(\mathscr{C}) = \mathscr{C} = \overleftarrow{\sigma}(\mathscr{C})$. Then $\mathscr{C} \in \mathscr{L}_{\{\leq_\sigma\}}$, by Theorem 3.6.

($\subseteq$): If $\mathscr{C} \in \mathscr{L}_{\{\leq_\sigma, ext\}}$, then $\mathbb{F}_\sigma \subseteq \mathscr{C}$ and $\mathscr{C} \subseteq \xi_{\{\leq\}}(\mathscr{C}) \subseteq \overleftarrow{\sigma}(\mathscr{C}) \subseteq \mathscr{C}$, by Theorem 3.6 and Lemma 3.3. Thus $\mathscr{C} = \xi_{\{\leq\}}(\mathscr{C})$ and $\mathscr{C} \in \mathscr{L}_{\{\leq, ext\}}$. □

Next corollary follows immediately.

Corollary 3.8. *If σ is a radical then* $\mathscr{L}_{\{\leq_\sigma, ext, \oplus\}} = \{\mathscr{C} \in \mathscr{L}_{\{\leq, ext, \oplus\}} \mid \mathscr{C} \supseteq \mathbb{F}_\sigma\}$.

Theorem 3.9. *If σ is a radical then*

$$\mathscr{L}_{\{\leq_\sigma, ext, \twoheadrightarrow_\sigma\}} = \{\mathscr{C} \in \mathscr{L}_{\{\leq, ext, \twoheadrightarrow\}} \mid \mathscr{C} \supseteq \mathbb{F}_\sigma\}.$$

Proof. ($\supseteq$): It suffices to show that a class $\mathscr{C}$ belonging to the left class is cohereditary. If $M \twoheadrightarrow N$ is an epimorphism with $M \in \mathscr{C}$, then $\sigma(N) \in \mathscr{C}$. From the exact sequence

$$0 \to \sigma(N) \to N \to N/\sigma(N) \to 0,$$

where $N/\sigma(N) \in \mathbb{F}_\sigma \subseteq \mathscr{C}$, we see that $N \in \mathscr{C}$.

($\subseteq$): Suppose $\mathscr{C}$ is a module class with the following properties: hereditary, cohereditary, closed under extensions and containing $\mathbb{F}_\sigma$. We want to prove that $\mathscr{C}$ is σ-hereditary and σ-cohereditary. First, we show that it is σ-hereditary. If $N \leq M$ with $M \in \mathscr{C}$, then $\sigma(N) \leq N \leq M$, thus $\sigma(N) \in C$. Now we are going to see that $\mathscr{C}$ is also σ-cohereditary. If $M \twoheadrightarrow N$ is an epimorphism with $M \in \mathscr{C}$, then N belongs to $\mathscr{C}$ and so does $\sigma(N)$. □

Corollary 3.10. *If σ is a radical, then*

$$R\text{-}(\sigma\text{-tors}) = \{\mathscr{C} \in R\text{-tors} \mid \mathscr{C} \supseteq \mathbb{F}_\sigma\}.$$

Proof. As R-(σ-tors) $:= \mathscr{L}_{\{\leq_\sigma, \twoheadrightarrow_\sigma, \oplus, ext\}}$, we have that R-(σ-tors) consists of the module classes belonging to

$$\{\mathscr{C} \in \mathscr{L}_{\{\leq, ext, \twoheadrightarrow\}} \mid \mathscr{C} \supseteq \mathbb{F}_\sigma\}$$

which are closed under direct sums. Thus the result follows immediately from the preceding theorem. □

Example 3.11. If $R = S \times T$ with S and T two rings. Define $\sigma \in R$-pr by $\sigma(M) = eM$, where $e = (1, 0)$. If $(\mathbb{T}, \mathbb{F})$ is a torsion theory in S-Mod, then $(\mathbb{T} \times T\text{-Mod}, \mathbb{F} \times T\text{-Mod})$ is a σ-torsion theory in R-Mod.

Lemma 3.12. *Let σ be a preradical. Then* $\mathscr{L}_{\leq_\sigma} \supseteq \{\overleftarrow{\sigma}(\mathscr{C}) \mid \mathscr{C} \in \mathscr{L}_{\leq}\}$.

Proof. Let $\mathscr{C}$ be a hereditary module class, we want to show that $\overleftarrow{\sigma}(\mathscr{C})$ is a σ-hereditary class. If $N \to M$ is a monomorphism with $M \in \overleftarrow{\sigma}(\mathscr{C})$ then $\sigma(N)$ embeds in $\sigma(M)$. As $\sigma(M) \in \mathscr{C}$. then $\sigma(N)$ and $\sigma(\sigma(N))$ belong to $\mathscr{C}$. Therefore $\sigma(N) \in \overleftarrow{\sigma}(\mathscr{C})$. □

Theorem 3.13. *Let σ be an exact preradical. Then*

$$R\text{-}(\sigma\text{-TORS}) = \{\overleftarrow{\sigma}(\mathscr{C}) \mid \mathscr{C} \in R\text{-TORS}\}.$$

Proof. ($\supseteq$): Let us assume $\mathscr{C} \in R$-TORS, we are going to show that $\overleftarrow{\sigma}(\mathscr{C}) \in R$-($\sigma$-TORS).

If $M \overset{f}{\twoheadrightarrow} N$ is an epimorphism with $M \in \overleftarrow{\sigma}(\mathscr{C})$, let us see that $\sigma(N) \in \overleftarrow{\sigma}(\mathscr{C})$.

As $M \in \overleftarrow{\sigma}(\mathscr{C})$, then $\sigma(M) \in \mathscr{C}$. As f is an epimorphism, σ is a radical and $\mathscr{C}$ is closed under quotients, then $\sigma(M) \in \mathscr{C}$. As σ is idempotent, then $\sigma(N) = \sigma(\sigma(N)) \in \mathscr{C}$. Hence $\sigma(N) \in \overleftarrow{\sigma}(\mathscr{C})$.

To show that $\overleftarrow{\sigma}(\mathscr{C})$ is closed under extensions, let us assume that

$$0 \to A \to B \to C \to 0$$

is an exact sequence with $A, B \in \overleftarrow{\sigma}(\mathscr{C})$. Then

$$0 \to \sigma(A) \to \sigma(B) \to \sigma(C) \to 0$$

is also exact with $\sigma(A), \sigma(C) \in \mathscr{C}$. As $\mathscr{C} \in R$-TORS, then $\sigma(B) \in \mathscr{C}$. I.e. $B \in \overleftarrow{\sigma}(\mathscr{C})$.

If $\{M_i\}_I$ is a family in $\overleftarrow{\sigma}(\mathscr{C})$ then $\sigma(\bigoplus M_i) = \bigoplus \sigma(M_i) \in \mathscr{C}$, thus $\bigoplus M_i \in \overleftarrow{\sigma}(\mathscr{C})$.

($\subseteq$): If $\mathscr{C} \in R$-(σ-TORS), we are going to show that $\mathscr{C} = \overleftarrow{\sigma}(\sigma^*(\mathscr{C}))$, and that $\sigma^*(\mathscr{C}) \in R$-TORS.

Clearly, $\mathscr{C} \subseteq \overleftarrow{\sigma}(\sigma^*(\mathscr{C}))$. If $\sigma(M) = \sigma(C)$ with $C \in \mathscr{C}$, then $\sigma(C) \in \mathscr{C}$. As $\sigma(M), M/\sigma(M)$ both belong to $\mathscr{C}$ because σ is a radical and $\mathbb{F}_\sigma \subseteq \mathscr{C}$, from $0 \to \sigma(M) \to M \to M/\sigma(M) \to 0$ we obtain that $M \in \mathscr{C}$.

It remains to show that $\sigma^*(\mathscr{C}) \in R$-($\sigma$-TORS).

If $M \in \mathscr{C}$ and $\sigma(M) \twoheadrightarrow N$ is an epimorphism, from the diagram

$$\begin{array}{ccc} \sigma(M) & \xrightarrow{\;f\;} & N \\ {\scriptstyle Id}\uparrow & & \uparrow \\ \sigma(\sigma(M)) & \xrightarrow{f|^{\restriction}_{\downharpoonright}} & \sigma(N), \end{array}$$

we get that $\sigma(N) = N \in \mathscr{C}$. Hence $N = \sigma(N) \in \sigma^*(\mathscr{C})$.

If $0 \to \sigma(A) \to \sigma(B) \to \sigma(C) \to 0$ is an exact sequence with $A, C \in \mathscr{C}$, then we have also $\sigma(A), \sigma(C) \in \mathscr{C}$ and we obtain that $\sigma(B) \in \mathscr{C}$. Then $\sigma(B) = \sigma(\sigma(B)) \in \sigma^*(\mathscr{C})$.

If $\{\sigma(M_i)\}_I$ is a family with $M_i \in \mathscr{C} \in R$-($\sigma$-TORS), $\forall i \in I$, then also $\sigma(M_i) \in \mathscr{C}, \forall i \in I$. Then $\sigma(\oplus M_i) = \oplus\sigma(M_i) \in \mathscr{C}$, with $\oplus M_i \in \mathscr{C}$. Hence $\oplus\sigma(M_i) \in \sigma^*(\mathscr{C})$. □

Recall that for each $\mathscr{C} \in R$-Mod, if we define

$$l(\mathscr{C}) = \{M \in R\text{-Mod} \mid \mathrm{Hom}_R(M, E(N)) = 0, \forall\ N \in \mathscr{C}\} \text{ and}$$

$$r(\mathscr{C}) = \{N \in R\text{-Mod} \mid \mathrm{Hom}_R(M, E(N)) = 0, \forall\ M \in \mathscr{C}\},$$

then $l(\mathscr{C})$ is a hereditary torsion class and $r(\mathscr{C})$ is its corresponding hereditary torsion free class (see [14] Chap. VI).

Recall that R-Simp denotes a set of representatives of isomorphism classes of simple modules.

