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ON I-FINITE LEFT QUASI-DUO RINGS

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Abstract. A ring is called left quasi-duo (left QD) if every maximal left ideal is a right ideal, and it is called I-finite if it contains no infinite orthogonal set of idempotents. It is shown that a ring is I-finite and left QD if and only if it is a generalized upper-triangular matrix ring with all diagonal rings being division rings except the lower one, which is either a division ring or it is I-finite, left QD and left 'soclin' (left QDS). Here a ring is called left soclin if each simple left ideal is nilpotent. The left QDS rings are shown to be finite direct products of indecomposable left QDS rings, in each of which $1 = f_1 + \cdots + f_m$ where the f_i are orthogonal primitive idempotents, with $f_k \approx f_l$ for all k, l, and $\approx$ is the block equivalence on $\{f_1, \ldots, f_m\}$.

A ring is shown to be left soclin if and only if every maximal left ideal is left essential, if and only if the left socle is contained in the left singular ideal. These left soclin rings are proved to be a Morita invariant class; and if a ring is semilocal and non-semisimple, then it is left soclin if and only if the Jacobson radical is essential as a left ideal.

Left quasi-duo elements are defined for any ring and shown to constitute a subring containing the centre and the Jacobson radical of the ring. The 'width' of any left QD ring is defined and applied to characterize the semilocal left QD rings, and to clarify the semiperfect case.

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1. Introduction

Throughout this paper every ring R is associative with nonzero unity, all modules are unitary, and module morphisms are written opposite the scalars. We write $J(R)$, $C(R)$, $U(R)$ and $I(R)$, respectively, for the Jacobson radical, the centre, the unit group and the set of idempotents of R; we write $S_l(R)$ and $S_r(R)$ for the left and right socles of R; and we write $Z_l(R)$ and $Z_r(R)$ for the left and right singular ideals of R. We shall abbreviate these as J, S_l, S_r, Z_l and Z_r, respectively, when no confusion results. The ring of $n \times n$ matrices over R will be denoted by $M_n(R)$, and we denote the integers by $\mathbb{Z}$ and write $\mathbb{Z}_n$ for the integers modulo n. Annihilators are written $\mathtt{l}(X)$ and $\mathtt{r}(X)$, and $A \lhd R$ signifies that A is a two-sided ideal of R. If $N \subseteq M$ are modules we write $N \subseteq^{max} M$, $N \subseteq^{ess} M$, and $N \subseteq^{\oplus} M$, respectively, to mean that N is a maximal (essential, direct summand) submodule of M. For phrases p and q, $p =: q$ means 'q is defined to be p'. For a left ring-theoretic condition $\mathfrak{c}$, a ring is called a $\mathfrak{c}$-ring if it is both a left and right $\mathfrak{c}$-ring.

A ring R is left duo [7] if every left ideal is an ideal. Our interest is in:

Definition 1.1. A ring R is called **left quasi-duo** (**left QD**) if every maximal left ideal is an ideal.

These rings were introduced (and named) in 1995 by Yu [23] and then studied for the next 10 years, notably in [9], [11], [12], and [14]. But the idea (not the name) had also arisen in papers by Burgess and Stephenson [3] in 1979 and (independently) by Nicholson [17] in 1997. Commutative, local, and left duo rings are all left QD, but the converse is not true: If a ring D is division, $\begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$ is left QD with none of these properties.

Section 1. Call an ideal $A \lhd R$ **left-max** if A is maximal in ${}_RR$, and call a simple module ${}_RK$ **ideal-simple** if $\mathtt{l}(k) = \mathtt{l}(K)$ for all $0 \neq k \in K$. These notions lead to quick proofs of many left QD properties, making the paper virtually self-contained. For example, $M_n(S)$ is never left QD. A ring is called **I-finite**[1] if it contains no infinite set of orthogonal idempotents, and the I-finite, semiprime, left QD rings are described. The left-max ideals in a split-null extension $\begin{bmatrix} R & V \\ 0 & S \end{bmatrix}$ are identified, a result that is used frequently. Using the remarkable Lam-Dugas characterization of left QD rings [14, Theorem 3.2], left QD *elements* are defined in any ring R and shown to comprise a subring $Q(R)$ of R containing $C(R)$ and $J(R)$. When $R = M_2(D)$, D division, the ring $Q(R)$ is described.

Section 2. The **width** of a left QD ring is defined, used to classify the semilocal left QD rings, determine the left-max ideals in a semiperfect ring R, and prove the first main theorem of the paper:

Triangular Theorem. (Theorem 3.27) *A ring R is left QD and I-finite if and only if R is an $n \times n$ generalized upper triangular matrix ring where the diagonal rings are all division except for the lower right one, which is I-finite, left QD and left 'soclin'.*

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1 Also called *orthogonally finite*.

Here we call a ring **left soclin** if every simple left ideal is contained in the Jacobson radical. This focuses attention on characterizing the I-finite, left QD, left soclin rings (left **QDS rings**).

Section 3. The left soclin rings are characterized in several ways, and proved to constitute a Morita invariant class. No nonzero left soclin ring is semisimple; and it is shown that a semilocal, non-semisimple ring is left soclin if and only if the Jacobson radical is essential as a left ideal. An example is given of a left soclin ring that is not right soclin.

Section 4. An I-finite, indecomposable, left QDS ring B is called a left **brick** if B contains a set $F = \{f_1, f_2, \dots, f_m\}$ of orthogonal, primitive idempotents where $\Sigma_{i=1}^m f_i = 1_B$ and $f_i \approx f_j$ for all i, j, where $\approx$ is the block equivalence for B induced by F. Furthermore, each corner f_iBf_i of B is a left QD ring that is either a division ring or left soclin (not both), and in which the only idempotents are 0 and f_i.

All left QDS rings are I-finite so we can use the block decomposition theorem [1, Theorem 7.9] to refine the triangular theorem (Theorem 3.27) into the second main theorem of the paper:

Structure Theorem. (Theorem 5.15) *A ring* R is I-finite and left QD if and only if the diagonal corners are all division except the lower right one, which is $B_1 \times B_2 \times \cdots \times B_m$ where each B_k is a left brick.

2. Left quasi-duo rings

We begin with some new approaches to left quasi-duo rings, yielding new results and quick proofs of the basic properties we need.

Definition 2.1. An additive subgroup A of a ring R is called **left-max** in R if $A \lhd R$ is an ideal and A is maximal as a left ideal of R.

Thus, R is left QD if and only if every maximal left ideal of R is left-max. A ring R is local if and only if J is left-max (or right-max) in R.

Proposition 2.2. *Let R be a ring and let $B \lhd R$. Then*:

(a) *If R is left QD then R/B is also left QD.*

(b) *R is left QD if and only if R/J is left QD.*

(c) *If $B \subseteq J$:* (i) *A is left-max in R $\Leftrightarrow$ A/B is left-max in R/B.*

(ii) *R is left QD if and only if R/B is left QD.*

Proof. If $B \subseteq A \subseteq R$ then A is an ideal (respectively a maximal left ideal) of R if and only if A/B has the same relationship to R/B. Since $J \subseteq A$ for any left-max ideal A of R, Proposition 2.2 follows. $\square$

Lemma 2.3. Left-max Lemma *Let $A \lhd R$ be an ideal in a ring R. Then*:

$$A \text{ is left-max in } R \quad \Leftrightarrow \quad R/A \text{ is a division ring.}$$

Proof. Let $A \lhd R$ be left-max. Suppose $X \neq 0$ is a left ideal of R/A, say $X = L/A$ where L is a left ideal of R. As $A \subseteq^{max} {}_R R$ it follows that $L = R$. Hence $X = R/A$, so R/A is a division ring.

Conversely, suppose R/A is a division ring, so ${}_{R/A}(R/A)$ is simple. The R- and R/A-actions on ${}_R(R/A)$ agree:

$$r \cdot (x + A) = rx + A = (r + A) \cdot (x + A).$$

It follows that ${}_R(R/A)$ is simple, so $A \subseteq^{max} {}_R R$. □

If $\{A_i \mid i \in I\}$ are ideals in a ring R such that $\cap_{i\in I} A_i = 0$, we say that R is a **subdirect product** of its images R/A_i. A ring R is said to be **reduced** if it has no nonzero nilpotent elements.

Proposition 2.4. *Let R be a left QD ring. Then*:

(a) [9, Corollary 2] *R/J is a subdirect product of division rings.*

(b) [23, Lemma 2.3] *R/J is reduced, so all nilpotents of R are in J.*

Proof. (a) Let $\{A_i \mid i \in I\}$ be the left-max ideals of R, so each $A_i \lhd R$ and R/A_i is division by Lemma 2.3. The map $R \to \Pi_i R/A_i$ given by $r \mapsto < r + A_i >$ is a ring morphism with kernel J. Thus R/J is a subdirect product of its images R/A_i. This proves (a).

(b) As R/J is reduced by (a), the rest is clear. □

A ring R is **semiprime** if it has no nonzero nilpotent ideals.

Proposition 2.5. *If a ring $R \neq 0$ is left QD, I-finite and semiprime, then*

$$R \cong D_n \times \cdots \times D_1 \times S \quad \textit{for some } n \geq 1$$

where each D_i is a division ring, and either $S = 0$ or S is left QD, semiprime and satisfies $\mathtt{r}(A) \subseteq A$ for every left-max ideal A of S.

Proof. Let A be left-max in R. As $\big(A \cap \mathtt{r}(A)\big)^2 \subseteq A\,\mathtt{r}(A) = 0$, we have $A \cap \mathtt{r}(A) = 0$. If $\mathtt{r}(A) \nsubseteq A$ we have $R = \mathtt{r}(A) \oplus A \cong D_1 \times R_1$ where $D_1 = \mathtt{r}(A) \cong R/A$ is division by Lemma 2.3, and $R_1 \cong A$ is left QD and semiprime. Now suppose that $R_1 \neq 0$ and A_2 is left-max in R_1 with $\mathtt{r}(A_2) \nsubseteq A_2$. Then we obtain $R \cong D_1 \times D_2 \times R_2$ where D_2 is division and R_2 is left QD and semiprime. As R is I-finite this process cannot continue, and the Proposition follows. □

Lemma 2.6. *The following are equivalent for a module $Rm \neq 0$:*
(1) $\mathtt{l}(m) \lhd R$. (2) $rm = 0$, $r \in R \Rightarrow rRm = 0$. (3) $\mathtt{l}(m) = \mathtt{l}(Rm)$.
In this case $R/\mathtt{l}(m) \cong \mathrm{end}(Rm)$ as rings.

Proof. The proofs of (1)⇒(2)⇒(3)⇒(1) are omitted. For $a \in R$ define

$$\alpha_a : Rm \to Rm \quad \text{by} \quad (rm)\alpha_a = ram \text{ for all } r \in R.$$

Then α_a is well defined by (2), and α_a is R-linear. With this, define

$$\theta : R \to end(Rm) \quad \text{by} \quad \theta(a) = \alpha_a \text{ for all } a \in R.$$

Then θ is a ring morphism, and $\ker(\theta) = \mathtt{l}(m)$ by (3). Finally, θ is onto. In fact, if $\alpha \in end(Rm)$, and $m\alpha = am$, $a \in R$, then $\alpha = \alpha_a$ because both maps are R-linear and $m\alpha_a = am = m\alpha$.[2] □

Proposition 2.7. [9, Proposition 1] *For a ring R, the following are equivalent:*
(1) *R is left QD.*
(2) *Every left primitive factor ring R/P is division.*
So if R is left QD : R is left primitive ⇔ R is simple ⇔ R is division.

Proof. (1)⇒(2) Write $S = R/P$. As S is a left primitive ring, let Sm be a simple, faithful left S-module. Hence $\mathtt{l}_S(m) \lhd S$ because S is left QD by (1) and Proposition 2.2. But $\mathtt{l}_S(m) = \mathtt{l}_S(Sm)$ by Lemma 2.6, so $\mathtt{l}_S(m) = 0$ because Sm is faithful. Hence $S \cong S/\mathtt{l}_S(m) \cong Sm$, and it follows that S is division. This proves (2).

(2)⇒(1) If $M \subseteq^{max} {}_RR$, write $P = \mathtt{l}(R/M) = \{b \in R \mid bR \subseteq M\}$, a left primitive ideal of R. Hence R/P is division by (2), so ${}_RP \subseteq^{max} {}_RR$. But $P \subseteq M$ and so $M = P \lhd R$, proving (1).

The last statement means showing any left QD, left primitive ring is division. But this follows by the proof of (1)⇒(2) with $P = 0$. □

Example 2.8. If F is a field define the **Weyl algebra** $W(F) = F[x, y]$, where x and y are indeterminants over F and $xy - yx = 1$. Then $W(F)$ is a simple, noetherian domain [15, Page 19]. But $W(F)$ is neither left nor right QD by Proposition 2.7 because it is not a division ring.

If R and S are rings and ${}_RV_S$ is a bimodule, write $\Lambda = \begin{bmatrix} R & V \\ 0 & S \end{bmatrix}$, a ring with matrix operations, called the **split-null extension** of $R \times S$ over V. If $U \in M_2(\mathbb{Z})$ is invertible then $detU = \pm 1$ so $U^{-1}\Lambda U$ is defined and the map $\Lambda \mapsto U^{-1}\Lambda U$ is a

[2] Via $(am)(bm) = abm$, Rm becomes a ring isomorphic to $\mathrm{end}(Rm)$.

ring isomorphism. We call $\Lambda \mapsto U^{-1}\Lambda U$ a **virtual** isomorphism, and say $U^{-1}\Lambda U$ a **virtual copy** of Λ. Taking $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ then $U^{-1} = U$ so

$$\begin{bmatrix} R & V \\ 0 & S \end{bmatrix} = \Lambda \cong U\Lambda\, U = \begin{bmatrix} S & 0 \\ V & R \end{bmatrix}.$$

By Proposition 2.2, $R \times S$ is left QD if and only if R and S are left QD. This extends to the following result which plays a basic role later.

Proposition 2.9. *Let* $\Lambda = \begin{bmatrix} R & V \\ 0 & S \end{bmatrix}$ *be split-null. Then*:

(a) [11, Proposition 10]. Λ *is left QD* $\Leftrightarrow$ R *and* S *are left QD.*

(b) $J(\Lambda) = \begin{bmatrix} J(R) & V \\ 0 & J(S) \end{bmatrix}$ *and* $\Lambda/J(\Lambda) \cong R/J(R) \times S/J(S)$.

(c) *If R is left QD, the left-max ideals of Λ are* $M_A = \begin{bmatrix} A & V \\ 0 & S \end{bmatrix}$ *or* $M_B = \begin{bmatrix} R & V \\ 0 & B \end{bmatrix}$ *where A and B are left-max in R and S respectively. Also, $\Lambda/M_A \cong R/A$ and $\Lambda/M_B \cong S/B$ as rings.*

Proof. (a) and (b) The mapping $\begin{bmatrix} r & v \\ 0 & s \end{bmatrix} \mapsto \big(r + J(R), s + J(S)\big)$ is an onto ring morphism $\Lambda \to R/J(R) \times S/J(S)$ which has kernel $\begin{bmatrix} J(R) & V \\ 0 & J(S) \end{bmatrix} = J(\Lambda)$. This gives (b), then (a) using Proposition 2.2.

(c) Let X be left-max in Λ. As $X \lhd \Lambda$, using a (virtual) copy of Goodearl [8, Proposition 4.1(c)] there exist $A \lhd R$, $B \lhd S$ and a sub-bimodule ${}_RP_S \subseteq {}_RV_S$ such that $X = \begin{bmatrix} A & W \\ 0 & B \end{bmatrix}$ and $AV + VB \subseteq W$. This latter condition implies that if either $A = R$ or $B = S$ then $W = V$. But X is a *maximal* left ideal of Λ, so either $A \neq R$ and $B = S$, or $B \neq S$ and $A = R$. It follows that there are two cases:

$$\text{(i)}\quad X = \begin{bmatrix} A & V \\ 0 & S \end{bmatrix} \qquad \text{or} \qquad \text{(ii)}\quad X = \begin{bmatrix} R & V \\ 0 & B \end{bmatrix}.$$

In Case (i) $R/A \cong \Lambda/X$ is division by Lemma 2.3, so A is left-max in R (again by Lemma 2.3). Hence, $X = M_A$ in the notation of (c). Similarly, in Case (ii) $X = M_B$. This proves (c). □

However, 'Left QD' is not a Morita invariant. To see why requires the next lemma (we omit the proof).

Lemma 2.10. *If $n \geq 1$ and R is a ring, denote $M_n(R) = \Lambda$, and regard R^n as rows. If ${}_RM$ is any left module, let $\mathcal{L}(M)$ denote the lattice of submodules of M. Define maps Φ and Θ as follows*:

$$\Phi : \mathcal{L}({}_\Lambda\Lambda) \to \mathcal{L}({}_RR^n) \quad by \quad \Phi(L) = \left\{ X \in R^n \;\middle|\; \begin{bmatrix} X \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in L \right\}$$

for all left ideals $L \subseteq \Lambda$.

$$\Theta : \mathcal{L}({}_RR^n) \to \mathcal{L}({}_\Lambda\Lambda) \quad by \quad \Theta(X) = \begin{bmatrix} X \\ X \\ \vdots \\ X \end{bmatrix}$$

for all submodules $X \subseteq R^n$. [3]

Then Φ and Θ are mutually inverse, lattice isomorphisms.

Theorem 2.11. *No matrix ring $M_n(R)$ is left QD if $n \geq 2$. So 'left QD' is not a Morita invariant.*

Proof. If $M \subseteq^{max} {}_RR$ let $\overline{M}$ denote the set of $n \times n$ matrices with every entry in column 1 from M, and the other columns arbitrary. Then $\overline{M}$ is a maximal left ideal of $M_n(R)$ by Lemma 2.10, but it is not a right ideal because $n \geq 2$. □

A ring R is called **semilocal** if R/J is a semisimple ring. Assume that $R/J \cong \Pi_{i=1}^n M_{n_i}(D_i)$ where D_i is a division ring for each i. If R is left QD, then each $M_{n_i}(D_i)$ is left QD by Proposition 2.2, so each $n_i = 1$ by Theorem 2.11 and we have $R/J \cong \Pi_{i=1}^n D_i$. This proves (1)⇒(2) in the following proposition, and (2)⇒(1) is clear by Proposition 2.2.

Proposition 2.12. [14, Corollary 4.8(1)] *If R is a ring then the following are equivalent:*

(1) *R is semilocal and left QD.*

(2) *R/J is a finite direct product of division rings.*

We have a more general version of this result in Theorem 3.8 below.

Theorem 2.11 shows that 'left QD' is not a Morita invariant, but we do have the following result from [9]. Because this will be used repeatedly below, we include a shorter (and simpler) proof.

Theorem 2.13. [9, Theorem 3] *If R is left QD so also is eRe for any $e^2 = e \in R$.*

Proof. Write $S = eRe$ and let $X \subseteq^{max} {}_SS$. Then $RX \subseteq Re$, and $RX \neq Re$ because $RX = Re$ implies that

$$S = eRe = e(RX) = eR(eX) = SX = X, \text{ a contradiction.}$$

3 In words: $\Phi(L)$ is the set of rows of matrices in L, and $\Theta(X)$ is the set of matrices with rows from X.

Hence, by Zorn's lemma, choose ${}_RM$ such that $RX \subseteq M \subseteq^{max} Re$. But then $X = SX = eReX = eRX \subseteq eM \subset S$ as $e \notin M$. Thus $X = eM$ because eM is a left ideal of S. Now write $\bar{M} = M \oplus R(1-e)$, and observe

$$\frac{R}{\bar{M}} = \frac{Re \oplus R(1-e)}{M \oplus R(1-e)} \cong \frac{Re}{M}.$$

Hence $\bar{M} \subseteq^{max} {}_RR$, so $\bar{M} \lhd R$ by hypothesis. Since $\bar{M}e = Me$, we have $\bar{M}S = MS$, and so obtain

$$XS = (eM)S = eMSe = e(\bar{M}S)e \subseteq e\bar{M}e = eMe = Me = X.$$

This shows that X is a right ideal of S, as required. $\square$

A ring R is called **directly finite**[4] (**DF**) if the following equivalent conditions are satisfied:

$$(1)\, ab = 1 \Rightarrow ba = 1. \;\; (2)\, aR = R \Rightarrow Ra = R. \;\; (3)\, Ra = R \Rightarrow aR = R.$$

The following result seems to have been first mentioned in [14].

Lemma 2.14. *Every left QD ring is directly finite. The converse fails.*

Proof. If $aR = R$ and $Ra \neq R$ let $Ra \subseteq A$ where $A \subseteq^{max} {}_RR$. Then $A \lhd R$, so $R = R^2 = R(aR) \subseteq A$, a contradiction. For the converse, if F is a field the ring $M_2(F)$ is DF but it is not left QD by Theorem 2.11.[5] $\square$

A ring R is left **morphic**, [18], if $R/Ra \cong \mathtt{l}(a)$ for all $a \in R$. Examples: local rings and [18, Example 4] **unit-regular** rings $\big($that is if $a \in R$ then $a = aua$ with $u \in U(R)\big)$.

Proposition 2.15. *The following are equivalent for a ring R:*

(1) *R is left morphic, left QD, and semiperfect.*

(2) *R is a finite direct product of local rings.*

Proof. (1)⇒(2) By [18, Theorem 29], $R \cong \Pi_{i=1}^{k} M_{n_i}(R_i)$ where each $M_{n_i}(R_i)$ is left morphic and $R_i \cong e_iRe_i$ for some local $e_i^2 = e_i \in R$ (that is e_iRe_i is a local ring). But R is left QD by (1), so each $n_i = 1$ and consequently $R \cong \Pi_{i=1}^{k} R_i \cong \Pi_{i=1}^{k} e_iRe_i$. This proves (2).

(2)⇒(1) Left morphic rings are closed under direct products. $\square$

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[4] Also called Dedekind finite.

[5] $M_2(\mathbb{Z}_2)$ is directly finite and *exchange*, but it is not left QD.

A ring is **abelian**[6] if all idempotents are central. For left QD rings, the main result here involves the exchange rings. Crawley and Jónsson [5] defined the exchange property for modules. Warfield [22] showed that $_RR$ has the exchange propery if and only if the same is true of R_R, and called R an **exchange ring** in this case. In 1979, we had the following result:

Theorem 2.16. **Burgess and Stephenson** [3] *Every abelian exchange ring is left QD.*

Note added in Proof: The proof of Theorem 2.16 uses sheaf-theoretical techniques. A short direct proof using [16] was accepted by the Canadian Mathematical Bulletin on August 6, 2018.

Note: $\mathbb{Z}$ is abelian and QD, but not exchange; and $\begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ is exchange and QD but not abelian.

A ring R is called **clean** if each $a \in R$ has the form $a = e + u$ where $e^2 = e$ and $u \in U(R)$. By [16, Proposition 1.8] every clean ring R is exchange; conversely if R is abelian. (See [23, Theorem 4.2].)

Proposition 2.17. (1) *Every clean ring is exchange.*

(2) *If R is exchange and left QD then R is clean.*

Proof. (1) is [16, Proposition 1.8]. As to (2), let R be exchange and left QD. Then R/J is also exchange by [16, Proposition 1.5] and left QD by Proposition 2.2(b). But then R/J is reduced by Proposition 2.4(b), and so is abelian. Hence R/J is clean again by [16, Proposition 1.8]. $\square$

For another characterization of the left QD rings, a module $_RM$ is called **very semisimple** (**VSS**) [17] if Rm is simple for all $0 \neq m \in M$. These modules are all homogeneous and semisimple, and it is proved that a ring is left QD if and only if every homogeneous, semisimple left module is VSS. We give a much shorter proof.

Lemma 2.18. [17, Lemma 1] *If $_RM$ is semisimple, then the following are equivalent:*

(1) *M is very semisimple.*

(2) *If Rm_1 and Rm_2 are simple, $m_1, m_2 \in M$, and $m_1 + m_2 \neq 0$, then $R(m_1 + m_2)$ is simple.*

[6] This term was introduced in 1968 by Kaplansky in his "Rings of Operators". In view of the usage in group theory, a better term would be *Abel*, Drazin [6], 1958.

Proof. (1)⇒(2) is clear. Given (2) and $0 \neq m \in M$, let $m \in \oplus_{i=1}^{k} Rx_i$. each Rx_i simple. Write $m = m_1 + m_2 + \cdots + m_k$ where $m_j \in Rx_j$ for each j. We may assume that each $m_j \neq 0$, so $Rm_j = Rx_j$ is simple. If $k = 1$ then $Rm_1 = Rx_1$ is simple. If $k = 2$ then, as $Rm_1 \oplus Rm_2$ is direct, $m_1 + m_2 \neq 0$ so $Rm = R(m_1 + m_2)$ is simple by (2).

If $k = 3$ then $Rm = R\big((m_1 + m_2) + m_3\big)$ is simple in the same way because $(m_1 + m_2) + m_3 \neq 0$. Continuing in this way proves (1). □

Lemma 2.19. [17, Proposition] *Every VSS module is homogeneous.*[7]

Proof. Assume ${}_RM$ is VSS. If Rm_1 and Rm_2 are simple, $m_1, m_2 \in M$, we show that $Rm_1 \cong Rm_2$. We may assume $Rm_1 \neq Rm_2$, so $Rm_1 \oplus Rm_2$ is direct. Hence $m_1 + m_2 \neq 0$, so $R(m_1 + m_2)$ is simple by Lemma 2.18.

Let $\pi_i : Rm_1 \oplus Rm_2 \to Rm_i$ be the projection, $i = 1, 2$. Then write

$$\alpha_i : R(m_1 + m_2) \to Rm_i \;\text{ for the restriction of }\; \pi_i.$$

Then $m_i\alpha_i = m_i$ for each i, so each α_i is an isomorphism by Schur's lemma. Hence $Rm_1 \cong R(m_1 + m_2) \cong Rm_2$. □

The following well known lemma will be needed below.

Lemma 2.20. *Let $A, B \lhd R$ where ${}_R(R/A) \cong {}_R(R/B)$. Then $A = B$.*

Theorem 2.21. *The following are equivalent for a ring R:*

(1) *R is left QD.*

(2) *Every homogeneous, semisimple left R-module is VSS.*

(3) *If $K \cong N$ are simple left R-modules then $K \oplus N$ is VSS.*

(4) *If K is a simple left R-module then $K \oplus K$ is very semisimple.*

Proof. It is clear that (2)⇒(3)⇒(4).

(1)⇒(2) Let ${}_RM$ be homogeneous and semisimple, and let Rm_1 and Rm_2 be simple, $m_i \in M$, where $m_1 + m_2 \neq 0$. By Lemma 2.18 it suffices to show that $R(m_1 + m_2)$ is simple. Observe that $Rm_1 \cong Rm_2$ because M is homogeneous, so $R/\mathtt{l}(m_1) \cong R/\mathtt{l}(m_2)$. But $\mathtt{l}(m_i) \lhd R$ for each i by (1), so Lemma 2.20 implies that $\mathtt{l}(m_1) = \mathtt{l}(m_2)$. Hence $\mathtt{l}(m_1 + m_2) \supseteq \mathtt{l}(m_1) \cap \mathtt{l}(m_2) = \mathtt{l}(m_1)$. As $\mathtt{l}(m_1) \subseteq^{max} {}_RR$ we have $\mathtt{l}(m_1 + m_2) = \mathtt{l}(m_1)$ is a maximal left ideal, as required.

(4)⇒(1) Let $L \subseteq^{max} {}_RR$, and consider ${}_RX = R/L \oplus R/L$. Given $r \in R$, write $x = (1 + L, r + L) \in X$. To show that $Lr \subseteq L$ it suffices to show that $Lx = 0$. Suppose on the contrary that $Lx \neq 0$. Since X is homogeneous and semisimple, (4)

[7] The converse fails as $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ is semisimple, but $\mathbb{Z}m$ is not simple if $m = (1 + 2\mathbb{Z}, 1 + 3\mathbb{Z})$.

implies that Rx is simple. As $Lx \neq 0$, it follows that $Lx = Rx$, whence $x = tx$ for some $t \in L$. This means that $(1 + L, r + L) = (t + L, tr + L)$, so $1 + L = t + L = L$, a contradiction. $\square$

Remark. This proves again that $R = M_n(D)$ cannot be left QD if D is division and $n \geq 2$. In fact, ${}_RR$ is semisimple but column 1 in R is not simple.

Left QD elements and the Lam-Dugas condition

To this point we have used only Definition 1.1 to study left QD rings. We begin with an observation stemming from a remarkable theorem of Lam and Dugas [14, Theorem 3.2].

Lemma 2.22. *If R is a ring, the following conditions are equivalent for any element $q \in R$:*

QD1. *$Mq \subseteq M$ for every maximal left ideal M of R.*

QD2. *$R = Ra + R(1 - aq)$ for any $a \in R$.*

Then by Definition 1.1, R is left QD $\Leftrightarrow$ QD2 holds for every $q \in R$.

Proof. QD1$\Rightarrow$QD2. Assume QD1. If $a \in R$ and $Ra + R(1 - aq) \neq R$, let $Ra + R(1 - aq) \subseteq M$, where $M \subseteq^{max} {}_RR$. Then $aq \in Mq \subseteq M$ by QD1. But $1 - aq \in M$ too, a contradiction.

QD2$\Rightarrow$QD1. Assume QD2. If $M \subseteq^{max} {}_RR$ and $Mq \nsubseteq M$, then $M + Mq = R$, say $c + aq = 1$ where $c, a \in M$. Hence

$$Ra + R(1 - aq) = Ra + Rc \subseteq M, \text{ contradicting QD2.}$$ $\square$

Definition 2.23. If R is a ring, $q \in R$ is called a **left QD element** of R if both QD1 and QD2 hold.

Condition QD2 provides a completely new perspective on left QD rings. As an illustration, if R is left QD, Lam and Dugas give a proof [14, Remark 4.4] that $R/J(R)$ is reduced using only QD2. Here is a similar proof that every left QD ring R is directly finite.

> *Suppose $aR = R$, say $ab = 1$, $a, b \in R$. As b is left QD,* QD2 *holds with $q = b$, so $R = Ra + R(1 - ab) = Ra$, as required.*

Definition 2.24. If R is a ring, write $Q(R) = \{q \mid q \text{ is left QD in } R\}$.

Lemma 2.25. *Let R be a ring, $Q = Q(R)$, $J = J(R)$ and $C = C(R)$.*

(a) *Q is a unital subring of R.*

(b) *R is left QD if and only if $Q = R$.*

(c) $C \subseteq Q$ *and* $J \subseteq J(Q)$.

(d) *However, a left QD subring of* R *need not be contained in* $Q(R)$.

Proof. (a) This follows from QD1.

(b) R is left QD $\;\Leftrightarrow\;$ Every $q \in R$ is left QD $\;\Leftrightarrow\;$ $R = Q$.