An R-module M is semiartinian if and only if M is of $\xi(R$-Simp)-torsion if and only if each one of its nonzero homomorphic images has a nonzero socle. R is a left semiartinian ring if it is semiartinian as a left R-module (See [9], Chap. 36).

Theorem 3.14. ([9], Chap. 36, Prop. 36.4) *The following conditions are equivalent for a ring R.*

(1) *R is a left semiartinian ring.*
(2) *Each hereditary torsion theory in* R-Mod *is generated by a family of simple modules.*
(3) R-tors *is a boolean lattice.*

Remark 3.15. If R-tors $= Skel(\mathscr{L}_{\{\leq, \twoheadrightarrow\}})$ (which happens if and only if R is left semiartinian), then for each $\mathbb{T} \in R$-tors there exists $\mathfrak{a} \subseteq R$-Simp such that $\mathbb{T} = \mathbb{T}_{\chi(\mathfrak{a})}$. Thus, if R-(σ-tors) $= Skel(\mathscr{L}_{\{\leq_\sigma, \twoheadrightarrow_\sigma\}})$, then R-(σ-tors) $= \{\overleftarrow{\sigma}(\mathbb{T}_{\chi(\mathfrak{a})}) \mid \mathfrak{a} \subseteq R\text{-Simp}\}$.

Furthermore, for each $\mathfrak{a} \subseteq R$-Simp we have that $\mathbb{T}_{\xi(\mathfrak{a})} = \mathbb{T}_{\chi(R\text{-Simp}\setminus\mathfrak{a})}$.

Remark 3.16. For each centrally splitting preradical σ, we have that $\mathbb{T}_\sigma = \sigma(R)$-Mod.

For each $M \in \mathbb{T}_\sigma$ there exists an epimorphism $R^{(X)} \to M$ for some set X. As σ is centrally splitting we have that $\sigma(R)^{(X)} \to \sigma(M)$ is an epimorphism, thus $M = \sigma(M) \in \sigma(R)$-Mod. So $\mathbb{T}_\sigma \subseteq \sigma(R)$-Mod.

For the other inclusion, notice that for each $M \in \sigma(R)$-Mod there exists an epimorphism $g : \sigma(R)^{(X)} \to M$ for some set X, thus $M = g(\sigma(R^{(X)})) \leq \sigma(M)$. Hence $M \in \mathbb{T}_\sigma$. Therefore $\sigma(R)$-Mod $\subseteq \mathbb{T}_\sigma$.

Furthermore, for each $\mathscr{C} \in R$-(σ-tors)we have that $\sigma^*(\mathscr{C}) \subseteq \sigma(R)$-Mod and $\sigma^*(\mathscr{C}) \in \sigma(R)$-tors.

Proposition 3.17. *Let σ be a centrally splitting preradical. The assignment σ^* : R-(σ-tors) $\to$ R-tors satisfies $\sigma^*(\mathscr{C} \cap \mathscr{D}) = \sigma^*(\mathscr{C}) \cap \sigma^*(\mathscr{D})$ for each $\mathscr{C}, \mathscr{D} \in R$-($\sigma$-tors). Then $\sigma^* : R$-(σ-tors) $\to \sigma(R)$-tors is a $\wedge$-isomorphism.*

Proof. Take $\mathscr{C}, \mathscr{D} \in R$-($\sigma$-tors). We have that $\sigma^*(\mathscr{C} \cap \mathscr{D}) \subseteq \sigma^*(\mathscr{C}) \cap \sigma^*(\mathscr{D})$. If $M \in \sigma^*(\mathscr{C}) \cap \sigma^*(\mathscr{D})$, then $M \in \sigma^*(\mathscr{C})$ and $M \in \sigma^*(\mathscr{D})$. Hence there exist $C \in \mathscr{C}$ and $D \in \mathscr{D}$ such that $M = \sigma(C)$ and $M = \sigma(D)$, besides $M = \sigma(C) \in \mathscr{C}$ and $M = \sigma(D) \in \mathscr{D}$. Hence $M \in \mathscr{C} \cap \mathscr{D}$, which implies that $M = \sigma(M) \in \sigma^*(\mathscr{C} \cap \mathscr{D})$. We conclude that $\sigma^*(\mathscr{C} \cap \mathscr{D}) = \sigma^*(\mathscr{C}) \cap \sigma^*(\mathscr{D})$.

If $\mathbb{T} \in \sigma(R)$-tors, then we have that $\sigma^*(\overleftarrow{\sigma}(\mathbb{T})) \subseteq \mathbb{T}$. If $M \in \mathbb{T}$, then $\sigma(M) \in \mathbb{T}$, this implies that $M \in \overleftarrow{\sigma}(\mathbb{T})$, from this we obtain $\sigma(M) \in \sigma^*(\overleftarrow{\sigma}(\mathbb{T}))$. As $M/\sigma(M) \in \mathbb{F}_\sigma$ we have that $M/\sigma(M) \in \sigma^*(\overleftarrow{\sigma}(\mathbb{T}))$ because $\mathbb{F}_\sigma \subseteq \sigma^*(\overleftarrow{\sigma}(\mathbb{T}))$. We get $M \in \sigma^*(\overleftarrow{\sigma}(\mathbb{T}))$, because $\mathbb{F}_\sigma \subseteq \sigma^*(\overleftarrow{\sigma}(\mathbb{T}))$ is closed under taking extensions. Hence $\mathbb{T} \subseteq \sigma^*(\overleftarrow{\sigma}(\mathbb{T}))$. We conclude that $\sigma^*(\overleftarrow{\sigma}(\mathbb{T})) = \mathbb{T}$.

Analogously it can be shown that $\mathscr{C} = \overleftarrow{\sigma}(\sigma^*(\mathscr{C}))$ for all $\mathscr{C} \in R$-(σ-tors). We conclude that $\sigma^* : R$-(σ-tors) $\to \sigma(R)$-tors is a $\wedge$-isomorphism. □

The following result is a generalization of Theorem 3.13.

Theorem 3.18. *Let $I \leq R$ an ideal generated by a central idempotent in R and take the preradical σ defined as $\sigma(M) = IM$. R-(σ-tors) $= Skel(\mathscr{L}_{\{\leq_\sigma, \twoheadrightarrow_\sigma\}})$ if and only if I is a semiartinian ring.*

Proof. From Remark 2.1 σ is an exact radical.

($\Rightarrow$): Suppose that R-(σ-tors) $= Skel(\mathscr{L}_{\{\leq_\sigma, \twoheadrightarrow_\sigma\}})$. As R-Mod $\in R$-(σ-tors) we have that $\mathbb{T}_\sigma = \sigma^*(R\text{-Mod}) \in R$-tors and $\sigma^*(R\text{-Mod})$ is the largest class in $Im(\sigma^*) = \{\sigma^*(\mathbb{T}) \mid \mathbb{T} \in R\text{-}(\sigma\text{-tors})\}$ (see Remark 2.2). Moreover, from Proposition 3.17, $Im(\sigma^*) = I$-tors, this implies, by Proposition 3.17 that I-tors is a boolean lattice and I is a semiartinian ring.

($\Leftarrow$):) Let us take a semiartinian factor I of R. We show that R-(σ-tors) $=$ $Skel(\mathscr{L}_{\{\leq_\sigma, \twoheadrightarrow_\sigma\}})$. As I is semiartinian, we have that for each $M \in R$-Mod that $\sigma(M)$ is semiartinian, this implies that I-tors $= Skel(\mathscr{L}^I_{\{\leq, \twoheadrightarrow\}})$. By Proposition 3.17 we have that R-(σ-tors) $= Skel(\mathscr{L}_{\{\leq_\sigma, \twoheadrightarrow_\sigma\}})$. □

3.1. σ-torsion theories. Since their introduction by Dickson [8] of torsion theories for Abelian categories, there have been defined several generalizations. We introduce a new extension of this concept.

Definition 3.19. Let σ be a preradical. Let us define the assignments $L_\sigma, R_\sigma : \wp(R\text{-Mod}) \to \wp(R\text{-Mod})$ as

(1) $L_\sigma(\mathscr{C}) = \{M \in R\text{-Mod} \mid \operatorname{Hom}_R(\sigma(M), \sigma(N)) = 0,\ \forall\ N \in \mathscr{C}\}$,
(2) $R_\sigma(\mathscr{C}) = \{N \in R\text{-Mod} \mid \operatorname{Hom}_R(\sigma(M), \sigma(N)) = 0,\ \forall\ M \in \mathscr{C}\}$.

It is immediate that for $\mathscr{C} \subseteq R$-Mod we have that $L_\sigma(\mathscr{C}) = \overleftarrow{\sigma}\ (L(\sigma^*(\mathscr{C})))$ and $R_\sigma(\mathscr{C}) = \overleftarrow{\sigma}\ (R(\sigma^*(\mathscr{C}))$.

Notice that L_σ and R_σ are order reversing assignments and that $L_\sigma R_\sigma$ and $R_\sigma L_\sigma$ are closure operators. Besides, $L_\sigma R_\sigma L_\sigma = L_\sigma$ and $R_\sigma L_\sigma R_\sigma = R_\sigma$.

When $\sigma = 1_{R\text{-Mod}}$ (see [14] Chap. VI), we have that

$$L_\sigma(\mathscr{C}) = L(\mathscr{C}) = \{M \in R\text{-Mod} \mid \operatorname{Hom}_R(M, N) = 0,\ \forall\ N \in \mathscr{C}\},$$

$$R_\sigma(\mathscr{C}) = R(\mathscr{C}) = \{N \in R\text{-Mod} \mid \operatorname{Hom}_R(M, N) = 0,\ \forall\ M \in \mathscr{C}\}.$$

It is known that there exists a bijective correspondence between torsion theories and idempotent radicals, then for all $\mathscr{C} \subseteq R$-Mod we have that $L(\mathscr{C}) = \mathbb{T}_\tau$ and $R(\mathscr{C}) = \mathbb{F}_\nu$ for some idempotent radicals τ, ν, respectively.