(c) If $c \in C$ then $Ra + R(1 - ca) = R$ for all a, so $c \in Q$ by QD2. If $b \in J$ then $Ra + R(1 - ab) = R$ for all a, so $J \subseteq Q$. But then J is a quasi-regular ideal of Q, so $J \subseteq J(Q)$.

(d) If F is a field write $R = M_2(F)$ and $S = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$. Then S is a left QD subring of R, but $S \nsubseteq Q(R)$ because

$$Q(R) = C(R) = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \middle| \, a \in F \right\}\text{—see Example 2.27(b) below.} \qquad \square$$

Proposition 2.26. *Let* R *be a ring,* $Q = Q(R)$*. Given* $B \lhd R$ *:*

Define $\quad \varphi : Q \to Q(R/B) \quad$ *by* $\quad \varphi(q) = q + B$ *for all* $q \in Q$.

(a) φ *is a ring morphism, and* ker $\varphi = B$.

(b) *If* $B \subseteq J(R)$ *then* φ *is onto.*

(c) *If* $B = J(R)$ *:* (i) $\varphi : Q \to Q\big(R/J(R)\big)$ *is onto with kernel* J.
(ii) $J(Q) = J$.

Proof. Write $\bar{R} = R/B$, $\bar{r} = r + B$ for $r \in R$, $J = J(R)$ and $Q = Q(R)$.

(a) We need only show φ is well defined, that is $\bar{q} \in Q(\bar{R})$ when $q \in Q$. Given $\bar{a} \in \bar{R}$ we have $Ra + R(1 - aq) = R$ by QD2, so we have $\bar{R}\bar{a} + \bar{R}(\bar{1} - \bar{a}\bar{q}) = \bar{R}$. Hence $\bar{q} \in Q(\bar{R})$, as required.

(b) Assume $B \subseteq J$. If $x \in Q(\bar{R})$, say $x = \bar{y}$, $y \in R$, we prove φ is onto by showing that $y \in Q$. To see this, fix $a \in R$. Since $\bar{y} = x \in Q(\bar{R})$, we have $\bar{R}\bar{a} + \bar{R}(\bar{1} - \bar{a}\bar{y}) = \bar{R}$, say $ra + s(1 - ay) - 1 =: b \in B$, $r, s \in R$. Because $B \subseteq J$ we have $u =: 1 + b \in U(R)$, so $ra + s(1 - ay) = u$. Hence we obtain $(u^{-1}r)a + (u^{-1}s)(1 - ay) = 1$. As $a \in R$ was arbitrary, this shows $y \in Q(R)$, proving (b).

(c) For c(i), take $B = J$ in (b). For c(ii): $J \subseteq J(Q)$ by Lemma 2.25(c). Then $J = J(Q)$ as $J(Q)/J$ is a quasi-regular ideal of R/J. $\square$

Question 1. *Is the converse true in* Proposition 2.26(b)?

Example 2.27. Let D be a division ring, and write $\Lambda = M_2(D)$. Then:

(a) There exists $u \in D$, $u \neq 0$, such that $Q(\Lambda) = \left\{ \begin{bmatrix} a & 0 \\ 0 & u^{-1}au \end{bmatrix} \middle| \, a \in D \right\}$.

(b) In particular $Q(\Lambda) = C(\Lambda)$ if D is a field.

Proof. As (a)⇒(b), we prove (a). Write $M_1 = \begin{bmatrix} 0 & D \\ 0 & D \end{bmatrix}$ and $M_2 = \begin{bmatrix} D & 0 \\ D & 0 \end{bmatrix}$ for the 'standard' maximal left ideals of Λ. Suppose $\lambda = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. It is routine to check that

$$M_1\lambda \subseteq M_1 \quad \Leftrightarrow \quad c = 0, \qquad \text{and} \qquad M_2\lambda \subseteq M_2 \quad \Leftrightarrow \quad b = 0.$$

So if $\lambda \in Q(\Lambda)$ then $\lambda = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ is diagonal.

If M is a maximal left ideal of Λ then $M = \begin{bmatrix} X \\ X \end{bmatrix}$ by Lemma 2.10 where X is a maximal D-submodule of D^2, written as rows. Hence $\dim_D(X) = 1$ so $X = Dv$ where $v = (p, q) \neq 0$. If $p = 0$ or $q = 0$ we obtain $M = M_1$ or $M = M_2$ respectively. But if $p \neq 0 \neq q$, assume $v = (1, u)$ where $0 \neq u \in D$, so $X = Dv = \{(d, du) \mid d \in D\}$, and we obtain a *third* maximal left ideal $M_3 = \left\{ \begin{bmatrix} e & eu \\ f & fu \end{bmatrix} \middle| e, f \in D \right\}$. So if $\lambda = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \in Q(\Lambda)$, it remains to show that

$$M_3\lambda \subseteq M_3 \quad \Leftrightarrow \quad d = u^{-1}au.$$

If $M_3\lambda \subseteq M_3$ then $\begin{bmatrix} 1 & u \\ 1 & u \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} e & eu \\ f & fu \end{bmatrix}$ for some $e, f \in D$, and so (top row) $a = e$ and $ud = eu$, whence $ud = au$ and $d = u^{-1}au$.

Conversely, if $\lambda = \begin{bmatrix} a & 0 \\ 0 & u^{-1}au \end{bmatrix}$ then

$$\begin{bmatrix} e & eu \\ f & fu \end{bmatrix} \lambda = \begin{bmatrix} e & eu \\ f & fu \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & u^{-1}au \end{bmatrix} = \begin{bmatrix} ea & eau \\ fa & fau \end{bmatrix} \in M_3$$

for all $e, f \in D$. Hence $M_3\lambda \subseteq M_3$, and the proof is complete. □

Question 2. *If $\Lambda = M_n(F)$, F a field, is $Q(\Lambda) = C(\Lambda)$?*

For any $n \geq 2$, if $\Lambda = M_n(D)$, D a division ring, then $Q(\Lambda)$ consists of diagonal matrices as in paragraph 1 of the proof of Example 2.27.

Question 3. *Describe the units and idempotents of $Q(R)$.*

Every left QD ring is directly finite (Lemma 2.14), but not conversely—$M_2(\mathbb{R})$ is I-finite but it is not left QD by Theorem 2.11. However the **left unimodularity** property in Condition 2 of the following result—stronger than directly finite—actually characterizes the left QD rings.

Theorem 2.28. Lam-Dugas [14, Theorem 3.2] *The following conditions are equivalent for a ring R:*

(1) *R is left QD.*

(2) $a_1R + \cdots + a_nR = R$ *implies* $Ra_1 + \cdots + Ra_n = R$.

(3) $aR + bR = R$ *implies* $Ra + Rb = R$.

(4) *If* $RXR = R$ *where* X *is a finite set, then* $RX = R$.

In this case: (a) $RaR = R$, $a \in R$, *implies* a *is a unit.*

(b) $ReR = R$, $e^2 = e \in R$, *implies* $e = 1$.

Proof. (1)⇒(2) Let $a_1R + \cdots + a_nR = R$. If $Ra_1 + \cdots + Ra_n \neq R$, let

$$Ra_1 + \cdots + Ra_n \subseteq M \text{ where } M \subseteq^{max} {}_RR.$$

Then $M \lhd R$ by (1) so $R = a_1R + \cdots + a_nR \subseteq M$, a contradiction.

(2)⇒(3) This is clear.

(3)⇒(1) Always $aR + (1 - aq)R = R$, so $Ra + R(1 - qa) = R$ by (3). Thus QD2 holds for R, so R is left QD by Lemma 2.22.

(2)⇒(4) Let $RXR = R$ and write $1 = \Sigma_i r_i x_i s_i$, $r_i, s_i \in R$, $x_i \in X$. Thus $R = \Sigma r_i x_i R$ so (2) gives $R = \Sigma R r_i x_i \subseteq RX$.

(4)⇒(2) If $\Sigma_{i=1}^n a_i R = R$ then $R = RXR$ where $X = \{a_1, \ldots, a_n\}$. By (4), $R = RX = \Sigma_{i=1}^n Ra_i$.

Finally, with (4) and Lemma 2.14, (a) and then (b) are routine. □

The enigmatic condition QD2 is a first order statement, which plays a basic role in another remarkable result of Lam and Dugas:

Theorem 2.29. [14, Corollary 3.6] *If $\mathcal{Q}$ is the class of left QD rings, then*:

(a) *$\mathcal{Q}$ is closed under direct products, direct limits and ultraproducts.*

(b) *A direct product $\Pi_{i\in I} R_i$ is in $\mathcal{Q}$ ⇔ each R_i is in $\mathcal{Q}$.*

(c) *Let R be a finite subdirect product $R \hookrightarrow \Pi_{i=1}^n R/A_i$, $A_i \lhd R$.*

Then R is in $\mathcal{Q}$ ⇔ each R/A_i is in $\mathcal{Q}$.

3. Width and the triangular theorem

In this section we introduce the ideal-simple left modules, a 'dual' of the left-max ideals. This leads to a new classification of the left QD rings, and to a description of the left-max ideals in a semiperfect ring. Finally, we present our first structure theorem for I-finite left QD rings in terms of generalized upper triangular matrix rings.

Ideal-simple modules

If R is a ring and ${}_RK$ is simple, Lemma 2.6 shows **IS1** ⇔ **IS2** where:

$$\begin{aligned} &\textbf{IS1:}\ \ \mathtt{l}(k) \lhd R \text{ for all } k \in K. \\ &\textbf{IS2:}\ \ \mathtt{l}(k) = \mathtt{l}(K) \text{ for all } 0 \neq k \in K. \end{aligned} \qquad (\S)$$

Definition 3.1. ${}_RK$ is **ideal-simple** if ${}_RK$ is simple and (§) holds.

If ${}_RK$ is ideal-simple and ${}_RM \cong {}_RK$, then ${}_RM$ is ideal-simple too.

These ideal-simple modules have two virtues for us. The first is that they provide a new way to think about the left QD rings:

Theorem 3.2. *The following conditions are equivalent for a ring R:*

(1) *R is left quasi-duo.*

(2) *Every simple left R-module is ideal-simple.*

Proof. (1)⇒(2) If ${}_RK$ is simple and $0 \neq k \in K$ then $\mathtt{l}(k)$ is a maximal left ideal of R. Hence $\mathtt{l}(k) \lhd R$ by (1), so (§) gives $\mathtt{l}(k) = \mathtt{l}(K)$.

(2)⇒(1) Let M be a maximal left ideal of R. By (2), the simple module R/M is ideal-simple. As $R/M = R(1+M)$, it follows that $M = \mathtt{l}(1+M) = \mathtt{l}(R/M) \lhd R$ by (§). This proves (1). □

The second virtue of the ideal-simple modules is that they serve as 'duals' of the left-max ideals. More precisely: While Lemma 2.3 characterizes the *ideals* that are left-max; condition (a) in the following Lemma characterizes the *maximal left ideals* that are left-max.

Lemma 3.3. *Let R be any ring. Then*:

(a) *Let A be a maximal left ideal of R. Then A is left-max in R if and only if ${}_R(R/A)$ is ideal-simple.*

(b) *Let ${}_RK$ be a simple module. Then ${}_RK$ is ideal-simple if and only if $\mathtt{l}(K)$ is left-max in R.*

Proof. Let R denote a ring.

(a) Here ${}_R(R/A)$ is simple. We show: $A \lhd R \Leftrightarrow R/A$ is ideal-simple.

(⇒). First, $A = \mathtt{l}(R/A)$ because $A \lhd R$. If $0 \neq k \in R/A$ we have $A = \mathtt{l}(R/A) \subseteq \mathtt{l}(k) \neq R$. But $A \subseteq^{max} {}_RR$ so $\mathtt{l}(k) = A = \mathtt{l}(R/A)$. Hence R/A is ideal-simple by (§).

(⇐). Here $A = \mathtt{l}(1+A)$ and $0 \neq 1 + A \in R/A$. As R/A is ideal-simple, $A \lhd R$ by (§).

(b) Now ${}_RK$ is simple. We show: ${}_RK$ is ideal-simple $\Leftrightarrow$ $\mathtt{l}(K)$ is left-max.

(⇒) Let ${}_RK$ be ideal-simple. If $0 \neq k \in K$ then $\mathtt{l}(k) = \mathtt{l}(K) \lhd R$, so $R/\mathtt{l}(K) \cong Rk = K$. Hence $\mathtt{l}(K) \subseteq^{max} {}_RR$ and so is left-max.

(⇐) Let $\mathtt{l}(K)$ be left-max in R. If $0 \neq k \in K$, then $\mathtt{l}(K) \subseteq \mathtt{l}(k) \neq R$, so $\mathtt{l}(K) = \mathtt{l}(k)$ as $\mathtt{l}(K) \subseteq^{max} {}_RR$. By (§), ${}_RK$ is ideal-simple. □

Our next application of these ideas is to define the:

Width of a left Quasi-duo ring

For a ring R, the isomorphism equivalence $\cong$ partitions the class of all simple left R-modules. The equivalence class of ${}_RK$, written

$$\text{class}_R K = \{{}_RX \mid {}_RX \cong {}_RK\},$$

is called the **isomorphism class** of K.

Definition 3.4. Let R be a left QD ring, and write:

$\mathcal{A}(R) = \{A \mid A \text{ is left-max in } R\}$.

$\mathcal{C}(R) = \{\text{class}_R K \mid {}_RK \text{ is ideal-simple}\}$.

Write $|X|$ for the cardinality of a set X.

Theorem 3.5. Width Theorem *If R is left QD then* $|\mathcal{A}(R)| = |\mathcal{C}(R)|$.

Proof. Write $\mathcal{A}(R) = \mathcal{A}$ and $\mathcal{C}(R) = \mathcal{C}$. Lemma 3.3 shows that $\mathcal{A}$ is nonempty if and only if $\mathcal{C}$ is nonempty. So we have two cases: $|\mathcal{A}| = 0$ and $|\mathcal{C}| = 0$, and $|\mathcal{A}| \neq 0$ and $|\mathcal{C}| \neq 0$. In the first case, $|\mathcal{A}| = |\mathcal{C}|$ is clear. In the second case define: $\Phi : \mathcal{A} \to \mathcal{C}$ and $\Psi : \mathcal{C} \to \mathcal{A}$ by:

$\Phi(A) = \text{class}(R/A)$ for all left-max $A \in \mathcal{A}$.

$\Psi(\text{class}\, K) = \mathtt{l}(K)$ for any ideal-simple module ${}_RK \in \mathcal{C}$.

Claim. Ψ is well defined.

Proof. Suppose $\text{class}(K) = \text{class}(N)$ where ${}_RK$ and ${}_RN$ are ideal-simple, $K \cong N$. Choose $0 \neq k \in K$ and $0 \neq n \in N$, so $\mathtt{l}(k) = \mathtt{l}(K)$ and $\mathtt{l}(n) = \mathtt{l}(N)$. Then

$$R/\mathtt{l}(K) = R/\mathtt{l}(k) \cong Rk = K \quad \text{and} \quad R/\mathtt{l}(N) = R/\mathtt{l}(n) \cong Rn = N.$$

Hence $R/\mathtt{l}(K) \cong K \cong N \cong R/\mathtt{l}(N)$ as left R-modules. But then $\mathtt{l}(K) = \mathtt{l}(N)$ by Lemma 2.20. This proves the Claim.