Remark 3.20. Let σ be a preradical and $M \in R$-Mod, then $M \in L_\sigma(R_\sigma(\{M\}))$ because

$$R_\sigma(\{M\}) = \{N \in R\text{-Mod} \mid \operatorname{Hom}_R(\sigma(M), \sigma(N)) = 0\}, \text{ and}$$

$$L_\sigma(R_\sigma(\{M\})) = \{L \in R\text{-Mod} \mid \operatorname{Hom}_R(\sigma(L), \sigma(N)) = 0, \forall\ N \in R_\sigma(\{M\})\}.$$

If σ is idempotent, then $\sigma(M) \in L_\sigma(R_\sigma(\{M\}))$ because for each $N \in R_\sigma(\{M\})$ we have that $\operatorname{Hom}_R(\sigma(\sigma(M)), \sigma(N)) = \operatorname{Hom}_R(\sigma(M), \sigma(N)) = 0$.

Analogously, $N \in R_\sigma(L_\sigma(\{N\}))$ and if σ is an idempotent preradical, then $\sigma(N) \in R_\sigma(L_\sigma(\{N\}))$.

Proposition 3.21. *Let σ be a preradical. For each $\mathscr{C} \subseteq R$-Mod, $R_\sigma(\mathscr{C}) \in \mathscr{L}_{\{\leq_\sigma, \prod\}}$.*

Proof. As $R_\sigma(\mathscr{C}) = \overleftarrow{\sigma}\ (R(\sigma^*(\mathscr{C})))$ and $R(\sigma^*(\mathscr{C}))$ is a torsion free class, then by Lemma 3.12 we obtain that $R_\sigma(\mathscr{C}) \in \mathscr{L}_{\{\leq_\sigma\}}$.

Take $\{N_\alpha\}_{\alpha \in X} \subseteq R_\sigma(\mathscr{C})$ and $M \in \mathscr{C}$, then $\operatorname{Hom}_R(\sigma(M), \sigma(N_\alpha)) = 0$ for each $\alpha \in X$, and $\sigma(\prod\limits_{\alpha \in X} N_\alpha) \leq \prod\limits_{\alpha \in X} \sigma(N_\alpha)$. Thus, we have a monomorphism $\operatorname{Hom}_R(\sigma(M), \sigma(\prod\limits_{\alpha \in X} N_\alpha)) \to \operatorname{Hom}_R(\sigma(M), \prod\limits_{\alpha \in X} \sigma(N_\alpha)) = 0$ with

$$\operatorname{Hom}_R(\sigma(M), \prod_{\alpha \in X} \sigma(N_\alpha)) \cong \prod_{\alpha \in X} \operatorname{Hom}_R(\sigma(M), \sigma(N_\alpha)) = 0.$$

We conclude that $\operatorname{Hom}_R(\sigma(M), \sigma(\prod\limits_{\alpha \in X} N_\alpha)) = 0$, thus $\prod\limits_{\alpha \in X} N_\alpha \in R_\sigma(\mathscr{C})$. Therefore $R_\sigma(\mathscr{C}) \in \mathscr{L}_{\{\leq_\sigma, \prod\}}$. □

Proposition 3.22. *Let σ be a left exact preradical. For each $\mathscr{C} \subseteq R$-Mod, $R_\sigma(\mathscr{C}) \in \mathscr{L}_{\{\leq_\sigma, \prod, ext\}}$.*

Proof. From Proposition 3.21 we have that $R_\sigma(\mathscr{C}) \in \mathscr{L}_{\{\leq_\sigma, \prod\}}$. Let us see that it is also closed under extensions.

Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence with $N', N'' \in R_\sigma(\mathscr{C})$ and take $M \in \mathscr{C}$. Then $0 \to \sigma(N') \to \sigma(N) \to \sigma(N'')$ is an exact sequence, thus the sequence

$$0 \to \mathrm{Hom}_R(\sigma(M), \sigma(N')) \to \mathrm{Hom}_R(\sigma(M), \sigma(N)) \to \mathrm{Hom}_R(\sigma(M), \sigma(N''))$$

is exact with $\mathrm{Hom}_R(\sigma(M), \sigma(N')) = 0, \mathrm{Hom}_R(\sigma(M), \sigma(N'')) = 0$. This implies that $\mathrm{Hom}_R(\sigma(M), \sigma(N)) = 0$. Hence $N \in R_\sigma(\mathscr{C})$. Therefore $R_\sigma(\mathscr{C}) \in \mathscr{L}_{\{\leq_\sigma, \prod, ext\}}$. □

Proposition 3.23. *Let σ be an exact preradical. For each $\mathscr{C} \subseteq R$-Mod, $L_\sigma(\mathscr{C}) \in R$-tors.*

Proof. Assume that $\mathscr{C} \subseteq R$-Mod. As $L_\sigma(\mathscr{C}) = \overleftarrow{\sigma}\,(L(\sigma^*(\mathscr{C})))$ and $L(\sigma^*(\mathscr{C})) \in R$-tors and $L(\sigma^*(\mathscr{C})) \supseteq \mathbb{F}_\sigma$ then the conclusion follows from Corollary 3.10. □

Definition 3.24. Let σ be an exact preradical. A σ-torsion theory is a pair of R-module classes $(\mathbb{T}, \mathbb{F})$ such that $\mathbb{T} = L_\sigma(\mathbb{F})$ and $\mathbb{F} = R_\sigma(\mathbb{T})$.

When $\sigma = 1_{R\text{-Mod}}$ the $1_{R\text{-Mod}}$-torsion theories are the usual torsion theories.

4. σ-retractable modules

Definition 4.1. A left R-module M is called **retractable** if for each $0 \neq N \leq M$ we have that $\mathrm{Hom}_R(M, N) \neq 0$.

In [10], it is proved that the class of mod-retractable commutative rings coincides with the class of commutative semiartinian rings. It is shown in [13] that every projective module over a right V-ring is retractable.

We mention some examples: free modules and semisimple modules are retractable. Any direct sum of modules of the form $\mathbb{Z}_{p^i}$ is retractable, where p is a prime number. The $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is not retractable.

Let us recall that there is a one to one correspondence between the class of left exact radicals and the class of hereditary torsion theories (See [14] Chap. VI).

The following theorem is proved in [10], we include a proof as an illustration.

Theorem 4.2. ([10]) R-tors $= R$-TORS *if and only if each R-module is retractable.*

Proof. ($\Rightarrow$): Assume that R-tors $=$ R-TORS and take $\mathbb{T} \in R$-tors. Take $M \in \mathbb{T}$ and $0 \neq N \leq M$.

Let $\mathbb{T}$ be the least torsion containing M, then $N \in \mathbb{T}$ implies that $N \notin \mathbb{F}$ hence $\mathrm{Hom}_R(M, N) \neq 0$. We conclude that M is retractable.

($\Leftarrow$): Suppose that each R-module is retractable and take $\mathbb{T}_\sigma \in R$-TORS. Take $M \in \mathbb{T}_\sigma$ and $N \leq M$.

If $N \notin \mathbb{T}_\sigma$, let us take the exact sequence $0 \to \sigma(N) \to N \to N/\sigma(N) \to 0$. In the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \sigma(N) & \longrightarrow & N & \longrightarrow & N/\sigma(N) & \longrightarrow & 0 \\ & & & & & & \uparrow{\scriptstyle f} & & \\ & & & & M & \xrightarrow{\pi} & M/\sigma(N) & & \end{array}$$

we have that $M/\sigma(N) \in \mathbb{T}_\sigma$ and $N/\sigma(N) \leq M/\sigma(N)$. Then there exists a nonzero $f : M/\sigma(N) \to N/\sigma(N)$. This implies that $f(M/\sigma(N)) \in \mathbb{T}_\sigma$.

Let us take $N'/\sigma(N) = f(M/\sigma(N))$. We have the exact sequence

$$0 \to \sigma(N) \to N' \to N'/\sigma(N) \to 0$$

with $\sigma(N), N'/\sigma(N) \in \mathbb{T}_\sigma$. From this, we have that $N' \in \mathbb{T}_\sigma$ con $N' \leq N$. This implies that $N' = \sigma(N') \leq \sigma(N)$, thus $N' = \sigma(N)$ (because σ is left exact). It follows that $f(M/\sigma(N)) = N'/\sigma(N) = 0$, contradicting that $f \neq 0$. Hence $\sigma(N) = N$.

Hence $N \in \mathbb{T}_\sigma$, and $\mathbb{T}_\sigma$ is closed under taking submodules. □

Definition 4.3. Let σ be a preradical. An R-module M is called left **σ-retractable** if for each $N \leq M$ with $\sigma(N) \neq 0$, one has that $\mathrm{Hom}_R(M, \sigma(N)) \neq 0$. A ring R will be called **σ-(R-Mod)-retractable** if each R-module is σ-retractable.

Remark 4.4. Notice that each retractable R-module is σ-retractable, but a non retractable R-module M can be σ-retractable for some $\sigma \in R$-pr.

As an example, let $t \in \mathbb{Z}$-pr denote the torsion functor and take the $\mathbb{Z}$-module $\mathbb{Q}$. For each $N \leq \mathbb{Q}$ we have that $t(N) = N \cap t(\mathbb{Q}) = N \cap 0 = 0$. Hence by vacuity, $\mathbb{Q}$ is t-retractable, but it is not retractable, because for $\frac{a}{b}\mathbb{Z} \hookrightarrow \mathbb{Q}$, with $a, b \in \mathbb{Z}$ and $a, b \neq 0$ it happens that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \frac{a}{b}\mathbb{Z}) = 0$.

Remark 4.5. For each preradical σ, each $M \in R$-Mod and $N \leq M$, if $\sigma(M/N) = 0$, then $\sigma(M) \leq N$ (see [5] Prop. I.1.1).

Theorem 4.6. *Let σ be a left exact preradical. If each R-module is σ-retractable, then R-(σ-tors) $=$ R-(σ-TORS).*

Proof. Let us assume that $\mathbb{T} \in R$-(σ-TORS) and $M \in \mathbb{T}$. Assume also that $0 \neq \sigma(N) \leq N \leq M$. We will show that $\sigma(N) \in \mathbb{T}$.