To see that $|\mathcal{A}| = |\mathcal{C}|$ we show Φ and Ψ are mutually inverse. To show $\Psi \circ \Phi = 1_{\mathcal{A}}$, let $A \in \mathcal{A}$. Then we have:

$$\Psi\big(\Phi(A)\big) = \Psi\big(\text{class}(R/A)\big) = \mathtt{l}(R/A) = A \quad \text{because } A \lhd R.$$

To prove $\Phi \circ \Psi = 1_{\mathcal{C}}$, let ${}_RK$ be ideal-simple and then choose $0 \neq k \in K$ where $\mathtt{l}(k) = \mathtt{l}(K)$. Then:

$$\Phi\big(\Psi(\text{class}\, K)\big) = \Phi\big(\mathtt{l}(K)\big) = \text{class}\big(R/\mathtt{l}(k)\big) = \text{class}(Rk) = \text{class}\, K,$$

as required. This proves Theorem 3.5. $\square$

Definition 3.6. Define the (left) **width** $\omega(R)$ of a left QD ring R by

$$\omega(R) = |\mathcal{A}(R)| = |\mathcal{C}(R)|.$$

Lemma 3.7. (a) *If R is left QD then $\omega(R) = 1$ if and only if R is local.*
(b) *If $R \cong S$ are left QD, then $\omega(R) = \omega(S)$.*
(c) *If R is left QD then $\omega(R) = \omega(R/J)$ where $J = J(R)$.*
(d) *If $R \cong D_1 \times \cdots \times D_n$ where each D_i is a division ring, then $\omega(R) = n$.*

Proof. (a) If $\omega(R) = 1$ then $|\mathcal{A}(R)| = 1$ so R is local (being left QD). The converse is clear.

(b) Ring isomorphisms preserve left-max ideals.

(c) The map $A \mapsto A/J$ from $\mathcal{A}(R) \to \mathcal{A}(R/J)$ is a bijection.

(d) The maximal ideals of R are $D_1 \times \cdots \times \hat{D}_i \times \cdots \times D_n$ (where D_i is omitted). □

The following theorem refines the characterization (in Proposition 2.12) of the semilocal left QD rings as the rings R such that R/J is a finite direct product of division rings.

Theorem 3.8. *Let R be left QD. If $n \leq 1$, the following are equivalent:*

(1) *R is semilocal of width n.*
(2) *$R/J(R)$ is a direct product of n division rings.*
(3) *R has n maximal left ideals (respectively maximal right ideals).*
(4) *R has n isomorphism classes of simple left modules (respectively simple right modules).*

Proof. Write $J(R) = J$ and $R/J = \bar{R}$.

(1)⇒(2) Given (1), $\bar{R} \cong \oplus_{l=1}^{m} M_{n_i}(D_i)$ where each D_i is a division ring. As $\bar{R}$ is left QD so also is each $M_{n_i}(D_i)$. Hence $n_i = 1$ for each i by Theorem 2.11, so $\bar{R}$ is a product of m division rings. Now (2) follows because, using Lemma 3.7, $m = \omega(\bar{R}) = \omega(R) = n$.

(2)⇒(3) By (2), $\omega(\bar{R}) = n$ by Lemma 3.7(d), so $|\mathcal{A}(R)| = \omega(R) = n$. As R is QD, this proves (3).

(3)⇒(4) As R is left QD, this is by Theorem 3.5.

(4)⇒(1) By (4) and Theorem 3.2 we have $|\mathcal{C}(R)| = n$, so $|\mathcal{A}(R)| = n$ by Theorem 3.5. If $\{A_1, A_2, \ldots, A_n\}$ are the distinct left-max ideals of R, then each R/A_i is a division ring by Lemma 2.3 (as R is left QD). The map $r \mapsto \langle r + A_i \rangle$ from $R \to \Pi_{i=1}^{n} R/A_i$ is a ring morphism with kernel J because R is left QD, and it is onto by the Chinese remainder theorem because $A_i + A_j = R$ whenever $i \neq j$. Hence $R/J \cong \Pi_{i=1}^{n} R/A_i$ is semilocal. Now $\omega(R) = n$ follows by Lemma 3.7(d). □

Corollary 3.9. *A ring R is left QD of width 1 if and only if R is local.*

Question 4. *Describe the left* QD *rings of width* 2.

Here are two width 2 examples:

1) If D and B are division, any split-null extension of $D \times B$ is artinian with $J^2 = 0$.

2) The following example is a noetherian PID with $J \supset J^2 \supset \cdots$.

Example 3.10. If $p \neq q$ are primes in $\mathbb{Z}$ write

$$\mathbb{Z}_{(p,q)} =: \{\tfrac{n}{m} \in \mathbb{Z} \mid p \nmid m \text{ and } q \nmid m\}.$$

Then $\mathbb{Z}_{(p,q)}$ is a commutative (so quasi-duo), noetherian, semilocal, PID with $R/J \cong \mathbb{Z}_p \times \mathbb{Z}_q$.

Proposition 3.11. [14, Question 7.7] *The following are equivalent:*

(1) *Every left QD ring is also right QD.*

(2) *Every left primitive, right quasi-duo ring is a division ring.*

Question 5. Lam-Dugas *Are the statements in Proposition* 3.11 *true*?

This question appears to be very difficult. However, Theorem 3.12 gives an affirmative answer for semilocal rings:

Theorem 3.12. *A semilocal ring is left QD* $\Leftrightarrow$ *it is right QD.*

Proof. If R is semilocal and left QD then R/J is a finite product of division rings. In particular, R/J is *left* QD, so R is *right* QD. ☐

I-finite rings and frames

The set $I(R)$ of **all idempotents** in a ring R is partially ordered by:

$$e \leq f \quad \Leftrightarrow \quad e \in fRf.$$

Lemma 3.13. [19, Lemma B.6] *For any ring R, the following are equivalent:*

(1) *R is I-finite (no infinite orthogonal set of idempotents).*

(2) *R has the ACC (equivalently the DCC) on idempotents.*

(3) *R has the ACC (equivalently the DCC) on direct summands of ${}_RR$ (equivalently of R_R).*

Thus left (or right) artinian, noetherian, and finite dimensional rings are all I-finite. If $R/J(R)$ is I-finite so also is R; conversely if idempotents lift modulo $J(R)$ [19, Lemma B.7]. The condition 'I-finite' passes to subrings and corners. A ring R is **I-free** if $I(R) = \{0, 1\}$. Minimal idempotents in $I(R) \smallsetminus \{0\}$ are called **primitive idempotents** in R, and we have:

$$0 \neq e \in I(R) \textit{ is primitive} \quad \textit{if and only if} \quad eRe \text{ is I-free.}$$

Definition 3.14. For $n \geq 1$, a **frame** for a ring R is a set $E = \{e_1, \ldots, e_n\}$ in R of nonzero, orthogonal idempotents such that $1 = e_1 + e_2 + \cdots + e_n$ (to emphasize n, it is called an n-**frame**). The rings e_iRe_i are the **corners** of E. A frame of primitive idempotents is a **primitive frame.**

Lemma 3.15. *I-finite rings have primitive frames. The converse fails.*[8]

Proof. As R has the DCC on idempotents, choose some minimal nonzero idempotent $e_1 \neq 0$ in R, so e_1 is primitive. If $e_1 = 1$ then $\{1\}$ is a primitive frame. Otherwise choose e_2 minimal in $(1-e_1)R(1-e_1)$. Then $\{e_1, e_2\}$ is orthogonal and $e_1 < e_1 + e_2$. Write $f_2 = e_1 + e_2$. If $f_2 = 1$ we are done. If not continue in this way to obtain idempotents $e_1 < f_2 < g_3 < \cdots$, violating the ACC for $I(R)$.

As to the converse, Shepherdson [21] presents a domain D for which $M_2(D)$ is not directly finite, and so not I-finite by Jacobson [10]. But $M_2(D)$ has a primitive frame $\{e_{11}, e_{22}\}$. $\square$

Semilocal rings are I-finite (R/J is artinian); not conversely ($\mathbb{Z}$). So we cannot replace 'semilocal' by 'I-finite' in Theorem 3.8(1) as the following example shows. The socles of R are denoted S_l and S_r.

Example 3.16. Let $R = \left\{ \begin{bmatrix} n & x & y \\ 0 & n & 0 \\ 0 & 0 & z \end{bmatrix} \middle| \, n \in \mathbb{Z};\ x, y, z \in \mathbb{Q} \right\}$ where, for clarity, we write R as a split-null extension $R = \begin{bmatrix} S & V \\ 0 & \mathbb{Q} \end{bmatrix}$ where

$$S = \left\{ \begin{bmatrix} n & x \\ 0 & n \end{bmatrix} \middle| \, n \in \mathbb{Z},\ x \in \mathbb{Q} \right\} \text{ and } {}_SV_{\mathbb{Q}} = \begin{bmatrix} \mathbb{Q} \\ 0 \end{bmatrix}.$$

(a) R is QD (left and right) and I-finite (in fact left noetherian), but R is not semilocal.

(b) The left-max ideals of R are $M = \begin{bmatrix} S & V \\ 0 & 0 \end{bmatrix}$ and, for various primes $p \in \mathbb{Z}$, $M_p = \left\{ \begin{bmatrix} pn & x & y \\ 0 & pn & 0 \\ 0 & 0 & z \end{bmatrix} \middle| \, n \in \mathbb{Z};\ x, y, z \in \mathbb{Q} \right\}$.

(c) $\omega(R)$ is not finite.

(d) $S_l = 0$.

(e) $S_r = \begin{bmatrix} 0 & 0 & \mathbb{Q} \\ 0 & 0 & 0 \\ 0 & 0 & \mathbb{Q} \end{bmatrix}$ is projective, homogeneous, and length 2 as a right R-module.

(f) J is not essential in R_R.

[8] The converse holds for exchange rings by [4, Proposition 3]

Proof. Define $\phi : R \to \mathbb{Z} \times \mathbb{Q}$ by $\phi\left(\begin{bmatrix} n & x & y \\ 0 & n & 0 \\ 0 & 0 & z \end{bmatrix}\right) = (n, z)$, and observe that ϕ is an onto ring morphism. As $J(\mathbb{Z}\times\mathbb{Q}) = 0$ it follows that $\ker(\phi) = J = \begin{bmatrix} 0 & \mathbb{Q} & \mathbb{Q} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. In particular, $R/J \cong \mathbb{Z} \times \mathbb{Q}$.

(a) As $R/J \cong \mathbb{Z} \times \mathbb{Q}$ is I-finite and QD, so also is R by Proposition 2.9. Clearly R is not semilocal. Finally, R is noetherian because R/J is noetherian and J has $\mathbb{Q}$-dimension 2.

(b) and (c) By Proposition 2.9 the left-max ideals of S are M and the M_p where $p \in \mathbb{Z}$ is a prime. Now (b) follows, and then (c) is clear.

(d) As $S_l \subseteq \mathtt{r}(J)$, we only show that $soc\big({}_R\mathtt{r}(J)\big) = 0$. First, if $\begin{bmatrix} n & x & y \\ 0 & n & 0 \\ 0 & 0 & z \end{bmatrix} \in \mathtt{r}(J)$ then $0 = \begin{bmatrix} 0 & \mathbb{Q} & \mathbb{Q} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} n & x & y \\ 0 & n & 0 \\ 0 & 0 & z \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{Q}n & \mathbb{Q}z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
Hence $\mathtt{r}(J) \subseteq \begin{bmatrix} 0 & \mathbb{Q} & \mathbb{Q} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = J$.[9] Let $R\gamma \subseteq \mathtt{r}(J)$ be simple where we write $\gamma = \begin{bmatrix} 0 & u & v \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $u, v \in \mathbb{Q}$. Then $R\gamma = \left\{ \begin{bmatrix} 0 & nu & nv \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \, n \in \mathbb{Z} \right\}$. Also, if $n\gamma = 0$, $n \in \mathbb{Z}$, then $n = 0$ (because one of u, v is nonzero in $\mathbb{Q}$). Hence $0 \subset 2R\gamma \subset R\gamma$, a contradiction. This proves that $S_l = 0$.

(e) If $i \neq j$ write $\varepsilon_{ij} \in R$ for the (i, j)-matrix unit, so
$\varepsilon_{12}R = \begin{bmatrix} 0 & \mathbb{Q} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\varepsilon_{13}R = \begin{bmatrix} 0 & 0 & \mathbb{Q} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\varepsilon_{33}R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbb{Q} \end{bmatrix}$.
For convenience, write $P_R = \varepsilon_{13}R \oplus \varepsilon_{33}R = \begin{bmatrix} 0 & 0 & \mathbb{Q} \\ 0 & 0 & 0 \\ 0 & 0 & \mathbb{Q} \end{bmatrix}$. One verifies that $\varepsilon_{13}R$ and $\varepsilon_{33}R$ are both simple so $P_R \subseteq S_r$. Note that $\varepsilon_{33}R$ is projective (a summand of R_R), and that $\varepsilon_{33}R \xrightarrow{\varepsilon_{13}\cdot} \varepsilon_{13}R$ is an R-isomorphism by Schur's lemma. Hence P_R is projective and homogeneous of length 2. It remains to show that $P_R = S_r$. To this end, one verifies

$$S_r \subseteq \mathtt{l}(J) = \begin{bmatrix} 0 & \mathbb{Q} & \mathbb{Q} \\ 0 & 0 & 0 \\ 0 & 0 & \mathbb{Q} \end{bmatrix} = \varepsilon_{12}R \oplus \varepsilon_{13}R \oplus \varepsilon_{33}R = \varepsilon_{12}R \oplus P_R.$$

[9] In fact $\mathtt{r}(J) = J$ because J is nilpotent,

But $\varepsilon_{12}R$ is *not* simple $\big($because $0 \neq 2(\varepsilon_{12}R) \subset \varepsilon_{12}R\big)$, so $\varepsilon_{12} \in \mathtt{l}(J) \smallsetminus S_r$. This means that $P_R \subseteq S_r \subset \mathtt{l}(J)$, where $dim_{\mathbb{Q}}(P_R) = 2$ and $dim_{\mathbb{Q}}\big(\mathtt{l}(J)\big) = 3$. [10] It follows that $P_R = S_r$, proving (e).

(f) $\varepsilon_{33}R$ is a simple right ideal of R, and $J \cap \varepsilon_{33}R = 0$. □

Semiperfect left quasi-duo rings

Note that Example 3.16 is left QD and I-finite, but *not* semilocal. Another important example is $\mathbb{Z}_{(p,q)}$ in Example 3.10, which is a semilocal, noetherian, quasi-duo, PID, but idempotents do not lift modulo J. A ring R is **semiperfect** if it is semilocal and idempotents lift modulo J; equivalently if R has a frame with local corners. A ring R is **semipotent** if, for any left (or right) ideal $L \nsubseteq J$, we have $0 \neq e^2 = e \in L$.

Lemma 3.17. *Let R be a ring. Then*:

(a) *R is semiperfect $\Rightarrow$ R is exchange $\Rightarrow$ R is semipotent.*

(b) *R is semiperfect $\Leftrightarrow$ R is exchange and I-finite*
$\Leftrightarrow$ R is semipotent and I-finite.