If there exists a nonzero homomorphism $f : M \to \sigma(N)$, then $\sigma(f(M)) \in \mathbb{T}$, so there exists a nonzero submodule L of $\sigma(N)$ with $L \in \mathbb{T}$. Let us denote $\mathcal{U} = \{L \leq \sigma(N) \mid L \in \mathbb{T}\}$. We have that $\bigoplus_{L \in \mathcal{U}} L \in \mathbb{T}$ and that there is an epimorphism $\bigoplus_{L \in \mathcal{U}} L \twoheadrightarrow \sum_{L \in \mathcal{U}} L$. Hence $\sigma(\sum_{L \in \mathcal{U}} L) \in \mathbb{T}$ and $\sigma(\sum_{L \in \mathcal{U}} L)$ is the largest submodule of $\sigma(N)$ belonging to $\mathbb{T}$. Let us denote $V = \sigma(\sum_{L \in \mathcal{U}} L)$.

We have that $\sigma(N)/V \leq N/V \leq M/V$. We have two cases: $\sigma(\sigma(N)/V) = 0$ and $\sigma(\sigma(N)/V) \neq 0$.

In the former case, $\sigma(\sigma(N)/V) = 0$, we have that $\sigma(N) = \sigma(\sigma(N)) \leq V$ (see Remark 4.5), this implies that $\sigma(N) = V \in \mathbb{T}$.

If $\sigma(\sigma(N)/V) \neq 0$, then $\sigma(M/V) \neq 0$ and we have that there exists a nonzero homomorphism $h : M/V \to \sigma(\sigma(N)/V)$. From this, we have that $h(M/V) = \sigma(h(M/V)) \in \mathbb{T}$. Taking $U \leq \sigma(N)$ such that $U/V = h(M/V)$, we obtain the exact sequence $0 \longrightarrow V \longrightarrow U \longrightarrow U/V \longrightarrow 0$ which has V, $U/V \in \mathbb{T}$. We get that $U \in \mathbb{T}$.

Then U is a submodule of $\sigma(N)$ such that $U \in \mathbb{T}$, this implies that $U \leq V$. We conclude that $U = V$ and from this, that $0 = U/V = h(M/V)$, a contradiction. Hence $\sigma(N) = V \in \mathbb{T}$ and $\mathbb{T}$ is σ-hereditary. □

Proposition 4.7. *Let σ be a left exact preradical. If R-(σ-TORS) $= R$-(σ-tors), then for each $\mathbb{T} \in R$-(σ-TORS), $\sigma^*(\mathbb{T})$ is a hereditary torsion class.*

Proof. Take $\mathbb{T} \in R$-(σ-TORS), $M \in \sigma^*(\mathbb{T})$ and $N \leq M$. We will show that $N \in \sigma^*(\mathbb{T})$.

By hypothesis there exists $L \in \mathbb{T}$ such that $\sigma(L) = M$. As $\sigma(L) \in \mathbb{T}$, $N \leq M = \sigma(L)$ and σ is left exact, we have that $N = \sigma(N) \in \mathbb{T}$, because it is a σ-hereditary class, this implies that $N = \sigma(N) \in \sigma^*(\mathbb{T})$. Hence $\sigma^*(\mathbb{T})$ is hereditary. □

Proposition 4.8. *Let σ be an exact preradical. If R-(σ-TORS) $= R$-(σ-tors) then each R-module $M \in \mathbb{T}_\sigma$ is σ-retractable.*

Proof. Let us assume that R-(σ-TORS) $= R$-(σ-tors), take $M \in \mathbb{T}_\sigma$ and $N \leq M$. We show that M is a σ-retractable module. Let us suppose that $\mathrm{Hom}_R(M, \sigma(N)) = 0$. We will show that $\sigma(N) = 0$.

From Proposition 3.23 we have that R-(σ-TORS) $= \{L_\sigma(\mathscr{C}) \mid \mathscr{C} \subseteq R\text{-Mod}\}$. Then $L_\sigma(R_\sigma(\{M\})) \in R$-($\sigma$-TORS) and $\sigma(M) = M \in L_\sigma(R_\sigma(\{M\}))$ (see Remark 3.20). As $L_\sigma(R_\sigma(\{M\}))$ is a σ-hereditary class, we have $\sigma(N) \in L_\sigma(R_\sigma(\{M\}))$, and $\mathrm{Hom}_R(\sigma(M), \sigma(\sigma(N))) = \mathrm{Hom}_R(\sigma(M), \sigma(N)) = 0$ implies that $\sigma(N) \in R_\sigma(\{M\})$.

This means that $\sigma(N) \in L_\sigma(R_\sigma(\{M\})) \cap R_\sigma(\{M\}) = \mathbb{F}_\sigma$, from this $\sigma(N) = \sigma(\sigma(N)) = 0$ follows. It follows that M is a σ-retractable module. □

Example 4.9. Let $I \leq R$ be a two sided pure ideal and let $t(\alpha_I^R) \in R$-pr be defined by $t(\alpha_I^R)(M) = \{m \in M \mid Im = 0\}$, the annihilator of I on M. Notice that there is a natural isomorphism $t(\alpha_I^R) \cong R/I \otimes_R -$. Then $t(\alpha_I^R)$ is an exact preradical (R/I is flat). Thus, for all $M \in R$-Mod we have that $t(\alpha_I^R)(M) \cong M/IM$. Notice that

(1) $t(\alpha_I^R)(M) = 0 \Leftrightarrow M = IM$.
(2) $t(\alpha_I^R)(M) = M \Leftrightarrow IM = 0 \Leftrightarrow M \in R/I$-Mod.

M is a $t(\alpha_I^R)$-retractable module if for all $N \leq M$ with $IN \neq N$, we have that $\mathrm{Hom}_R(M, N/IN) \neq 0$. By Proposition 4.6, if each R-module M is $t(\alpha_I^R)$-retractable, then R-($t(\alpha_I^R)$-TORS) $= R$-($t(\alpha_I^R)$-tors). By Proposition 4.8, if R-($t(\alpha_I^R)$-TORS) $= R$-($t(\alpha_I^R)$-tors), then each R/I-module is $t(\alpha_I^R)$-retractable.

Example 4.10. If $R = S \times T$ with S and T two rings. Define $\sigma \in R$-pr by $\sigma(M) = eM$, where $e = (1, 0)$. Then R-(σ-TORS) $= R$-(σ-tors) if and only if each R-module σ-retractable, which is equivalent to each S-module be retractable and S-TORS $= S$-tors.

5. σ-open and σ-stable classes

The big lattice of σ-open classes is denoted by $\mathscr{L}_{\{\leq_\sigma, \twoheadrightarrow_\sigma\}}$.

Remark 5.1. If σ is an idempotent preradical, then for each $M, N \in R$-Mod and each epimorphism $g : \sigma(M) \twoheadrightarrow N$ we have that $\sigma(N) = N$, because

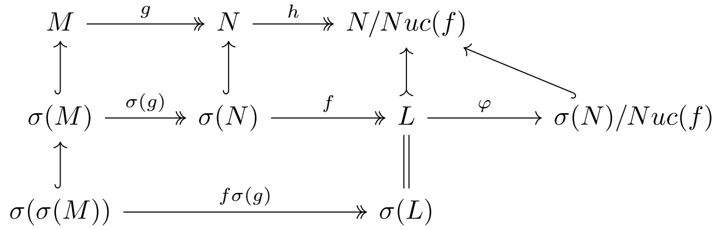
$$N = g(\sigma(M)) = g(\sigma(\sigma(M))) \leq \sigma(N) \leq N.$$

Proposition 5.2. *Let σ be a cohereditary idempotent preradical. There is an assignment $\rho_{\twoheadrightarrow_\sigma} : \mathscr{L}_{\{\leq_\sigma, \twoheadrightarrow_\sigma\}} \to R$-($\sigma$-TORS) defined by*

$$\rho_{\twoheadrightarrow_\sigma}(\mathscr{C}) = \{M \in R\text{-Mod} \mid \forall\ M \twoheadrightarrow L,\ (\sigma(L) \in \mathscr{C} \Rightarrow \sigma(L) = 0)\} \cup \mathbb{F}_\sigma.$$

Proof. If $\mathscr{C} \in \mathscr{L}_{\{\leq_\sigma, \twoheadrightarrow_\sigma\}}$, we will show that $\rho_{\twoheadrightarrow_\sigma}(\mathscr{C}) \in R$-($\sigma$-TORS). To get started, we show that if $M \in \rho_{\twoheadrightarrow_\sigma}(\mathscr{C})$, then $\sigma(M) \in \rho_{\twoheadrightarrow_\sigma}(\mathscr{C})$.

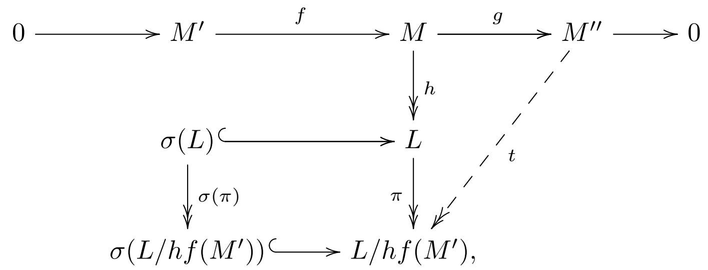
Let us take $M \in \rho_{\twoheadrightarrow_\sigma}(\mathscr{C})$ and $g : M \twoheadrightarrow N$. We will show $\sigma(N) \in \rho_{\twoheadrightarrow_\sigma}(\mathscr{C})$. Further take $f : \sigma(N) \twoheadrightarrow L$ and let us suppose that $\sigma(L) \in \mathscr{C}$ and $\sigma(L) \neq 0$. We have that $\sigma(L) = L$ (see Remark 5.1). We have the following commutative diagram:



where φ is an isomorphism. As σ is cohereditary and $\bar{f}$ is an epimorphism then $\sigma(\bar{f}) : \sigma(N) \to \sigma(N/Nuc(f))$ is an epimorphism, $\sigma(N/Nuc(f)) = \sigma(N)/Nuc(f) \cong L \in \mathscr{C}$ and $0 \neq L \cong \sigma(N/Nuc(f))$. This is a contradiction since $\bar{f} \circ g : M \twoheadrightarrow N/Nuc(f)$ is an epimorphism with $M \in \rho_{\twoheadrightarrow_\sigma}(\mathscr{C})$. Therefore $\sigma(L) \in \mathscr{C}$ implies $\sigma(L) = 0$, then $\sigma(N) \in \rho_{\twoheadrightarrow_\sigma}(\mathscr{C})$.

Now, let $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ be an exact sequence with $M', M'' \in \rho_{\twoheadrightarrow_\sigma}(\mathscr{C})$. We are going to show that $M \in \rho_{\twoheadrightarrow_\sigma}(\mathscr{C})$. As σ is idempotent and cohereditary, then it is an idempotent radical and $\mathbb{F}_\sigma$ is closed under taking extensions.

If $h : M \twoheadrightarrow L$ is an epimorphism with $\sigma(L) \neq 0$, we are going to show that $\sigma(L) \notin \mathscr{C}$. We have the following commutative diagram:



where π denotes the natural epimorphism, and $t : M'' \twoheadrightarrow L/hf(M')$ is an epimorphism. Notice now that $hf : M' \twoheadrightarrow hf(M')$ is an epimorphism with $M' \in \rho_{\twoheadrightarrow_\sigma}(\mathscr{C})$.

Let us first consider the case where $M' \notin \mathbb{F}_\sigma$ and $M'' \notin \mathbb{F}_\sigma$. If $0 \neq \sigma(L) \in \mathscr{C}$, we have the exact sequence

$$0 \to hf(M') \to L \to L/hf(M') \to 0,$$

from which we obtain the exact sequence

$$\sigma(hf(M')) \to \sigma(L) \to \sigma(L/hf(M')) \to 0.$$

Since $\sigma(L) \in \mathscr{C}$, σ is idempotent and $\mathscr{C} \in \mathscr{L}_{\{\twoheadrightarrow_\sigma\}}$, then $\sigma(L/hf(M')) \in \mathscr{C}$. But since $L/hf(M')$ is a quotient of M'', then $\sigma(L/hf(M'))$ has to be 0. Thus $\sigma(hf(M')) = \sigma(L)$, contradicting than $M' \in \rho_{\twoheadrightarrow_\sigma}(\mathscr{C})$.

Now consider the case $M' \in \mathbb{F}_\sigma$ and $M'' \notin \mathbb{F}_\sigma$. As in the previous case, we obtain $\sigma(L/hf(M')) = \sigma(L) \neq 0$, in contradiction to the hypothesis that $M' \in \mathbb{F}_\sigma$

If $M' \notin \mathbb{F}_\sigma$ and $M'' \in \mathbb{F}_\sigma$ then $\sigma\left(L/hf\left(M'\right)\right) = 0$, so $\sigma\left(L\right) \subseteq hf\left(M'\right) \subseteq L$. Using that σ is idempotent, we obtain that $\sigma\left(L\right) = \sigma\left(hf\left(M'\right)\right) = 0$. This contradicts the hypothesis.

Finally, if $M', M'' \in \mathbb{F}_\sigma$, as in the previous case, we have $\sigma\left(L\right) = \sigma\left(hf\left(M'\right)\right) = 0$, since $M' \in \mathbb{F}_\sigma$. This is a contradiction. We conclude that $M \in \rho_{\twoheadrightarrow_\sigma}(\mathscr{C})$.

Now, take a family $\{M_\alpha\}_{\alpha\in X} \in \rho_{\twoheadrightarrow_\sigma}(\mathscr{C})$. We are going to show $\bigoplus\limits_{\alpha\in X} M_\alpha \in \rho_{\twoheadrightarrow_\sigma}(\mathscr{C})$. Let us suppose there is an epimorphism $\bigoplus\limits_{\alpha\in X} M_\alpha \twoheadrightarrow L$ with $\sigma(L) \in \mathscr{C}$ and $\sigma(L) \neq 0$.

Note that we always have an epimorphism $h : \bigoplus\limits_{l\in L} Rl \twoheadrightarrow L$, then $\sigma(h) : \bigoplus\limits_{l\in L} \sigma(Rl) \twoheadrightarrow \sigma(L)$ is an epimorphism. As $\sigma(L) \neq 0$, there exists $0 \neq l \in \sigma(L)$ such that $\sigma(Rl) \neq 0$, furthermore $\sigma(Rl) \in \mathscr{C}$.

$$\begin{array}{ccccc}
\bigoplus\limits_{\alpha\in X} M_\alpha & \twoheadrightarrow & L & & \\
\uparrow & & \uparrow & & \\
\bigoplus\limits_{\alpha\in X} \sigma(M_\alpha) & \twoheadrightarrow & \sigma(L) & \hookleftarrow & \sigma(Rl)
\end{array}$$

We have that $l = m_{\alpha_1} + m_{\alpha_2} + \cdots + m_{\alpha_k}$ with $m_{\alpha_j} \in M_{\alpha_j}$ con $1 \leq j \leq k$. Then for each j there exists an epimorphism $M_{\alpha_j} \twoheadrightarrow Rm_j$ with $\sigma(Rm_j) \in \mathscr{C}$, this implies $\sigma(Rm_j) = 0$ for each j. Then $\sigma(L) = 0$ (because of the epimorphism $0 = \bigoplus\limits_{j=1}^{k} \sigma(Rm_{\alpha_j}) = \sigma(\bigoplus\limits_{j=1}^{k} Rm_{\alpha_j}) \twoheadrightarrow \sigma(L)$), contradicting that $\sigma(L) \neq 0$. Hence $\sigma(L) \in \mathscr{C}$ implies that $\sigma(L) = 0$. We conclude that $\bigoplus\limits_{\alpha\in X} M_\alpha \in \rho_{\twoheadrightarrow_\sigma}(\mathscr{C})$. □

Proposition 5.3. *Let σ be an idempotent preradical. We have the assignment $\rho_{\twoheadrightarrow_\sigma} : \mathscr{L}_{\{\leq_\sigma, \sigma E\}} \to \mathscr{L}_{\{\leq_\sigma\}}$.*

Proof. Take $M \in \rho_{\twoheadrightarrow_\sigma}(\mathscr{C})$ and $N \leq M$. We prove that $\sigma(N) \in \rho_{\twoheadrightarrow_\sigma}(\mathscr{C})$. If $g : \sigma(N) \twoheadrightarrow L$ is an epimorphism, with $\sigma(L) \in \mathscr{C}$ and $\sigma(L) \neq 0$, then we have that $\sigma(L) = L$ (see Remark 5.1). We have the following commutative diagram:

$$\begin{array}{ccccc}
 & \sigma(N) & \hookrightarrow & N \hookrightarrow & M \\
\swarrow & \downarrow & & & \downarrow \\
\sigma(L) = & L & \overset{h}{\rightarrowtail} & & Q
\end{array}$$

Let $Q' \leq Q$ be a pseudocomplement of $h(L) \leq Q$, then $h(L) + Q' \leq_e Q$ and there exists an essential monomorphism $L \overset{ess}{\rightarrowtail} Q/Q'$, then there also exists a monomorphism $Q/Q' \rightarrowtail E(L)$.

$$\begin{array}{ccccc} \sigma(L) & = & L & \overset{h}{\dashrightarrow} & Q \\ & & \big\downarrow & \overset{ess}{\searrow} & \big\downarrow \\ & & E(L) & \longleftarrow & Q/Q' \end{array}$$

Thus, we have that $E(L) \cong E(Q/Q')$. As $\sigma(L) \in \mathscr{C}$, it follows that $\sigma(E(Q/Q')) \in \mathscr{C}$ because $\sigma(E(Q/Q')) \cong \sigma(E(L)) = \sigma(E(\sigma(L)))$. As $\sigma(Q/Q') \leq \sigma(E(Q/Q'))$, we have that $\sigma(Q/Q') = \sigma(\sigma(Q/Q')) \in \mathscr{C}$, thus as $M \in \rho_{\twoheadrightarrow_\sigma}(\mathscr{C})$ and $\sigma(Q/Q') \in \mathscr{C}$ we have that $\sigma(Q/Q') = 0$ because $M \twoheadrightarrow Q/Q'$ is an epimorphism. Hence $\sigma(Q/Q') = 0$ implies that $\sigma(Q) \leq Q'$. As $h(L) \leq \sigma(Q)$, we have that $h(L) \cap Q' = 0$ implies that $L \cong h(L) = 0$, contradiction. Hence $\sigma(L) \in \mathscr{C}$ implies $\sigma(L) = 0$. We conclude that $\sigma(N) \in \rho_{\twoheadrightarrow_\sigma}(\mathscr{C})$ and $\rho_{\twoheadrightarrow_\sigma}(\mathscr{C}) \in \mathscr{L}_{\{\leq_\sigma\}}$. □

From Propositions 5.2 and 5.3 we have the following result:

Corollary 5.4. *If σ is a cohereditary preradical, we have the following assignments.*

(1) $\rho_{\twoheadrightarrow_\sigma} : \mathscr{L}_{\{\leq_\sigma, \twoheadrightarrow_\sigma\}} \to R\text{-}(\sigma\text{-TORS})$.
(2) $\rho_{\twoheadrightarrow_\sigma} : \mathscr{L}_{\{\leq_\sigma, \twoheadrightarrow_\sigma, \sigma E\}} \to R\text{-}(\sigma\text{-tors})$.
(3) *If besides σ is a left exact stable preradical, then we have the assignment* $\rho_{\twoheadrightarrow_\sigma} : \sigma\text{-}(R\text{-Nat}) \to R\text{-}(\sigma\text{-tors})$.