Proof. (a) Use [4, Corollary 12] and [16, Proposition 1.9].
(b) This follows from (a) and [19, Theorem B.9]. □

Definition 3.18. A frame $\{e_1, e_2, \ldots, e_n\}$ for a ring R is J-**central** if

$$e_k + J \in C(R/J) \quad \text{for each } k.$$

Theorem 3.19. *Given a ring R, the following conditions are equivalent*:

(1) *R is semiperfect and left quasi-duo.*

(2) *R has a J-central, local frame.*

(3) *R has a J-central frame $E = \{e_1, e_2, \ldots, e_n\}$ where each corner $(e_i + J)(R/J)(e_i + J)$ of R/J is a division ring.*

When this is the case, the following conditions hold for any J-central local frame $\{e_1, e_2, \ldots, e_n\}$ for R :

---

[10] $\mathtt{l}(J) = \begin{bmatrix} 0 & \mathbb{Q} & \mathbb{Q} \\ 0 & 0 & 0 \\ 0 & 0 & \mathbb{Q} \end{bmatrix}$ is a $\mathbb{Q}$-vector space via $\begin{bmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{bmatrix} \cdot q = \begin{bmatrix} 0 & xq & yq \\ 0 & 0 & 0 \\ 0 & 0 & zq \end{bmatrix}$.

(a) *The set of all distinct left-max ideals of* R *is* $\{A_1, \ldots, A_n\}$, [11]
where $A_i = R(1-e_i)R + J$. *In particular,* $\omega(R) = n$.

(b) *Write* $K_i = Re_i/Je_i$ *for* $k = 1, 2, \ldots, n$. *Then* $\{K_1, \ldots, K_n\}$ *is a system of distinct representatives of the isomorphism classes of ideal-simple left* R*-modules, and* $K_i \leftrightarrow A_i$ *is a bijection* $\{K_i \mid i = 1, 2, \ldots, n\} \to \{A_i \mid i = 1, 2, \ldots, n\}$.

Proof. Write $J = J(R)$, $\bar{R} = R/J$, and $\bar{r} = r + J$ when $r \in R$. Observe:

$$\text{If } e^2 = e \in R \quad \text{then} \quad \bar{e}\bar{R}\bar{e} \cong eRe/J(eRe) \quad (\text{via } x \leftrightarrow \bar{x}). \tag{*}$$

(1)⇒(2) As R is left QD, $\bar{R}$ is a finite product of division rings, say:

$$\bar{R} = G_1 \oplus \cdots \oplus G_n \quad \text{where each } G_k \lhd \bar{R} \text{ is a division ring.}$$

Hence there is a frame $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n\}$ for $\bar{R}$ where $G_k = \bar{R}\bar{e}_k$ for each k and $\bar{e}_k \in \bar{R}$ is a central idempotent in $\bar{R}$. As idempotents lift modulo J, we may assume that $\{e_1, \ldots, e_n\}$ is a frame for R [19, Proposition B.5]. By (*) we have $e_kRe_k/J(e_kRe_k) \cong \bar{e}_k\bar{R}\bar{e}_k = G_k$ is a division ring for each k. Thus e_kRe_k is local for each k, that is $\{e_1, \ldots, e_n\}$ is a local frame for R. This proves (2).

(2)⇒(3) By (2) let $\{e_1, \ldots, e_n\}$ be a central, local frame for R. As each e_kRe_k is local, (*) gives $\bar{e}_k\bar{R}\bar{e}_k \cong e_kRe_k/J(e_kRe_k)$ is a division ring. This proves (3).

(3)⇒(1) Given the situation in (3), the frame $\{e_1, \ldots, e_n\}$ in (3) is local by (*), so we obtain R is semiperfect. Moreover, $\{\bar{e}_1, \ldots, \bar{e}_n\}$ is a *central* frame for $\bar{R}$, so $\bar{R} \cong \bar{e}_1\bar{R}\bar{e}_1 \times \cdots \times \bar{e}\bar{R}\bar{e}$. Thus $\bar{R}$ is left QD because each $\bar{e}_k\bar{R}\bar{e}_k$ is division by (3). Hence R is left QD by Lemma 2.3, proving (1).

(a) By (3) let $\{e_1, e_2, \ldots, e_n\}$ be a local, J-central frame for R where each $\bar{e}_k\bar{R}\bar{e}_k$ is a division ring. For each k define $\varphi_k : R \to \bar{e}_k\bar{R}\bar{e}_k$ by $\varphi_k(r) = \bar{e}_k\bar{r}\bar{e}_k$. Then φ_k is an onto ring morphism ($\bar{e}_k$ is central in $\bar{R}$). Furthermore $\ker\varphi_i = R(1-e_i)R + J$ because

$$r \in \ker\varphi_i \quad \Leftrightarrow \quad \bar{e}_i\bar{r}\bar{e}_i = \bar{0} \quad \Leftrightarrow \quad \bar{r} \in \bar{R}(\bar{1} - \bar{e}_i)\bar{R} = \overline{R(1-e_i)R}.$$

Define $A_i = R(1-e_i)R + J$ for $i = 1, 2, \ldots, n$, so $A_i = \ker\varphi_i$.

Then each A_i is left-max in R by Lemma 2.3 because

$$R/A_i = R/\ker\varphi_i \cong \operatorname{im}\varphi_i = \bar{e}_k\bar{R}\bar{e}_k \text{ is a division ring.}$$

Thus $\{A_1, A_2, \ldots, A_n\} \subseteq \mathcal{A}(R)$. But $\omega(R) = \omega(\bar{R}) = n$ as $\bar{R} = \Pi_{i=1}^n \bar{e}_i\bar{R}\bar{e}_i$ (in the proof of (3)⇒(1)). Hence $|\mathcal{A}(R)| = n$, so it remains to show that these A_i are distinct.

[11] This shows that R is a *basic* semiperfect ring [13, Definition 25.5].

Suppose, if possible, that $A_i = A_k$, $i \neq k$. Observe $(1-e_i)R(1-e_i)$ has a frame $\{e_1, \ldots, \widehat{e_i}, \ldots, e_n\}$, where $\widehat{e_i}$ missing. As $k \neq i$ we have:

$$e_k = e_k(1-e_i) \in R(1-e_i)R \subseteq A_i = A_k.$$

But $1 - e_k \in A_k$ too, so $A_k = R$, a contradiction. So $A_i \neq A_k$ after all, proving (a).
(b) Let ${}_RK = Rk$ be any simple module. As $1 = \Sigma_{i=1}^n e_i$, we have $e_t k \neq 0$ for some t, so right multiplication $Re_t \xrightarrow{\cdot k} Re_t k = K$ is epic. Hence $\ker(\cdot k) \subseteq^{max} Re_t$. But e_t is a local idempotent so $\ker(\cdot k) = Je_t$ by [19, Proposition B.2]. Thus $K \cong Re_t/\ker(\cdot k) = Re_t/Je_t = K_t$.

With this, to prove (b) it remains to show that $K_i = K_j$ implies $i = j$; that is $Re_i/Je_i \cong Re_j/Je_j$ implies $i = j$. As $A_i = A_j$ implies $i = j$, this holds by Lemma 2.20 if we can prove the:

Claim: *$Re_i/Je_i \cong R/A_i$ as left modules for each $i = 1, 2, \ldots, n$.*

Proof. Define $\phi_i : R \to Re_i/Je_i$ by $\phi_i(r) = re_i + Je_i$ for all $r \in R$. Then ϕ_i is R-linear and epic, and :

$$r \in \ker(\phi_i) \Leftrightarrow Re_i = Je_i \Leftrightarrow r \in R(1-e_i) + J = R(1-e_i)R + J = A_i$$

because e_i is J-central. This proves the Claim, and so proves (b). □

Question 6. *If $1 \leq k \leq n$, let $M \subseteq {}_RR$ be maximal with respect to $e_k \notin M$. Must $M = A_k$?*

The Triangular Theorem

In this section, describing the structure of an I-finite left QD ring R is reduced to whether or not the left socle S_l of R is contained in the Jacobson radical J. We need some notation and terminology.

Let $R_1, R_2, \ldots, R_n$ be rings and let V_{ij} be an R_i-R_j bimodule whenever $i \neq j$. If conditions on the V_{ij} are such that the set

$$G_n(R_i; V_{ij}) = \begin{bmatrix} R_1 & V_{12} & \cdots & V_{1n} \\ V_{21} & R_2 & \cdots & V_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ V_{n1} & V_{n2} & \cdots & R_n \end{bmatrix}$$

is an associative ring with matrix operations, then we call $G_n(R_i; V_{ij})$ a **generalized $n \times n$ matrix ring** over the V_{ij}.[12] Our interest lies in:

Definition 3.20. Let $R_1, R_2, \ldots, R_n$ be rings and let V_{ij} be an R_i-R_j bimodule whenever $i \neq j$. The **generalized, $n \times n$ upper-triangular (UT) matrix ring**

12 The prototype is $end_R M$ where $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ are R-modules, $R_i = \text{end}(M_i)$ and $V_{ij} = \text{hom}(M_i, M_j)$.

$UT_n(R_i;V_{ij})$ is obtained from $G_n(R_i;V_{ij})$ by insisting that $V_{ij}=0$ whenever $i>j$. Here we write

$$UT_n(R_i;V_{ij}) = \begin{bmatrix} R_1 & V_{12} & \cdots & V_{1\,n-1} & V_{1\,n} \\ 0 & R_2 & \cdots & V_{2\,n-1} & V_{2\,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & R_{n-1} & V_{n-1\,n} \\ 0 & 0 & \cdots & 0 & R_n \end{bmatrix}.$$

If $n=1$ we identify $UT_1(R_1)=R_1$; and $UT_2(R_1)$ is the split-null extension of $R_1\times R_2$ over V_{12}. The following is a useful link between generalized UT rings and split-null extensions.

Lemma 3.21. $\begin{bmatrix} R & V & W \\ 0 & S & Z \\ 0 & 0 & T \end{bmatrix} \cong \begin{bmatrix} \begin{bmatrix} R & V \\ 0 & S \end{bmatrix} & \begin{bmatrix} W \\ Z \end{bmatrix} \\ [\ 0 \quad 0\ ] & T \end{bmatrix}$ *as rings.*

Proof. $\begin{bmatrix} r & v & w \\ 0 & s & z \\ 0 & 0 & t \end{bmatrix} \mapsto \begin{bmatrix} \begin{bmatrix} r & v \\ 0 & s \end{bmatrix} & \begin{bmatrix} w \\ z \end{bmatrix} \\ [\ 0 \quad 0\ ] & t \end{bmatrix}$ is a ring isomorphism. □

With Lemma 3.21 we can extend Proposition 2.9 to generalized UT matrix rings. Here (a), (b) and (c) are due to Yu [23, Proposition 2.1].

Proposition 3.22. *Let $R=UT_n(R_i;V_{ij})$ be an $n\times n$ generalized UT matrix ring. Then*:

(a) $J(R)=UT_n\big(J(R_i);V_{ij}\big)$.

(b) $R/J(R)\cong \Pi_i\big(R_i/J(R_i)\big)$.

(c) *R is left QD if and only if each R_i is left QD.*

(d) *If R is left QD, the left-max ideals of R are*

$$M_k = UT_n(A_i;V_{ij}) \text{ for } k=1,2,\ldots n,$$

where $A_i=R_i$ for all $i\neq k$ and A_k is a left-max ideal of R_k.

Proof. If $n=1$ there is nothing to prove. If $n\geq 2$ use induction on n, Propositions 2.5 and 2.9, and Lemma 3.21. □

There is another description for generalized UT matrix rings, using:

Definition 3.23. A frame $\{e_1,e_2,\ldots,e_n\}$ for R is **upper triangular** (**UT**) if $e_iRe_j=0$ whenever $i>j$.

The name arises because the Pierce decomposition takes the form

$$R \cong \begin{bmatrix} e_1Re_1 & e_1Re_2 & \cdots & e_1Re_{n-1} & e_1Re_n \\ 0 & e_2Re_2 & \cdots & e_2Re_{n-1} & e_2Re_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & e_{n-1}Re_{n-1} & e_{n-1}Re_n \\ 0 & 0 & \cdots & 0 & e_nRe_n \end{bmatrix} = UT_n(e_iRe_i;e_iRe_j).$$

R has a UT frame $\{e_1, e_2, \dots, e_n\} \quad \Leftrightarrow \quad R \cong UT_n(R_i; V_{ij})$ for some R_i and V_{ij}.

Lemma 3.24. *Suppose $E = \{e_1, e_2, \dots, e_n\}$ is a UT frame for a ring R. If $\{f_1, f_2\}$ is a UT frame for $e_n R e_n$ then $E' = \{e_1, e_2, \dots, e_{n-1}, f_1, f_2\}$ is another UT frame for R.*

Proof. If $n = 1$ then $e_1 = 1$ and there is nothing to prove. So assume $n \geq 2$. We have $f_j \in e_n R e_n$ for each j so $f_j e_n = f_j = e_n f_j$. The fact that E' is a UT frame for R, follows from:

- For all i and j : $f_j R e_i = (f_j e_n) R e_i = f_j\, 0 = 0$ as E is a UT frame for R;
- $f_2 R f_1 = (f_2 e_n) R (e_n f_1) = f_2 (e_n R e_n) f_1 = 0$ because $\{f_1, f_2\}$ is a UT frame for $e_n R e_n$. □

The left QD rings have a close connection with upper triangular rings, and Lemma 3.26 below is the key to understanding why. We need the following well-known result of Brauer from 1950 [2].

Lemma 3.25. Brauer's Lemma *If $K \subseteq R$ is a simple left ideal then $K^2 = 0$ or $K = Re$, $e^2 = e$.*

Proof. If $K^2 \neq 0$ let $Ka \neq 0$, $a \in K$. Then $Ka = K$ by simplicity, so $a = ea$, $0 \neq e \in K$. Then $e^2 - e \in B$ where $B = \{b \in K \mid ba = 0\}$. Since B is a left ideal and $B \subset K$ we have $B = 0$. But then $e^2 = e \neq 0$ and $Re \subseteq K$, so $Re = K$ by a third appeal to simplicity. □

Lemma 3.26. *Let $R \neq 0$ be a left QD ring which is not a division ring and where $S_l(R) \nsubseteq J(R)$. Then R has an upper triangular frame $\{e_1, e_2\}$ such that:*

$$e_1Re_1 \text{ is division, } e_2Re_2 \neq 0, \text{ and } R \cong \begin{bmatrix} e_1Re_1 & e_1Re_2 \\ 0 & e_2Re_2 \end{bmatrix}.$$

Proof. As $S_l \nsubseteq J$, there exists a simple left ideal K of R such that $K \nsubseteq J$. By Brauer's lemma we have $K = Re$ where $e^2 = e \in R$. Hence $R(1-e)$ is a maximal left ideal, so $R(1-e) \lhd R$ because R is left QD. In particular, $(1-e)R \subseteq R(1-e)$, so $(1-e)Re = 0$. Next, $eRe \cong \text{end}(Re)$ is a division ring by Schur's Lemma (Re is simple). Finally, $(1-e)R(1-e) \neq 0$ because otherwise $R = eRe$ is division, contrary to our hypothesis.

If we write $e_1 = e$ and $e_2 = 1 - e$ then $e_2Re_1 = 0$ so $\{e_1, e_2\}$ is an upper triangular frame for R. So the Pierce decomposition of R is

$$R \cong \begin{bmatrix} eRe & eR(1-e) \\ 0 & (1-e)R(1-e) \end{bmatrix} = \begin{bmatrix} e_1Re_1 & e_1Re_2 \\ 0 & e_2Re_2 \end{bmatrix}. \qquad \square$$

We can now prove the first Main Theorem of this paper.

Theorem 3.27. Triangular Theorem *Conditions* (1) *and* (2) *are equivalent for a ring* $R \neq 0$:

(1) *R is left QD and I-finite.*

(2) $R \cong UT_n(R_i; V_{ij})$ *is generalized upper triangular where* $n \geq 1$, *and either* (a) *or* (b) *holds*:

(a) R_i *is a division ring for each* $i = 1, 2, \ldots, n$;[13]

(b) R_i *is division if* $i < n$; *and* $R_n \neq 0$ *is I-finite, left QD, and satisfies* $S_l(R_n) \subseteq J(R_n)$.

Proof. (2)⇒(1) Given (2): R is I-finite and left QD because each R_i has these properties (the proof of I-finiteness is routine; for left QD use (c) of Proposition 3.22).

(1)⇒(2) We assume that (2) is false, and search for a contradiction.

The following terminology simplifies the exposition:

- An I-finite, left QD ring will be called an **IQD** ring.
- With an eye on Lemma 3.26, call a ring $T \neq 0$ **nice** if T is IQD, but T is not a division ring and $S_l(T) \nsubseteq J(T)$. [14]
- For a ring S, call a UT frame $\{e_1, e_2, \ldots, e_n\}$ **strong** if $e_i S e_i$ is division, $i < n$, and $e_n S e_n \neq 0$.

Claim 1. Every IQD ring $T \neq 0$ is nice.

Proof. If T is a division ring then (2a) holds for $n = 1$ and $R_1 = T$; if $S_l(T) \subseteq J(T)$ then (2b) holds for $n = 1$ and $R_1 = T$. These are both contradictions as we are assuming that (2) fails. It follows that T is not division and $S_l(T) \nsubseteq J(T)$. This proves Claim 1.

Claim 2. Suppose a ring S is IQD with a strong UT frame $\{e_1, e_2, \ldots, e_n\}$. Then S has another strong UT frame of the form $\{e_1, e_2, \ldots, e_{n-1}, f_1, f_2\}$.

Proof. Write $T = e_n S e_n$. Then T is IQD (it is a corner of S), and so is nice by Claim 1. Now Lemma 3.26 implies that T has a UT frame $\{f_1, f_2\}$ where $f_1 T f_1$ is division and $f_2 T f_2 \neq 0$. But then $\{e_1, e_2, \ldots, e_{n-1}, f_1, f_2\}$ is a UT frame for T by Lemma 3.24, proving Claim 2.

With these preliminaries, we can complete the proof of (1)⇒(2).

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[13] R semiperfect in this case.

[14] $T \neq 0$ because $S_l(0) = 0 = J(0)$.

First, R is nice by Claim 1, so Lemma 3.26 implies that R has a UT frame $\{e_1, e_2\}$ such that e_1Re_1 is division and $e_2Re_2 \neq 0$. In other words, $\{e_1, e_2\}$ is a strong UT frame for R. But then Claim 2 implies that R has a UT frame $\{e_1, f_1, f_2\}$. Again Claim 2 shows R has a UT frame $\{e_1, f_1, g_1, g_2\}$.

Once more, Claim 2 shows R has a UT frame $\{e_1, f_1, g_1, h_1, h_2\}$.

This process continues to create a sequence of strong UT frames for R each containing more orthogonal idempotents than those before. This contradicts the I-finiteness hypothesis on R, and so proves the theorem. $\square$

The answer to the Lam-Dugas Question is 'yes' for semilocal rings by Theorem 3.12. So we ask:

Question 7. *Is every I-finite left QD ring a right QD ring*?

Note that the answer to Question 5 is 'yes' by Theorem 3.27 if every I-finite, left QD ring with $S_l \subseteq J$ is right QD.

4. Left soclin rings

The Triangular Theorem focuses our attention on the I-finite, left QD rings R satisfying $S_l \subseteq J$. So the next step is to study these rings.

Definition 4.1. A ring R will be called **left soclin** if $S_l \subseteq J$.

Clearly every ring with $S_l = 0$ (hence every domain) is left soclin. We refer to semisimple rings as **SS-rings**. No nonzero SS-ring is left soclin because $S_l = R$ while $J = 0$. Hence:

Lemma 4.2. *The only ring R that is both SS and left soclin is $R = 0$.*

Even so, all non-SS, local rings are (left and right) soclin. However: If D is division the ring $\begin{bmatrix} D & D \\ 0 & D \end{bmatrix}$ is artinian and QD, but neither left nor right soclin; while the Weyl Example (Example 2.8) is a simple noetherian domain (and so soclin) but it is neither left nor right QD. Recall that a ring is I-free if 0 and 1 are the only idempotents (for example domains and local rings).

The set of left soclin rings with $S_l = 0$ is vast including (in addition to domains) semiprime rings, left nonsingular rings ($Z_l = 0$) and polynomial rings.[15]

[15] If $K = R[x]k$, $deg(k) = n$, then $K = R[x](x^{n+1}k)$ so $k = g\,(x^{n+1}k)$, $g \in R[x]$. Hence $n = \deg(k) \geq n + 1$.

Example 4.3. If R is an I-free ring then R is division or R is left soclin, but not both.

Proof. If R is not left soclin then $S_l \nsubseteq J$, so let $K \nsubseteq J$ be a simple left ideal of R. By Brauer's lemma $K = Re$, $e^2 = e \in R$. As $e \neq 0$ we have $e = 1$ because R is I-free. But then $R = Re = K$ and it follows that R is a division ring. The last statement is by Lemma 4.2. □

Theorem 4.4. *The following conditions are equivalent for a ring $R \neq 0$:*

(1) *R is left soclin.*

(2) *Every maximal left ideal of R is essential in ${}_RR$.*

(3) *If M is a maximal left ideal of R then $\mathtt{r}(M) \subseteq Z_l$.*

(4) *$S_l \subseteq Z_l$.*

(5) *No maximal left ideal of R is a direct summand of ${}_RR$.*

Proof. (1)⇒(2) Let $M \subseteq^{max} {}_RR$, and suppose ${}_RM \subseteq^{ess} {}_RR$ fails, say $M \cap L = 0$ where $L \neq 0$ is a left ideal of R. As M is maximal, $R = M \oplus L$, say $M = Re$ and $L = R(1-e)$, $e^2 = e \in R$. Thus ${}_RL$ is simple so $L \subseteq J$ by (1). But then $1 - e \in J$, so $e = 1$ and $M = R$, a contradiction.

(2)⇒(3) Let $a \in \mathtt{r}(M)$ so $M \subseteq \mathtt{l}(a)$. Hence (2) implies $\mathtt{l}(a) \subseteq^{ess} {}_RR$, that is $a \in Z_l$.

(3)⇒(4) Given (3), suppose if possible that $S_l \nsubseteq Z_l$, say $K \nsubseteq Z_l$ for some simple left ideal $K \subseteq R$. Write $K = Rk$, so $\mathtt{l}(k) \subseteq^{max} {}_RR$. By (3) this means $\mathtt{rl}(k) \subseteq Z_l$. But $k \in \mathtt{rl}(k)$ always holds, so $k \in \mathtt{rl}(k) \subseteq Z_l$. As $Z_l \lhd R$ this means $K = Rk \subseteq Z_l$, a contradiction.

(4)⇒(5) If (5) fails, let $Re \subseteq^{max} {}_RR$, $e^2 = e$. Then $e \neq 1$ so $R(1-e)$ is a simple left ideal. Hence $R(1-e) \subseteq Z_l$ by (4). But then $1 - e \in Z_l$, a contradiction as $\mathtt{l}(1-e)$ is not essential in ${}_RR$.

(5)⇒(1) If (1) fails let $K \nsubseteq J$ be a simple left ideal. By Brauer's lemma $K = Re$ for some $e^2 = e \in R$, so $R(1-e)$ is a maximal left ideal of R. This contradicts (5). □

An image of a left soclin ring need not be left soclin. Indeed, if R is a non-SS, local ring then R is soclin (left and right) but the division ring R/J is neither left nor right soclin. However we do have:

Proposition 4.5. *Let $R = S \times T$ be rings. Then*:

$$R \text{ is left soclin} \quad \Leftrightarrow \quad \text{both } S \text{ and } T \text{ are left soclin.}$$

Proof. ($\Rightarrow$) We prove it for S. Suppose ${}_SK \subseteq S$ is simple. Then $K \times 0$ is a simple left ideal of R so, by hypothesis, $K \times 0 \subseteq J(R) = J(S) \times J(T)$. Hence $K \subseteq J(S)$, proving that S is left soclin.

($\Leftarrow$) We work internally. Let $R = A \oplus B$ where $A \lhd R$, $B \lhd R$ and both A and B are left soclin as a rings. Then

$$S_l(R) = soc({}_RR) = \mathrm{soc}({}_RA) \oplus \mathrm{soc}({}_RB).$$ [16].

Observe: $\quad J(R) = J(A) \oplus J(B) \quad$ and $\quad S_l(R) = \mathrm{soc}({}_RA) \oplus \mathrm{soc}({}_RB)$.

We show $\mathrm{soc}({}_RA) \subseteq J(A)$. If ${}_RK \subseteq A$, ${}_RK$ simple, $0 \neq k \in K$, then

$$K = Rk = (A \oplus B\ )Rk = Ak \oplus Bk = Ak \quad \text{as } Bk \subseteq BA = 0. \qquad \square$$

While images of left soclin rings may not be left soclin, we do have

Theorem 4.6. *Being left soclin is a Morita invariant. More precisely*:

(a) *If a ring R is left soclin and $ReR = R$ where $e^2 = e \in R$, then eRe is left soclin. The condition $ReR = R$ is necessary.*

(b) *If a ring R is left soclin, so also is $M_n(R)$ for any $n \geq 1$.*

Proof. Let R denote a left soclin ring.

(a) If $e^2 = e \in R$ where $ReR = R$, write $Q = eRe$. If Qk is simple in Q, $k \in Q$, observe that $eRk = eR(ek) = Qk$. It suffices to show Rk is simple—then $Rk \subseteq J$, so $Qk = eRk = (eRk)e \subseteq eJe = J(Q)$.

So choose $0 \neq x \in Rk$, say $x = ak$, $a \in R$. As $ReR = R$ we have $0 \neq x \in ReRx$, so $ebx \neq 0$ for some $b \in R$. Then

$$0 \neq ebx = eb(ak) = eba(ek) \in Qk, \quad \text{so } Qebx = Qk$$

because Qk is simple in Q. Hence $k \in Rx$ so $Rk \subseteq Rx$, as desired.

For the last statement, let $R = M_2(\mathbb{Z}_2)$, $e = e_{11}$, and use Lemma 4.2.

(b) We prove it for $n = 2$; the general case is analogous. Writing $M_2(R) = \Lambda$, we must show that if ${}_\Lambda K \subseteq \Lambda$ is simple then $K \subseteq J(\Lambda)$. By Lemma 2.10, $K = \begin{bmatrix} X \\ X \end{bmatrix}$ where ${}_RX$ is a submodule of R^2 (written as rows). Observe that ${}_RX$ is simple because $Y \subset X$ implies $\begin{bmatrix} Y \\ Y \end{bmatrix} \subset \begin{bmatrix} X \\ X \end{bmatrix} = K$. Hence

$$X \subseteq \mathrm{soc}(R^2) = \mathrm{soc}(R \oplus R) = \mathrm{soc}(R) \oplus \mathrm{soc}(R) = S_l \oplus S_l.$$

But $S_l \subseteq J$ by hypothesis, so each row of X has every entry from J. It follows that $K = \begin{bmatrix} X \\ X \end{bmatrix} \subseteq M_2(J) = J(\Lambda)$, as required. $\square$

Turning to the semilocal case, the following result plays a role later.

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[16] In general if ${}_RM = {}_RP \oplus {}_RQ$ then $\mathrm{soc}({}_RM) = \mathrm{soc}({}_RP) \oplus \mathrm{soc}({}_RQ)$.

Theorem 4.7. *Let R be a semilocal, non-SS ring. Then the following are equivalent:*

(1) *R is left soclin.*

(2) *$J \subseteq^{ess} {}_RR$.*

Proof. As R is semilocal, let $J = M_1 \cap M_2 \cap \cdots \cap M_n$, $M_i \subseteq^{max} {}_RR$.

(1)⇒(2) Given (1), each $M_i \subseteq^{ess} {}_RR$ by Theorem 4.4(2), and (2) follows.

(2)⇒(1) By Theorem 4.4(2), $J \subseteq M_i$ for each i. □

Notes.
- (1) ⇒ (2) fails if R is merely I-finite—see Example 3.16(f).
- $J \subseteq^{ess} {}_RR$ can fail even if R is artinian, QD and $J \subseteq^{ess} R_R$, see Example 4.9(g) below.

Question 8. *If R is left QD, is $J \subseteq^{ess} {}_RR \quad \Leftrightarrow \quad R$ is left soclin?*

Example 4.9 below requires four well-known results:

Lemma 4.8. (a) *If R is semilocal then $S_l = \mathtt{r}(J)$ and $S_r = \mathtt{l}(J)$.*

(b) *If R is semilocal and $JM = 0$, M a left module, then M is semisimple.*

(c) *For any ring R, $S_lZ_l = 0$ and $Z_rS_r = 0$.*

(d) *If $M \subseteq^{max} {}_RR$, then either $M \subseteq^{ess} {}_RR$ or $M \subseteq^{\oplus} {}_RR$.*

A ring R can be left soclin but not right soclin, as the next example shows (among many other things).

Example 4.9. Let D be a division ring, and define a ring R as follows:

$$R = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & d \end{bmatrix} \middle| \, a, b, c, d \in D \right\}.$$

(a) R is a semilocal (in fact artinian), QD ring of width 2.

(b) $J = \begin{bmatrix} 0 & D & D \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and the only left-max ideals of R are

$$A_1 = \begin{bmatrix} 0 & D & D \\ 0 & 0 & 0 \\ 0 & 0 & D \end{bmatrix} \quad \text{and} \quad A_2 = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \, a, b, c \in D \right\}.$$

(c) R is left soclin but not right soclin—in fact $S_l = J$, and $S_r \supset J$.

Moreover: $\quad S_l = A_1 \subseteq^{ess} {}_RR \quad$ and $\quad S_r = \mathtt{l}(J) = A_1 \supset J$.

(d) $A_1 \subseteq^{ess} {}_RR$ and $A_2 \subseteq^{ess} {}_RR$. Also, $A_1 \subseteq^{ess} R_R$, but A_2 is *not* essential in R_R.

(e) $S_l = \mathtt{r}(J) = J \quad$ and $\quad S_r = \mathtt{l}(J) = A_1 \supset J$.

(f) $Z_l = J$ and $Z_r = \begin{bmatrix} 0 & D & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Both ${}_RZ_r$ and $(Z_r)_R$ are simple.

(g) $J \subseteq^{ess} {}_R R$ but J is not essential in R_R.

(h) $\mathtt{r}(A_1) = J \neq 0$ and $\mathtt{r}(A_2) = 0$.

Proof. Observe: $R \cong \begin{bmatrix} S & V \\ 0 & D \end{bmatrix}$ is the split-null extension of the local ring $S =: \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \middle| a, b \in D \right\}$ and the division ring D by ${}_S V_D = \begin{bmatrix} D \\ 0 \end{bmatrix}$.

(**a**) We have $J = \begin{bmatrix} 0 & D & D \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ by Proposition 2.9(b). Define

$$\varphi : R \to D \times D \ \text{ by } \ \varphi \begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & d \end{bmatrix} = (a, d).$$

Then φ is an onto ring morphism with $\ker(\varphi) = J$, so $R/J \cong D \times D$. Hence R is semilocal, and R is QD of width 2 (by Proposition 2.2 and Theorem 3.8). Finally, R is artinian because $\dim({}_D R) = 4$.