In [12] is given an assignment between R-Nat and R-tors. From the preceding proposition, for $\sigma = 1_{R\text{-Mod}}$, we have the following assignments.

Corollary 5.5. *There exist assignments:*

(1) $\rho_{\twoheadrightarrow} : \mathscr{L}_{\{\leq, \twoheadrightarrow\}} \to R$-TORS.
(2) $\rho_{\twoheadrightarrow} : \mathscr{L}_{\{\leq, \twoheadrightarrow, E\}} \to R$-tors.
(3) $\rho_{\twoheadrightarrow} : R$-Nat $\to R$-tors.
(4) *Furthermore, we have the commutative diagram:*

$$\begin{array}{ccc} \mathscr{L}_{\{\leq, \twoheadrightarrow, E\}} & \xrightarrow{\rho_{\twoheadrightarrow}} & R\text{-tors} \\ {\scriptstyle \xi_{nat}} \big\downarrow & \nearrow_{\rho_{\twoheadrightarrow}} & \\ R\text{-Nat.} & & \end{array}$$

Recall that we denote R-jtors the collection of all hereditary jansian torsion theories, i.e., the collection of hereditary torsion classes closed under taking products. Notice that R is a left perfect ring if and only if every hereditary **right** torsion class is closed under taking products and it is generated by a family of right simple modules (notice the change of side). Thus, it could happen that R-tors $\neq R$-jtors even if R is left perfect.

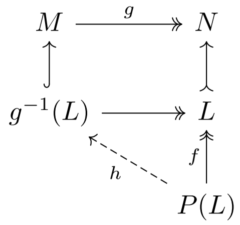
Proposition 5.6. *If R is a left perfect ring, then we have the following assignment $\rho_{\leq} : \mathscr{L}_{\{\leq,\twoheadrightarrow,P\}} \to R$-jtors, defined by*

$$\rho_{\leq}(\mathscr{C}) = \{M \in R\text{-Mod} \mid L \rightarrowtail M,\ L \in \mathscr{C} \Rightarrow L = 0\}.$$

Proof. Let $\mathscr{C} \in \mathscr{L}_{\{\leq,\twoheadrightarrow,P\}}$, we prove that $\rho_{\leq}(\mathscr{C}) \in R$-jtors. Further, take $M \in \rho_{\leq}(\mathscr{C})$ and $N \leq M$. We prove that $N \in \rho_{\leq}(\mathscr{C})$.

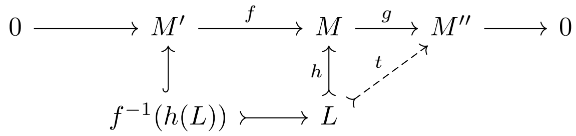
Let us assume that there exists a monomorphism $L \rightarrowtail N$ with $L \in \mathscr{C}$ and $L \neq 0$. Then we have that the composition $L \rightarrowtail N \rightarrowtail M$ is a monomorphism with $M \in \rho_{\leq}(\mathscr{C})$ and $L \in \mathscr{C}$. This implies that $L = 0$, a contradiction. It follows that $N \in \rho_{\leq}(\mathscr{C})$.

Now, take $M \in \rho_{\leq}(\mathscr{C})$ and $g : M \twoheadrightarrow N$. We are going to show that $N \in \rho_{\leq}(\mathscr{C})$. Suppose that there exists $f : L \rightarrowtail N$ with $L \in \mathscr{C}$ and $L \neq 0$. Suppose there is a projective cover $f : P(L) \twoheadrightarrow L$ of L, thus $P(L) \in \mathscr{C}$. We have the following commutative diagram



There is a homomorphism $0 \neq h : P(L) \to g^{-1}(L)$ making the diagram commutative, and we have that $0 \neq h(P(L)) \in \mathscr{C}$, as $h(P(L)) \leq M$ we have that $h(P(L)) = 0$, a contradiction. Hence $P(L) = 0$, thus $L = 0$, contradicting that $L \neq 0$. Hence $L \in \mathscr{C}$ implies that $L = 0$. It follows that $N \in \rho_{\leq}(\mathscr{C})$.

Now, let $0 \to M' \to M \to M'' \to 0$ be an exact sequence with $M', M'' \in \rho_{\leq}(\mathscr{C})$. We are going to show that $M \in \rho_{\leq}(\mathscr{C})$. Let us take a monomorphism $L \rightarrowtail M$ with $L \in \mathscr{C}$ and $L \neq 0$.



As $M' \in \rho_{\leq}(\mathscr{C})$ and $f^{-1}(h(L)) \leq M'$, then $f^{-1}(h(L)) = 0$, because $f^{-1}(h(L)) \cong L \in \mathscr{C}$. Then

$$f^{-1}(f(M') \cap h(L)) = M' \cap f^{-1}(h(L)) = M' \cap 0 = 0.$$

Hence there exists a monomorphism $t : L \rightarrowtail M''$ which implies that $L = 0$, a contradiction. We conclude that $L \in \mathscr{C}$ implies that $L = 0$, thus $M \in \rho_{\leq}(\mathscr{C})$.

Finally, suppose that $\{M_\alpha\}_{\alpha\in X} \subseteq \rho_{\leq}(\mathscr{C})$. We are going to show that $\prod_{\alpha\in X} M_\alpha \in \rho_{\leq}(\mathscr{C})$. Assume that $f : L \rightarrowtail \prod_{\alpha\in X} M_\alpha$ is a monomorphism with $L \in \mathscr{C}$ and $L \neq 0$. Take $\pi_\beta : \prod_{\alpha\in X} M_\alpha \twoheadrightarrow M_\beta$ such that $\pi_\beta \circ f \neq 0$, thus $0 \neq \pi_\beta(f(L)) \leq M_\beta$.

If $0 \neq Rl \leq L$, then $Rl \in \mathscr{C}$. Let us take $0 \neq l = m_{\alpha_1} + \cdots + m_{\alpha_k}$ with the least possible k. Then there exists a monomorphism $Rl \rightarrowtail M_{\alpha_1}$ with $Rm_{\alpha_1} \in \mathscr{C}$ (see Proposition 6 in [6]). This implies that $R_{\alpha_1} = 0$, and thus $m_{\alpha_1} = 0$, a contradiction to the choice of k. Hence $L \in \mathscr{C}$ and $L = 0$. We conclude that $\prod_{\alpha\in X} M_\alpha \in \rho_{\leq}(\mathscr{C})$. Hence $\rho_{\leq}(\mathscr{C}) \in R\text{-tors} \cap \mathscr{L}_{\{\prod\}}$. □

Let σ be a preradical, we have the assignment

$$\rho_{\leq_\sigma}(\mathscr{C}) = \{M \in R\text{-Mod} \mid L \rightarrowtail M, \sigma(L) \in \mathscr{C} \Rightarrow \sigma(L) = 0\}.$$

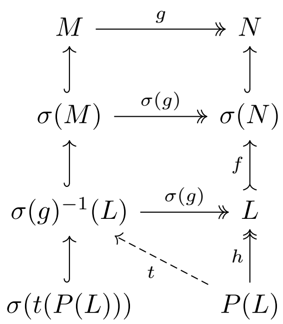
Notice that $\mathbb{F}_\sigma \subseteq \rho_{\leq_\sigma}(\mathscr{C})$. We denote $R\text{-}(\sigma\text{-jtors}) := R\text{-}(\sigma\text{-tors}) \cap \mathscr{L}_{\{\prod\}}$.

Proposition 5.7. *Let R be a left perfect ring. Let σ be an exact and costable preradical. We have an assignment $\rho_{\leq_\sigma} : \mathscr{L}_{\{\leq_\sigma, \twoheadrightarrow_\sigma, \sigma P\}} \to R\text{-}(\sigma\text{-jtors})$.*

Proof. Let $\mathscr{C} \in \mathscr{L}_{\{\leq_\sigma, \twoheadrightarrow_\sigma, \sigma P\}}$. We are going show that $\rho_{\leq_\sigma}(\mathscr{C}) \in R\text{-}(\sigma\text{-jtors})$. Take $M \in \rho_{\leq_\sigma}(\mathscr{C})$ and $N \leq M$. We are going to show that $\sigma(N) \in \rho_{\leq_\sigma}(\mathscr{C})$.

If $L \rightarrowtail \sigma(N)$ is a monomorphism with $\sigma(L) \neq 0$, then as $\sigma(M) \in \rho_{\leq_\sigma}(\mathscr{C})$ and the composition $\sigma(L) \rightarrowtail \sigma(N) \rightarrowtail \sigma(M)$ is a monomorphism with $\sigma(\sigma(L)) = \sigma(L) \neq 0$, we have that $\sigma(L) = \sigma(\sigma(L)) \notin \mathscr{C}$. Hence $\sigma(N) \in \rho_{\leq_\sigma}(\mathscr{C})$.

Now, take $M \in \rho_{\leq_\sigma}(\mathscr{C})$ and $g : M \twoheadrightarrow N$. We are going to show that $\sigma(N) \in \rho_{\leq_\sigma}(\mathscr{C})$. Let us suppose that $f : L \rightarrowtail \sigma(N)$ is a monomorphism with $\sigma(L) \in \mathscr{C}$ and $\sigma(L) \neq 0$. Notice that $\sigma(L) \cong \sigma(f(L)) = f(L) \cap \sigma(N) = f(L) \cong L$. Let $h : P(L) \to L$ be a projective cover of L. We have that $h(\sigma(P(L))) = \sigma(L) \neq 0$, from this it follows $\sigma(P(L)) \in \mathscr{C}$ and $\sigma(P(L)) \neq 0$. We have the following commutative diagram:



As $\sigma(P(L)) \neq 0$ we have that $\sigma(t(P(L))) \neq 0$ and $\sigma(t(P(L))) \in \mathscr{C}$, because $\sigma(t) : \sigma(P(L)) \to \sigma(t(P(L)))$ is an epimorphism. As $\sigma(M) \in \rho_{\leq_\sigma}(\mathscr{C})$ we have that

$\sigma(t(P(L))) \notin \mathscr{C}$, a contradiction. Hence, $\sigma(t(P(L))) = 0$, thus $\sigma(t) = 0$. This implies that $\sigma(P(L)) = 0$, a contradiction.