(**b**) Write $\lambda = \begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & d \end{bmatrix}$ and define $\varphi_1, \varphi_2 : R \to D$ given by $\lambda \mapsto a, d$ respectively. These maps are both onto ring morphisms, and the kernels are A_1 and A_2. Hence each $A_i \lhd R$, and so is left-max in R by Lemma 2.3. As $A_1 \neq A_2$, and R has width 2 by (**a**), it follows that $\mathcal{A}(R) = \{A_1, A_2\}$. But R is left QD by (**a**), and so (**b**) follows because $J = A_1 \cap A_2 = \begin{bmatrix} 0 & D & D \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(**c**) As R is semilocal, and in view of Lemma 4.8(a), we compute $\mathtt{r}(J)$ and $\mathtt{l}(J)$ using the following:

$$\begin{bmatrix} 0 & D & D \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & d \end{bmatrix} = \begin{bmatrix} 0 & Da & Dd \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{*}$$

and

$$\begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & d \end{bmatrix} \begin{bmatrix} 0 & D & D \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & aD & aD \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{**}$$

So $S_l = \mathtt{r}(J) = \begin{bmatrix} 0 & D & D \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = J$ and $S_r = \mathtt{l}(J) = \begin{bmatrix} 0 & D & D \\ 0 & 0 & 0 \\ 0 & 0 & D \end{bmatrix} = A_1$. As $A_1 \supset J$ this shows that R is not right soclin by Definition 4.1. Finally, each $A_i \subseteq^{ess} {}_R R$ by Theorem 4.4(2).

(**d**) As R is left soclin by (**c**), each $A_i \subseteq^{ess} {}_R R$ by Theorem 4.4(1). To see that $A_1 \subseteq^{ess} R_R$ we use the right version of Theorem 4.4(5) by showing that $A_1 \neq \phi R$ for any $0 \neq \phi^2 = \phi \in A_1$. Indeed, such a ϕ has the form $\phi = \begin{bmatrix} 0 & 0 & q \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ where $q \in D$, and $\phi R = \begin{bmatrix} 0 & 0 & qD \\ 0 & 0 & 0 \\ 0 & 0 & D \end{bmatrix} \neq A_1$.

Finally, A_2 is *not* essential in R_R as $A_2 = \varepsilon R$, $\varepsilon = \varepsilon^2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$—$A_2 \subseteq \varepsilon R$ because $\alpha = \varepsilon\alpha$ for all $\alpha \in A_2$ [17]. This proves (**d**).

(**e**) These were proved in (*) and (**) above.

(**f**) By (*), (**) and Theorem 4.4(4), we have $J = S_l \subseteq Z_l$. It follows that $Z_l = J$ because R is semipotent (being exchange) so $J \subset Z_l$ would contradict the fact that Z_l is I-free.

Turning to Z_r, write $E = \begin{bmatrix} 0 & D & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and observe $E = \varepsilon_{12}R$ where ε_{12} is the matrix-unit. Lemma 4.8(c) shows that $Z_rS_r = 0$, so $Z_r \subseteq \mathtt{l}(S_r) \overset{(\mathbf{e})}{=} \mathtt{l}(A_1) = E$ as is easily verified. One also verifies that both E_R and ${}_RE$ are simple, so $Z_r = 0$ or $Z_r = E$; we claim $Z_r = E$. As $E = \varepsilon_{12}R$, to prove this it is enough to show that that $\varepsilon_{12} \in Z_r$, that is that $\mathtt{r}(\varepsilon_{12}) \subseteq^{ess} R_R$. But $\mathtt{r}(\varepsilon_{12}) = A_1$ so this follows by (**d**) above.

(**g**) By (**e**) $J = S_l$, and $S_l \subseteq^{ess} {}_RR$ as R is left artinian. This proves $J \subseteq^{ess} {}_RR$. For the rest of (**g**), write $K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D \end{bmatrix}$. Then K is a simple right ideal of R and $J \cap K = 0$, so J is not essential in R_R. This proves (**g**).

(**h**) To see $\mathtt{r}(A_2) = 0$, let $A_2\xi = 0$, $\xi = \begin{bmatrix} x & y & z \\ 0 & x & 0 \\ 0 & 0 & w \end{bmatrix}$. For $a, b, c \in D$,

$$\begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x & y & z \\ 0 & x & 0 \\ 0 & 0 & w \end{bmatrix} = \begin{bmatrix} ax & ay + bx & az + cw \\ 0 & ax & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

It follows that $Dx = 0$, then $Dy = 0$, and finally $Dz + Dw = 0$. Hence $\xi = 0$ as required. One verifies that $\mathtt{r}(A_1) = J$. $\square$

Recall that the Lam-Dugas question has an affirmative answer for semilocal rings (Theorem 42). However the answer seems to be unknown when 'semilocal' is replaced by 'I-finite'. Hence we sharpen it:

Question 9. *If R is left QD, I-finite and left soclin, is R right QD?*

5. Left QDS rings

In this section the structure of the I-finite, left QD, left soclin rings is explored further, and the second Main Theorem of the paper (Theorem 5.15) is proved. We clearly need:

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[17] The nonzero idempotents in A_2 are $\phi_q = \begin{bmatrix} 1 & 0 & q \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $q \in D$, and $\phi_q R = A_2$.

Definition 5.1. A ring will be called a left **QDS ring** if it is I-finite, left QD and left soclin.

We begin with some basic properties of left QDS rings:

Lemma 5.2. (a) *The class of left QDS rings is closed under full corners* (*Theorems 2.13 and 4.6*), *under direct factors* (*Propositions 2.9*(a) *and 4.5*), *but* not *under images* ($\mathbb{Z} \to \mathbb{Z}_2$).

(b) *If* $R = \Pi_{i=1}^{n} R_i$, *then* R *is left QDS* $\Leftrightarrow$ *each* R_i *is left QDS* (*Propositions 2.9*(a) *and 4.5*).

(c) *Every left QDS ring has a primitive frame* (*Lemma 3.15*).

Because these left QDS rings are I-finite, the block decomposition theorem [1, Theorem 7.9] applies. The next three results include a brief proof of this latter theorem for clarity, completeness and reference. This involves the choice of a primitive frame E for the ring, and emphasizes the dependence upon E.

The E-Decomposition of a Ring

If E is a primitive frame for a ring R, define a relation $\sim$ on E by

$$e \sim f \quad \Leftrightarrow \quad eRg \neq 0 \text{ and } fRg \neq 0 \text{ for some } g \in E. \tag{†}$$

Then $\sim$ is reflexive and symmetric, so $\approx$ is an equivalence on E if:

$$e \approx f \quad \Leftrightarrow \quad e \sim g_1 \sim g_2 \sim \cdots \sim g_t \sim f \quad \text{for some } g_i \in E. \tag{††}$$

Definition 5.3. For a frame E, $\approx$ is the E-**equivalence** on E.[18]

Lemma 5.4. *If E is a primitive frame for R, and $e, f \in E$, then*:

(a) $e \sim e$; $\quad eRf \neq 0 \Rightarrow e \sim f$; $\quad$ *and* $e \sim f \Rightarrow e \approx f$.

(b) *Let* $c^2 = c \in C(R)$. *If* $e \in E$, $ec = 0$ *or* $ec = e$.

(c) *Let* $c^2 = c \in C(R)$. *If* $e \sim f$, *then*:

(i) $ec = 0 \Leftrightarrow fc = 0$ $\quad$ *and* $\quad$ (ii) $ec = e \Leftrightarrow fc = f$.

(d) *Condition* (c) *holds with* $\sim$ *replaced by* $\approx$.

Proof. (a) $eRe \neq 0$; $\quad eRf \neq 0$ and $fRf \neq 0$; $\quad$ and $\quad e \sim e \sim f$.

(b) This is because $(ec)^2 = ec \in eRe$, and e is primitive.

(c) We need only prove ($\Rightarrow$) in each case—interchange e and f. As $e \sim f$, we have $eRg \neq 0$ and $fRg \neq 0$ for some $g \in E$.

(i) ($\Rightarrow$) Let $ec = 0$. If $fc \neq 0$ then $fc = f$ by (b), so $0 \neq fRg = (fc)Rg = fR(gc)$. Thus $gc \neq 0$, so $gc = g$ again by (b). But then, $0 \neq eRg = eR(gc) = (ec)Rg = 0\,Rg$, a contradiction.

[18] This is usually called the block equivalence for R induced by E.

(ii) ($\Rightarrow$) Let $ec = e$. If $fc \neq f$, then $fc = 0$ by (b), from which $ec = 0$ by (i). This is the desired contradiction.

(d) As per (††), let $e \sim g_1 \sim g_2 \sim \cdots \sim g_t \sim f$ where $e, f, g_i \in E$. Using (c) repeatedly: $ec = 0 \;\Rightarrow\; g_1c = 0 \;\Rightarrow \cdots \Rightarrow fc = 0$. □

Definition 5.5. Let $\{E_1, E_2, \ldots, E_m\}$ be the partition of the frame E into $\approx$**-equivalence classes**. For $1 \leq p \leq m$, define:

$$c_p = \Sigma\{e \mid e \in E_p\}\text{—the sum of the idempotents in } E_p.$$

Because E is a frame for R, $\{c_1, c_2, \cdots, c_m\}$ is also a frame for R (called the E-**block-frame** for R), and the corners c_pRc_p (denoted $B_p = c_pRc_p$) are called the E-**blocks** of R.

Lemma 5.6. *Let $B_1, B_2, \ldots, B_m$ be the E-blocks of a ring R induced by a primitve frame E, and let $B_p = c_pRc_p$ for $1 \leq p \leq m$.*

(a) *E_p is a primitive frame for $B_p = c_pRc_p$, for all $p = 1, 2, \ldots, m$.* [20]

(b) *If $e, f \in E$ then $e \approx f \quad\Leftrightarrow\quad e$ and f lie in the same E-block.*

(c) $\{c_1,\, c_2, \cdots, c_m\} \subseteq C(R) \quad\Leftrightarrow\quad E_pRE_q = 0$ *when $p \neq q$.*[19]

(d) *Each idempotent c_k is central in R for $1 \leq p \leq m$.*

Proof. (a) Write $E_p = \{f_1, \ldots, f_k\}$, so $c_p = \Sigma_{i=1}^k f_i$. Given f_j, $1 \leq j \leq k$, we have $f_jc_p = \Sigma_{i=1}^k f_jf_i = f_j$. Similarly $c_pf_j = f_j$ so each $f_j \in B_p$. Hence E_p is a frame for B_p; it is primitive because $E_p \subseteq E$.

(b) ($\Rightarrow$) Let $e \approx f$, so $e, f \in E_p$ for some p. Use (a).

($\Leftarrow$) Suppose $e \not\approx f$, say $e \in E_{p_1}$ and $f \in E_{p_2}$, $p_1 \neq p_2$. By (a), $e \in c_{p_1}Rc_{p_1}$ and $f \in c_{p_2}Rc_{p_2}$, a contradiction.

(c) ($\Rightarrow$) Suppose $\{c_1,\, c_2, \cdots, c_m\} \subseteq C(R)$. If $e \in E_p$, $f \in E_q$, and $p \neq q$, then (a) shows that $eRf = (c_pe)R(fc_q) = eRfc_pc_q = 0$.

($\Leftarrow$) If $E_pRE_q = 0$ whenever $p \neq q$ then, using (a),

$$c_pRc_q \subseteq \Sigma_{p,q}\{eRf \mid e \in E_p, f \in E_q\} = 0$$

because $p \neq q$. $c_pr = (c_pr)1 = c_pr(c_1 + \cdots + c_m) = c_prc_p$ for $r \in R$. Similarly $rc_p = c_prc_p$, so $c_p \in C(R)$.

(d) By (c), we show that $E_pRE_q = 0$ when $p \neq q$. If $e \in E_p$ and $f \in E_q$, then $e \not\approx f$ because E_p and E_q are distinct $\approx$-equivalence classes. But then $eRf = 0$ by Lemma 5.4(a). □

[20] This is valid for *any* equivalence on E in place of $\approx$.

Theorem 5.7. E-**Block Decomposition**[21] *Let $E = \{e_1, e_2, \cdots, e_n\}$ be a primitive frame for a ring R, with E-block-frame $\{c_1, c_2, \ldots, c_m\}$.*

(a) *If $e, f \in E$, then: $e \approx f \Leftrightarrow e$ and f are in the same E-block B_p.*

(b) *Each E-block $B_p = c_pRc_p$ is indecomposable as a ring.*

(c) *Each c_p is central in R.*

Thus the (unique) decomposition of the ring R as a direct product of indecomposable rings is

$$R = c_1Rc_1 \dot{+} c_2Rc_2 \dot{+} \cdots \dot{+} c_mRc_m \cong B_1 \times B_2 \times \cdots \times B_m \qquad (\dagger\dagger\dagger)$$

where, for $p = 1, 2, \ldots m$, the $B_p = c_pRc_p$ are the E-blocks of R.

Proof. Again, let $E_1, E_2, \ldots, E_m$ be the $\approx$-**equivalence classes** in E.

(a) This is by Lemma 5.6(b).

(b) If $0 \neq c^2 = c$ is central in c_pRc_p, we show $c = c_p$. As $c = c_pc = \Sigma\{fc \mid f \in E_p\}$, we have $fc \neq 0$ for *some* $f \in E_p$ (as $c \neq 0$). Then $ec = e$ for *every* $e \in E_p$ by Lemma 5.4(c), applied to the ring c_pRc_p. Hence (b) follows from

$$c = c_pc = \Sigma\{ec \mid e \in E_p\} = \Sigma\{e \mid e \in E_p\} = c_p,$$

proving (b).

(c) This is by Lemma 5.6(d).

Finally $(\dagger\dagger\dagger)$ holds because ${}_RR = \oplus_{j=1}^m B_j$ and each $B_j \lhd R$. □

Definition 5.8. Condition $(\dagger\dagger\dagger)$ is called the E-**block decomposition** of a ring R with a primitive frame E.

Let R have a primitive frame E, so each E-block $B_p = c_pRc_p$ of R has a related primitive frame $E_p \subseteq E$ by Lemma 5.6. Our goal now is to understand how E-properties of R pass between E_p-properties of B_p. To describe this we use the following notation:

Definition 5.9. Given a primitive frame E for a ring R, we write:

$$e \sim f \text{ as } e \sim f \pmod{E} \qquad \text{and} \qquad e \approx f \text{ as } e \approx f \pmod{E}.$$

Proposition 5.11 below captures the nature of this relationship; the following lemma is critical.

Lemma 5.10. *Let R be a ring with primitive frame E, and E-equivalence classes $E_1, E_2, \ldots, E_m$. Fix $p \in \{1, 2, \ldots, m\}$ and let $e, f \in E_p$. Then:*

$$e \sim f \ (mod\ E_p) \text{ in } B_p \quad \Leftrightarrow \quad e \sim f \ (mod\ E) \text{ in } R.$$

[21] Often called simply the Block Decomposition

Proof. ($\Rightarrow$) Let $e \sim f$ (mod E_p), say $eB_pg \neq 0$ and $fB_pg \neq 0$ where $g \in E_p$. As $E_p \subseteq E$, we get $e \sim f$ (mod E).

($\Leftarrow$) Let $e \sim f$ (mod E), say $eRh \neq 0$ and $fRh \neq 0$ with $h \in E$. Then:

(i) $h \in B_p$. As $h \in E = \cup E_k$ we have $h \in E_q$ for *some* q. Note that $E_q \subseteq B_q$. If $q \neq p$, $e = e^2 \in E_qE_p \subseteq B_qB_p = (c_qRc_q)(c_pRc_p) = 0$, a contradiction. So $q = p$, and $h \in E_p \subseteq B_p$.