Then we have that $\sigma(L) \in \mathscr{C}$ implies that $\sigma(L) = 0$. We conclude that $\sigma(N) \in \rho_{\leq_\sigma}(\mathscr{C})$.

Now, let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be an exact sequence with $M', M'' \in \rho_{\leq_\sigma}(\mathscr{C})$. We show that $M \in \rho_{\leq_\sigma}(\mathscr{C})$.

Let $h : L \rightarrowtail M$ be a monomorphism with $\sigma(L) \neq 0$. We have the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{\quad f \quad} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\ & & \uparrow & & \uparrow h & & & & \\ & & h^{-1}(Im(h) \cap Im(f)) & \hookrightarrow & L & & & & \end{array}$$

If $Im(h) \cap Im(f) = 0$, then there exists a monomorphism $L \rightarrowtail M''$ with $\sigma(L) \neq 0$, thus $\sigma(L) \notin \mathscr{C}$. In this case we conclude that $M \in \rho_{\leq_\sigma}(\mathscr{C})$. If $Im(h) \cap Im(f) \neq 0$, then there exists $h^{-1}(Im(h) \cap Im(f)) \rightarrowtail M'$, a monomorphism with $0 \neq \sigma(h^{-1}(Im(h) \cap Im(f)))$. It follows that $\sigma(h^{-1}(Im(h) \cap Im(f))) \notin \mathscr{C}$, which implies that $\sigma(L) \notin \mathscr{C}$. In this case we conclude that $M \in \rho_{\leq_\sigma}(\mathscr{C})$.

Finally, take $\{M_\alpha\}_{\alpha \in X} \subseteq \rho_{\leq_\sigma}(\mathscr{C})$. Let us suppose that $f : L \rightarrowtail \prod\limits_{\alpha \in X} M_\alpha$ is a monomorphism with $\sigma(L) \in \mathscr{C}$ and $\sigma(L) \neq 0$. We will show that $\prod\limits_{\alpha \in X} M_\alpha \in \rho_{\leq_\sigma}(\mathscr{C})$.

We have that $\sigma(L) \overset{\sigma(f)}{\rightarrowtail} \sigma(\prod\limits_{\alpha \in X} M_\alpha) \overset{i}{\hookrightarrow} \prod\limits_{\alpha \in X} \sigma(M_\alpha)$ is a monomorphism. Let $\pi_\beta : \prod\limits_{\alpha \in X} \sigma(M_\alpha) \to \sigma(M_\beta)$ denote the canonical projection such that $\pi_\beta \circ (i \circ \sigma(f)) \neq 0$. For each $0 \neq Rl \in \sigma(L)$ such that $0 \neq \pi_\beta((i \circ \sigma(f))(Rl))$, as $Rl \to \pi_\beta(\sigma(f)(Rl))$ is an epimorphism with $Rl \in \mathscr{C}$ then $\pi_\beta((i \circ \sigma(f))(Rl)) \in \mathscr{C}$. This implies that $\pi_\beta((i \circ \sigma(f))(Rl)) = 0$. Thus $\pi_\beta \circ (i \circ \sigma(f)) \neq 0$ implies $Rl = \sigma(Rl) = 0$, a contradiction. Hence $\sigma(L) = 0$. We conclude that $\prod\limits_{\alpha \in X} M_\alpha \in \rho_{\leq_\sigma}(\mathscr{C})$. Hence $\rho_{\leq_\sigma}(\mathscr{C}) \in R$-($\sigma$-jtors). □

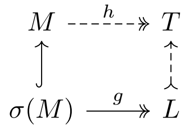
The lattice R-(σ-Conat) $= Skel(\mathscr{L}_{\{\twoheadrightarrow_\sigma\}})$, for an exact and costable preradical σ, is defined in [6] . Where the strong pseudocomplement of $\mathscr{C} \in \mathscr{L}_{\{\twoheadrightarrow_\sigma\}}$ is given by

$$\mathscr{C}^{\perp_{\{\twoheadrightarrow_\sigma\}}} = \{M \in R\text{-Mod} \mid \forall\ M \twoheadrightarrow L,\ \sigma(L) \in \mathscr{C} \Rightarrow \sigma(L) = 0\} \cup \mathbb{F}_\sigma.$$

Proposition 5.8. *Let R be a left perfect ring. Let σ be an exact and costable preradical. Then for each $\mathscr{C} \in \mathscr{L}_{\{\twoheadrightarrow_\sigma\}}$, one has that $\mathscr{C}^{\perp_{\{\twoheadrightarrow_\sigma\}}} \in \mathscr{L}_{\{\sigma P\}}$.*

Proof. Take $\mathscr{D} = \mathscr{C}^{\perp_{\{\twoheadrightarrow_\sigma\}}}$. Let us see that if $M \in \mathscr{D}$, then $\sigma(M) \in \mathscr{D}$.

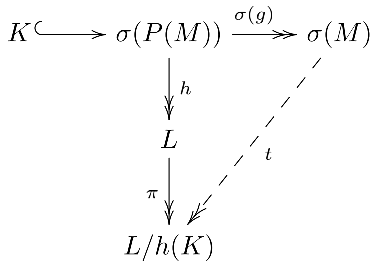
Let us suppose that $g : \sigma(M) \twoheadrightarrow L$ is an epimorphism with $\sigma(L) \in \mathscr{C}$ and $\sigma(L) \neq 0$. We can complete the diagram:



It follows that $\sigma(h) : \sigma(M) \twoheadrightarrow \sigma(T)$ is an epimorphism, thus $\sigma(T) = L = 0$ because $\sigma(L) \in \mathscr{C}$, a contradiction. Hence $\sigma(M) \in \mathscr{D}$.

Let $M \in \mathscr{D}$ and let $g : P(M) \to M$ be a projective cover of M. We are going to show that $\sigma(P(M)) \in \mathscr{D}$.

Notice that $\sigma(P(M))$ is a projective module. Let us suppose that $h : \sigma(P(M)) \twoheadrightarrow L$ is an epimorphism with $\sigma(L) \in \mathscr{C}$ and $\sigma(L) \neq 0$. Denote $K = \ker(\sigma(g))$, thus we have the following commutative diagram:



As σ is exact we have that $\sigma(L) = L$ and $\sigma(L/h(k)) = L/h(K)$, it follows that $\sigma(L) \in \mathscr{C}$ implies that $L/h(K) = \sigma(L/h(K)) \in \mathscr{C}$. As $M \in \mathscr{D}$ and $\sigma(L/h(K)) \in \mathscr{C}$, it follows $L/h(K) = \sigma(L/h(K)) = 0$. Thus $L = h(K)$. As $ker(h) + K = \sigma(P(M))$, we have that $ker(h) = \sigma(P(M))$, which implies that $h = 0$ and $L = 0$, a contradiction. Hence $\sigma(L) \in \mathscr{C}$ implies that $L = \sigma(L) = 0$. We conclude that $\sigma(P(M)) \in \mathscr{D}$. □

R is a left Max ring if and only if every conatural class is closed under direct sums (see [2], Theorem 30). Recall that a ring R is left perfect if each left R-modules has a projective cover. If R is a left perfect ring, then each conatural class in R-Mod is generated by a family of simple R-modules (see Corollary 43 of [1]).

Proposition 5.9. *Let R be a left perfect ring. Let σ be an exact and costable preradical. Then the following statements are equivalent:*

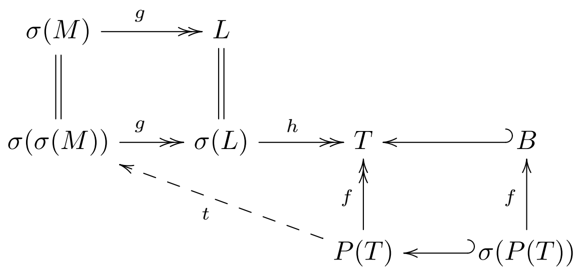
(1) $\mathscr{C} \in R$-(σ-Conat).
(2) $\mathscr{C} \in \mathscr{L}_{\{\twoheadrightarrow_\sigma, ext, \sigma P\}}$.
(3) $\mathscr{C} \in \mathscr{L}_{\{\twoheadrightarrow_\sigma, \oplus, ext, \sigma P\}}$.

Proof. (1) ⇒ (2) We have that if $\mathscr{C} \in R$-(σ-Conat), then $\mathscr{C} \in \mathscr{L}_{\{\twoheadrightarrow_\sigma, ext, \sigma P\}}$ (see Corollary 4 of [6]).

(2) ⇒ (1) Let $\mathscr{C} \in \mathscr{L}_{\{\twoheadrightarrow_\sigma, ext, \sigma P\}}$. We have that $\mathscr{C} \subseteq (\mathscr{C}^{\perp_{\{\twoheadrightarrow_\sigma\}}})^{\perp_{\{\twoheadrightarrow_\sigma\}}}$. Take $M \in (\mathscr{C}^{\perp_{\{\twoheadrightarrow_\sigma\}}})^{\perp_{\{\twoheadrightarrow_\sigma\}}}$. We are going to show that $M \in \mathscr{C}$.

We have that for each $M \in (\mathscr{C}^{\perp_{\{\twoheadrightarrow_\sigma\}}})^{\perp_{\{\twoheadrightarrow_\sigma\}}}$ and each epimorphism $0 \neq g : M \twoheadrightarrow L$ with $\sigma(L) \neq 0$, there exists an epimorphism $0 \neq h : \sigma(L) \twoheadrightarrow T$ with $\sigma(T) \in \mathscr{C}$ and $\sigma(T) \neq 0$. Firstly, we show that $\sigma(M) \in \mathscr{C}$.

Assume that $0 \neq g : \sigma(M) \twoheadrightarrow L$ is an epimorphism with $\sigma(L) \neq 0$, then there exists an epimorphism $0 \neq h : \sigma(L) \twoheadrightarrow T$ with $\sigma(T) \in \mathscr{C}$ and $\sigma(T) \neq 0$. As σ is exact, we have that $\sigma(L) = L$, because $L \leq g(\sigma(M)) \leq \sigma(L)$. Besides, $\sigma(T) = T$. Let $f : P(T) \to T$ be a projective cover, then as σ is costable, then $\sigma(P)$ is projective and there exists $f : P(T) \to \sigma(M)$ such that $f = hgt$:



We have that $\sigma(P(T)) \in \mathscr{C}$, then $t(\sigma(P(T))) = \sigma(t(\sigma((P(T))) \in \mathscr{C}$, besides $t(\sigma(P(T))) \leq \sigma(M)$. Let us take $B = f(\sigma(P(T))) \leq T$, it follows that $B = \sigma(B) \in \mathscr{C}$. We have that $B = hg(t(\sigma(P(T))))$, i. e., B is a quotient of $\sigma(M)$, and we get the exact sequence $0 \to t(\sigma(P(T)) \to \sigma(M) \to B \to 0$, with $t(\sigma(P(T)) \in \mathscr{C}$. This implies that $\sigma(M) \in \mathscr{C}$. Besides, we also have the exact sequence $0 \to \sigma(M) \to M \to M/\sigma(M) \to 0$, with $\sigma(M), M/\sigma(M) \in \mathscr{C}$ (because σ is a radical and $M/\sigma(M) \in \mathbb{F}_\sigma \subseteq \mathscr{C}$), which implies that $M \in \mathscr{C}$. We conclude that $(\mathscr{C}^{\perp_{\{\twoheadrightarrow_\sigma\}}})^{\perp_{\{\twoheadrightarrow_\sigma\}}} \subseteq \mathscr{C}$.

Hence $\mathscr{C} = (\mathscr{C}^{\perp_{\{\twoheadrightarrow_\sigma\}}})^{\perp_{\{\twoheadrightarrow_\sigma\}}}$. It follows that $\mathscr{C} \in R$-(σ-Conat). We conclude that R-(σ-Conat) $= \mathscr{L}_{\{\twoheadrightarrow_\sigma, ext, \sigma P\}}$.

(2) $\Rightarrow$ (3) Since every left perfect ring is left Max, in this case, every conatural class $\mathscr{C}$ is closed under direct sums, this implies that $\overleftarrow{\sigma}(\mathscr{C})$ is closed under direct sums (see [6], Proposition 4).

As R-(σ-Conat) $= \{\overleftarrow{\sigma}(\mathscr{C}) \mid \mathscr{C} \in R\text{-Conat}\}$ (see [6], Proposition 15) we have that all σ-conatural class is closed under direct sums.

(3) $\Rightarrow$ (2) It is clear. □

6. σ-V-rings and σ-Max-rings

We generalize the concept of Max-rings and V-rings.

Definition 6.1. Take σ an idempotent preradical. An R-module M is σ-**coatomic** if each quotient L of M with $\sigma(L) \neq 0$ has a simple quotient S with $\sigma(S) = S$.

Definition 6.2. Take σ an idempotent preradical. A ring R is left σ-**Max** if each R-module M is σ-coatomic.

Definition 6.3. Take an idempotent preradical σ. A ring R is a left σ-**V**-ring if each simple R-module in $\mathbb{T}_\sigma$ is injective.

Theorem 6.4. *Let σ be a left exact preradical. If R is a left σ-V-ring, then R is a left σ-Max-ring.*

Proof. Let $f : M \to N$ an epimorphism where $\sigma(N) \neq 0$. Let us take $0 \neq x \in \sigma(N)$, then Rx has a simple quotient S which is σ-torsion because σ is a left exact preradical. Thus, we have a diagram

$$\begin{array}{ccccc} Rx & \hookrightarrow & \sigma(N) & \hookrightarrow & N \\ \downarrow\!\!\!\downarrow & & & & \\ S & & & & \end{array}$$

, where ${}_RS$ is an injective simple module. Then there is a morphism $N \to S \neq 0$. Hence M is σ-coatomic and consequently, R is a σ-Max-ring. □

Given a class $\mathscr{C}$ of R-modules, we denote $\xi_{\text{conat}}(\mathscr{C})$ the least conatural class containing $\mathscr{C}$ (see [3]), where

$$\xi_{\text{conat}}(\mathscr{C}) = \{M \in R\text{-Mod} \mid \forall\ M \twoheadrightarrow N \neq 0,\ \exists\ N \twoheadrightarrow L \neq 0 \text{ with } L \text{ quotient of some element of } \mathscr{C}\}.$$

Proposition 6.5. *Let σ be an exact and costable preradical. If R is a left σ-Max-ring, then every σ-conatural class is closed under direct sums.*

Proof. Let $\mathscr{C}$ a σ-conatural class and let $\{M_i\}_{i\in I}$ be a class in $\mathscr{C}$. We are going to show that for each quotient L of $\bigoplus_{i\in I} M_i$ such that $\sigma(L) \neq 0$ there exists a quotient U of $\sigma(L)$ with $0 \neq U \in \mathscr{C}$.

Suppose that $\bigoplus_{i\in I} M_i \twoheadrightarrow L$ is an epimorphism with $\sigma(L) \neq 0$. As R is σ-Max, then L has a simple quotient S with $\sigma(S) = S$. As S is also quotient of $\bigoplus_{i\in I} M_i$, then S is quotient of one M_j, for some $j \in I$. As $M_j \in \mathscr{C}$, which is a σ-cohereditary class, and $\sigma(S) = S$, then $S \in \mathscr{C}$. Now $L \twoheadrightarrow S$, and σ-cohereditary imply that $\sigma(L) \twoheadrightarrow \sigma(S) = S$, with $0 \neq S \in \mathscr{C}$. □

Proposition 6.6. *Let σ be a left exact preradical. If each R-module M is σ-retractable, then R is a left σ-Max-ring.*

Proof. We have to prove that each module M is σ-coatomic. Take $M \in R$-Mod and let us suppose that L is a quotient of M such that $\sigma(L) \neq 0$. We will prove that L has a simple quotient S such that $\sigma(S) = S$.

Let $0 \neq Rx \leq \sigma(L)$ be a cyclic module and let us take an epimorphism $g : Rx \to S$, with a simple quotient S. As σ is left exact, then both of Rx and S are of σ-torsion, i.e., $S = \sigma(S)$. Notice that we have a commutative diagram

$$\begin{array}{ccccc} Rx & \hookrightarrow & \sigma(L) & \hookrightarrow & L \\ \downarrow g & & & & \downarrow \bar{g} \\ S & \rightarrowtail & & \longrightarrow & L/Nuc(g). \end{array}$$

As $L/Nuc(g)$ is σ-retractable and $\sigma(S) = S$ then there exists a nonzero morphism $L/Nuc(g) \to S$, which composed with $\bar{g}$ provides a nonzero R-morphism $f : L \to S$.

We conclude that M is σ-Max. □

Acknowledgement. We thank the referee for the attention devoted to the manuscript and for his detailed, precise and useful comments, which improved greatly the quality of this paper.

References

[1] A. Alvarado-García, H. Rincón-Mejía and J. Ríos-Montes, *On the lattices of natural and conatural classes in R-Mod,* Comm. Algebra, 29(2) (2001), 541-556.

[2] A. Alvarado-García, H. Rincón-Mejía and J. Ríos-Montes, *On some lattices of module classes,* J. Algebra Appl., 5(1) (2006), 105-117.

[3] A. Alvarado-García, H. Rincón-Mejía, J. Ríos-Montes and B. Tomé-Arreola, *On conatural classes and cotype submodules,* Int. Electron. J. Algebra, 11 (2012), 64-81.

[4] A. Alvarado-García, C. Cejudo-Castilla, H. Rincón-Mejía, F. Vilchis-Montalvo and M. Zorrilla-Noriega, *On boolean lattices of module classes,* Algebra Colloq., 25(2) (2018), 285-294.

[5] L. Bican, T. Kepka and P. Němec, Rings, Modules and Preradicals, Marcel Dekker, Inc., New York, 1982.

[6] E. Cerda-León and H. Rincón-Mejía, *Big lattices of module classes induced by preradicals*, São Paulo J. Math. Sci., 14 (2020), 185-206.

[7] J. Dauns and Y. Zhou, Classes of Modules, Chapman & Hall/CRC, Boca Raton, FL, 2006.

[8] S. E. Dickson, *A torsion theory for abelian categories,* Trans. Amer. Math. Soc., 121 (1966), 223-235.

[9] J. Golan, Torsion Theories, Longman Scientific & Technical, New York, 1986.

[10] T. Koşan and M. J. Žemlička, *Mod-retractable rings,* Comm. Algebra, 42 (2014), 998-1010.

[11] F. Raggi, J. Ríos, H. Rincón-Mejía, R. Fernández-Alonso and C. Signoret, *The lattice structure of preradicals,* Comm. Algebra, 30(3) (2002), 1533-1544.

[12] H. Rincón-Mejía and M. Zorrilla-Noriega, *On some relations between the lattices R-nat, R-conat and R-tors and the rings they characterize,* J. Algebra Appl., 12(5) (2013), 1250214 (13 pp).

[13] P. F. Smith, *Modules with many homomorphisms,* J. Pure Appl. Algebra, 197 (2005), 305-321.

[14] B. Stenström, Rings of Quotients, Springer-Verlag, New York Heidelberg, 1975.

Erwin Cerda-León and **Hugo Rincón-Mejía** (Corresponding Author)
Departamento de Matemáticas
Facultad de Ciencias
Universidad Nacional Autónoma de México
Circuito exterior, C.U., 04510
Ciudad de México, CDMX, México
e-mails: erwin_nietzsche@hotmail.com (E. Cerda-León)
hurincon@gmail.com (H. Rincón-Mejía)