(ii) $eB_ph \neq 0$. By the above, $h \in E_p \subseteq B_p$ so $h = c_ph$. Similarly

$e = ec_p$ so $0 \neq eRh = (ec_p)R(c_ph) = e(c_pRc_p)h = eB_ph$.

(iii) $fB_ph \neq 0$. The proof is similar to (ii).

Now, statements (ii) and (iii) show that $e \sim f$ (mod E_p). □

Proposition 5.11. *Let R be a ring with primitive frame E, and E-equivalence classes $E_1, E_2, \ldots, E_m$. Fix $p \in \{1, 2, \ldots, m\}$ and let $e, f \in E_p$.*

(a) *$e \approx f$ (mod E_p) in B_p $\Leftrightarrow$ $e \approx f$ (mod E) in R.*

(b) *$E_p = E \cap B_p$ for each $p = 1, 2, \ldots, m$.*

Proof. (a) Let $e \approx f$ (mod E_p). By (†), and then Lemma 5.10, we have

$$e \sim g_1 \sim \cdots \sim g_t \sim f \ (\text{mod} E_p) \quad \text{then} \quad e \sim g_1 \sim \cdots \sim g_t \sim f \ (\text{mod } E).$$

So $e \approx f$ (mod E) as required. The converse is similar.

(b) We have $E_p \subseteq E \cap B_p$ because $E_p \subseteq E$ by definition, and $E_p \subseteq B_p$ by Lemma 5.6(a). Conversely, let $e \in E \cap B_p$. Then we have $e \in E = \cup_k E_k$, say $e \in E_q \subseteq B_q$ for some q; we show $q = p$. But if $p \neq q$ then $e = e^2 \in B_pB_q = 0$, a contradiction. So $q = p$, as required. □

Left Bricks

We now return to the study of QDS rings, and of some "building blocks" that are indecomposable left QDS rings, and which we call left "bricks".

If R is left QDS with a primitive frame E, the core of Theorem 5.7 is that the E-block-frame $\{c_1, c_2, \ldots, c_m\}$ lies in the centre $C(R)$ of R. Hence we again obtain equation († † †) :

$$R \cong B_1 \times B_2 \times \cdots \times B_m \quad \text{where } c_pRc_p = B_p \text{ for each } p.$$

So describing the left QDS rings becomes describing the E-blocks B_p.

Properties of the E-blocks B_p which will be needed:

1) B_p is a corner of R by († † †).

2) B_p is I-finite (as R is I-finite).

3) B_p is left QD (by **1**) and Theorem 2.13).

4) B_p is left soclin (by **1**) and Theorem 4.6).

5) B_p is indecomposable as a ring by Theorem 5.7(b).

6) B_p has a primitive frame $E_p = \{f_1, f_2, \ldots, f_t\} \subseteq E$ by Lemma 5.6(a)—see Definition 5.5 and Lemma 5.6(a).

7) The E-equivalence for B_p induced by E_p is the restriction of $\approx$ to E_p by Proposition 5.11.

8) $f_k \approx f_l$ for all k, l, where $\approx$ is the E_p-equivalence for B_p induced by E_p (by Proposition 5.11).

9) Properties **2**), **3**), **4**), and **5**) show that each B_p is an indecomposable left QDS ring.

Definition 5.12. Let R be a ring with a primitive frame E, and let $\approx$ be the E-equivalence for R.

(a) Call R a left **brick** if it is an indecomposable left QDS ring and has a primitive, homogeneous E-frame.

(b) Here, we call E a **homogeneous frame** if $f \approx g$ for all $f, g \in E$.

Clearly the frame $\{1\}$ is always homogeneous. But if $e^2 = e$ then both e and $1 - e$ *cannot* belong to any homogeneous, primitive frame. [If $eg = (1 - e)g$ then $1 - 2e = 0$, so $g = 0$.] The E-blocks in Theorem 5.7 have homogeneous primitive frames by Theorem 5.7(b).

Example 5.13. Examples of left bricks :

(a) Every E-block B_p arising as in († † †) is a left brick by Theorem 5.7. Conversely every left brick B arises in this way (by Theorem 5.7 applied to $R = B$).

(b) Every left soclin ring that is not a division ring is a left brick (Example 4.3).

(c) No division ring is a left brick (left bricks are left soclin).

(d) All local rings R with $J(R) \neq 0$ *are* left bricks.

(e) Every non-soclin, QD domain is a left brick.

(f) All PIDs are bricks.

(g) The semilocal ring $\mathbb{Z}_{(p,q)}$ in Example 3.10 is a brick.

(h) In Example 5.17 below we present a left brick that is not a right brick.

These left bricks will be our chief concern in the sequel. One main reason for this is the following characterization of the left QDS rings.

Proposition 5.14. *The following are equivalent for a ring R:*

(1) *R is a left QDS ring.*

(2) *R is a finite direct product of left bricks.*

Proof. (1)⇒(2) Given (1), († † †) shows that $R \cong B_1 \times B_2 \times \cdots \times B_m$ where each B_p has the following properties:

- B_p is an indecomposable left QDS ring by **2**), **3**), **4**), and **5**), and
- B_p has a homogeneous frame E_p by **6**) and Theorem 5.7(b).

That is these B_p are all left bricks, proving (2).

(2)⇒(1) Suppose $R \cong B_1 \times B_2 \times \cdots \times B_m$ where each B_p is a left brick. Then R is: I-finite by **2**); left QD by **3**), and left soclin by Proposition 4.5. □

Of course, the primary goal of this paper is to determine the structure of *all* I-finite left QD rings. We began this task with the Triangular Theorem (Theorem 3.27–called the first Main Theorem). After incorporating Definition 5.1 of the left soclin rings, this theorem becomes:

A nonzero ring R is I-finite and left QD if and only if $R \cong UT_n(R_i; V_{ij})$ is generalized upper triangular, where $n \geq 1$ and either

(a) *R_i is a division ring for each $i = 1, 2, \ldots, n$; or*

(b) *R_i is a division ring for each $i < n$, and $R_n \neq 0$ is a left QDS-ring.*

Combining this with Proposition 5.14 yields our second Main Theorem:

Theorem 5.15. Structure Theorem *The following conditions are equivalent for a ring $R \neq 0$:*

(1) *R is left QD and I-finite.*

(2) *$R \cong UT_n(R_i; V_{ij})$ is generalized upper triangular where $n \geq 1$ and either* (a) *or* (b) *holds*:

(a) *R_i is division for each $i = 1, 2, \ldots, n$; or*

(b) *R_i is division if $i < n$; and $R_n \neq 0$ is a finite direct product of left bricks.*

Thus, the description of the I-finite, left QD rings amounts to describing the left bricks.

Here are some related remarks. Suppose B is a left brick B with a primitive, homogeneous, block frame

$$F = \{f_1, f_2, \ldots, f_t\}.$$

This leads to a representation of B as a generalized $t \times t$ matrix ring

$$B \cong \mathrm{end}_D\big(Df_1 \oplus Df_2 \oplus \cdots \oplus Df_t\big) \cong \begin{bmatrix} f_1Bf_1 & f_1Bf_2 & \cdots & f_1Bf_t \\ f_2Bf_1 & f_2Bf_2 & \cdots & f_2Bf_t \\ \vdots & \vdots & \ddots & \vdots \\ f_tBf_1 & f_tBf_2 & \cdots & f_tBf_t \end{bmatrix}.$$

Here each f_iBf_i is indecomposable (as f_i is primitive), it is left QD by Theorem 2.13, and the primitive frame $\{1, f_i\}$ is homogeneous ($1 \sim f_i$ because $1f_i = f_if_i$). Moreover, Example 4.3 shows that each f_iBf_i is either division or left soclin (not both). It follows that, after relabeling, there exists $k \in \mathbb{Z}$ with $1 \leq k \leq t$ where $f_1Bf_1, \cdots, f_kBf_k$ are division, and $f_{k+1}Bf_{k+1}, \cdots, f_tBf_t$ are left soclin left bricks.

This combines with the following fact: For $e^2 = e$, $f^2 = f$ in B,

$$eBf \cong hom_{eRe}(Be, Bf) \text{ via } a \mapsto \lambda_a : Re \to Rf, \text{ where } x\lambda_a = xa.$$

With this it remains to use the homogeneity of $\approx$ in B to discover more about the structure of the ring in (*).

Examples of these left soclin left bricks include:

- any I-free left soclin ring that is not division;
- all left QD domains (including all PIDs);
- the semilocal ring $\mathbb{Z}_{(p,q)}$ in Example 3.10; and
- local rings with $J \neq 0$ (the only semiperfect examples).

Question 10. *Describe the semiperfect left bricks:—see* [13, Theorem 22.6] *for the artinian case.*

A left QDS brick that is not a right QDS brick

Lemma 5.16. *Let* $\Gamma = \begin{bmatrix} R & V \\ 0 & S \end{bmatrix}$ *be a split-null extension where R and S are both I-free. Given $v, w \in V$, use the notations*:

$$\widehat{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \varepsilon_v = \begin{bmatrix} 1 & v \\ 0 & 0 \end{bmatrix} \text{ and } \phi_w = \begin{bmatrix} 0 & w \\ 0 & 1 \end{bmatrix}.$$

(a) $I(\Gamma) = \{\widehat{0}, \widehat{1}, \varepsilon_v, \phi_w\}$ *where v and w range independently over V.*

(b) $\widehat{1} - \varepsilon_v = \phi_{(-v)}$ *for all $v \in V$.*

(c) $\phi_w\varepsilon_v = \widehat{0}$ *for all $v, w \in V$.*

(d) $\varepsilon_v\phi_w = \widehat{0} \;\Leftrightarrow\; v + w = 0 \;\Leftrightarrow\; \phi_w = \widehat{1} - \varepsilon_v$.

(e) *Each ε_v and each ϕ_w is a primitive idempotent.*

(f) *The only frames for Γ are $\{\widehat{1}\}$, $\{\widehat{0}, \widehat{1}\}$, and for some $v \in V$,*
$\{\varepsilon_v, \widehat{1} - \varepsilon_v\} = \{\varepsilon_v, \phi_{-v}\}$.

(g) *If $V \neq 0$ the frame $\{\varepsilon_v, \widehat{1} - \varepsilon_v\} = \{\varepsilon_v, \phi_{-v}\}$ is homogeneous for each $v \in V$.*

(h) Γ *is indecomposable as a ring.*

Proof. (a) If $\varepsilon = \begin{bmatrix} e & v \\ 0 & f \end{bmatrix}$ in Γ, then $\varepsilon^2 = \varepsilon$ if and only if $e^2 = e$, $f^2 = f$ and $ev + vf = v$. As R and S are both I-free then $e = 0, 1$ in R and $f = 0, 1$ in S. These four cases lead to $I(\Gamma) \subseteq \{\widehat{0},\, \widehat{1},\, \varepsilon_v,\, \phi_w\}$. The other inclusion is because $\{\widehat{0},\, \widehat{1},\, \varepsilon_v,\, \phi_w\}$ consists of idempotents

(b), (c), and (d) Given (a), these are routine.

(e) Suppose $\lambda^2 = \lambda \le \varepsilon_v$ where $\lambda \ne \widehat{0}$. Then $\lambda\varepsilon_v = \lambda = \varepsilon_v\lambda$ and $\lambda \ne \widehat{1}$. So (c) implies $\lambda \ne \phi_w$ for all $w \in V$. Hence (a) shows that $\lambda = \varepsilon_{v'}$ for some $v' \in V$. But then ε_v is primitive because

$$\lambda = \lambda\varepsilon_v = \varepsilon_{v'}\varepsilon_v = \begin{bmatrix} 1 & v' \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & v \\ 0 & 0 \end{bmatrix} = \varepsilon_v.$$

Turning to ϕ_w, suppose $\mu^2 = \mu \le \phi_w$ where $\mu \ne \widehat{0}$. Again $\mu \ne \widehat{1}$, and $\mu \ne \varepsilon_v$ for all $v \in V$ by (c). So (a) implies that $\mu = \phi_{w'}$ with $w' \in W$, and so, as required, we obtain

$$\mu = \phi_w\mu = \phi_w\phi_{w'} = \begin{bmatrix} 0 & w \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & w' \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & w \\ 0 & 1 \end{bmatrix} = \phi_w.$$

(f) Let E be a frame for Γ, and assume that both $E \ne \{\widehat{1}\}$ and $E \ne \{\widehat{0}, \widehat{1}\}$. Observe that: $\widehat{0} \notin E$—as $\widehat{0}$ belongs to no frame; and $\widehat{1} \notin E$ as then $E = \{\widehat{1}\}$. Hence, by (a), $E = \{\varepsilon_{v_1}, \varepsilon_{v_2}, \dots,\ \phi_{w_1}, \phi_{w_2}, \dots\}$. But $\varepsilon_v\varepsilon_{v'} \ne 0$ whenever $v \ne v'$, and $\phi_w\phi_{w'} \ne 0$ when $w \ne w'$. It follows that $E = \{\varepsilon_v, \phi_w\}$ for some $v, w \in V$. Finally, this means that $\varepsilon_v + \phi_w = \widehat{1}$, so $\phi_w = \widehat{1} - \varepsilon_v$ by (b). This proves (f).

(g) We must show that $\varepsilon_v \approx \phi_{-v}$; that is we must find $\gamma \in \Gamma$ such that $\varepsilon_v\Gamma\gamma \ne 0$ and $\phi_{-v}\Gamma\gamma \ne 0$. It turns out that $\gamma = \phi_{-v}$ does it. Certainly $\phi_{-v}\Gamma\phi_{-v} \ne 0$ as it contains $\phi_{-v} \ne 0$. And $\varepsilon_v\Gamma\phi_{-v} \ne 0$ because, if $0 \ne x \in V$ by hypothesis, we have $\varepsilon_v \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \phi_{-v} = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \ne 0$.

(h) Suppose $\psi^2 = \psi = \begin{bmatrix} e & z \\ 0 & f \end{bmatrix} \in C(\Gamma)$, Then $e^2 = e$, $f^2 = f$, and $z = 0$ as ψ commutes with $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $e \in C(R)$, $f \in C(S)$ and $ev = vf$ for all $v \in R$ as ψ commutes with each $\begin{bmatrix} r & v \\ 0 & s \end{bmatrix}$. Moreover, as R and S are both I-free, ψ takes one of the following forms $\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & v \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & w \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & v \\ 0 & 1 \end{bmatrix}$. But then the fact that ψ is central shows easiy that $\psi = \widehat{0}$ or $\psi = \widehat{1}$, proving (h). □

Example 5.17. Let $R = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & d \end{bmatrix} \middle|\, a, b, c, d \in D \right\}$ from Example 4.9, where D is division ring. Then R is a left QDS brick that is not a right QDS brick.

Proof. Clearly $R = \begin{bmatrix} S & V \\ 0 & D \end{bmatrix}$ is the split-null extension of the local ring $S = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \middle| a, b \in D \right\}$ and the division ring D over the bimodule ${}_SV_D = \begin{bmatrix} D \\ 0 \end{bmatrix}$. Since both S and D are I-free, Lemma 5.16 applies to R. Hence we conclude:

- R is a QD ring by Example 4.9(a).
- R is left soclin by Example 4.9(c).
- R has a primitive, homogeneous frame by Lemma 5.16(e) and 5.16(g).
- R is indecomposable as a ring by Lemma 5.16(h).

Hence R is a left QDS brick by Definition 5.12. But R is not right soclin, again by Example 4.9(c), so it is not a right QDS brick. This completes the proof. $\square$

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