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SOME HARDY-TYPE INTEGRAL INEQUALITIES WITH SHARP CONSTANT INVOLVING MONOTONE FUNCTIONS

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Abstract. In this work, we present some Hardy-type integral inequalities for $0 < p < 1$ via a second parameter $q > 0$ with sharp constant. These inequalities are new generalizations to the inequalities given bellow.

1. Introduction

It is well-known that for L^p spaces with $0 < p < 1$, the Hardy inequality is not satisfied for arbitrary non-negative functions, but is satisfied for non-negative monotone functions. Moreover the sharp constant was found in the Hardy type-inequality for non-negative monotone functions (see [4] for more details). Namely the following statement was proved there.

Theorem 1. *Let* $0 < p < 1$*:*

- *If* $-\frac{1}{p} < \alpha < 1-\frac{1}{p}$*, then for all functions* f *non-negative and non-increasing on* $(0, +\infty)$

$$\|x^{\alpha}(Hf)(x)\|_{L^p(0,+\infty)} \leq \left(1-\frac{1}{p}-\alpha\right)^{-\frac{1}{p}} \|x^{\alpha}f(x)\|_{L^p(0,+\infty)}. \tag{1}$$

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- *If $\alpha < -\frac{1}{p}$, then for all functions f non-negative and non-decreasing on $(0,+\infty)$*

$$\|x^\alpha (Hf)(x)\|_{L^p(0,+\infty)} \leq (p\,\beta(p,-\alpha p))^{\frac{1}{p}} \|x^\alpha f(x)\|_{L^p(0,+\infty)} . \tag{2}$$

- *If $\alpha > 1-\frac{1}{p}$, then for all functions f non-negative and non-increasing on $(0,+\infty)$*

$$\left\|x^\alpha (\widetilde{H}f)(x)\right\|_{L^p(0,+\infty)} \leq (p\,\beta(p,\alpha p+1-p))^{\frac{1}{p}} \|x^\alpha f(x)\|_{L^p(0,+\infty)} . \tag{3}$$

Here

$$(Hf)(x) = \frac{1}{x}\int_0^x f(t)dt, \quad (\widetilde{H}f)(x) = \frac{1}{x}\int_x^\infty f(t)dt.$$

$\beta(u,v) = \displaystyle\int_0^1 t^{u-1}(1-t)^{v-1}dt$ *is the Euler -Beta function.*
The constants in the inequalities (1), (2), (3) are sharp.

In 2012 W.T. Sulaiman [5] extended Hardy's integral inequality as follows.

Theorem 2. *If $f \geq 0,\ g > 0,\ x^{-1}g(x)$ is non-decreasing $p > 1$, $0 < a < 1$ and $F(x) = \displaystyle\int_0^x f(t)dt$, then*

$$\int_0^\infty \left(\frac{F(x)}{g(x)}\right)^p dx \leq \frac{1}{a(p-1)(1-a)^{p-1}} \int_0^\infty \left(\frac{xf(x)}{g(x)}\right)^p dx, \tag{4}$$

in particular if $a = \frac{1}{p}$, $g(x) = x$, we obtain Hardy's inequality.

Moreover he proved the reverse inequality.

Theorem 3. *If $f \geq 0,\ g > 0,\ x^{-1}g(x)$ is non-increasing $0 < p < 1$, $a > 0$ and $F(x) = \displaystyle\int_0^x f(t)dt$, then*

$$\int_0^\infty \left(\frac{F(x)}{g(x)}\right)^p dx \geq \frac{1}{a(1-p)(1+a)^{p-1}} \int_0^\infty \left(\frac{xf(x)}{g(x)}\right)^p dx. \tag{5}$$

The following Lemmas were established in [4].

Lemma 1. *Let $0 < p < 1$, $-\infty < a < b \leq +\infty$ and f a non- negative non-increasing function on (a,b), then*

$$\left(\int_a^b f(x)dx\right)^p \leq p\int_a^b f^p(x)(x-a)^{p-1}dx. \tag{6}$$

Lemma 2. *Let $0 < p < 1$, $-\infty \leq a < b < +\infty$ and f a non- negative non-decreasing function on (a,b), then*

$$\left(\int_a^b f(x)dx\right)^p \leq p\int_a^b f^p(x)(b-x)^{p-1}dx. \tag{7}$$

The factor p is the best possible in inequalities (6) and (7).

About the Hardy inequality, its history and some related results one can consult [1], [2], [3], [6] and [7].

The aim of this work is includes two objectives, first the power weight function x^{α} in Theorem 1 is replaced by $g(x)$, where $x^{-\alpha}g(x)$ is non-decreasing or non-increasing function and we give a new some Hardy-type integral inequalities with sharp constant. The second objective is to present some generalizations for the weighted Hardy operator with $0 < p < 1$. Moreover we introduce a second parameter $q > 0$ for these generalizations.

2. Main Results

In this section, we present our results. We assume that f and g are non-negative Lebesgue measurable functions on $(0, +\infty)$.

Theorem 4. *Let $0 < p < 1$, $q > 0$, $g > 0$ and the function $x^{\alpha}g(x)$ is non-decreasing for $-\frac{1}{q} < \alpha < \frac{p-1}{q}$, then for all non-negative non-increasing function f we have*

$$\int_0^{\infty} \frac{(Hf)^p(x)}{g^q(x)} dx \leq \frac{p}{p - \alpha\, q - 1} \int_0^{\infty} \frac{f^p(x)}{g^q(x)} dx. \tag{8}$$

The constant in (8) is sharp.

Proof.
Since f is non-increasing, then by Lemma 1 we get

$$\begin{aligned}
\int_0^{\infty} \frac{(Hf)^p(x)}{g^q(x)} dx &= \int_0^{\infty} x^{-p} g^{-q}(x) \left(\int_0^x f(t)dt \right)^p dx \\
&\leq p \int_0^{\infty} x^{-p} g^{-q}(x) \left(\int_0^x f^p(t) t^{p-1} dt \right) dx \\
&= p \int_0^{\infty} t^{p-1} f^p(t) \left(\int_t^{+\infty} x^{-p} g^{-q}(x) dx \right) dt \\
&\leq p \int_0^{\infty} t^{p-1} f^p(t) \left(\frac{t^{-\alpha}}{g(t)} \right)^q \left(\int_t^{+\infty} x^{-p+\alpha q} dx \right) dt \\
&= \frac{p}{p - \alpha q - 1} \int_0^{\infty} t^{p-1} f^p(t) \frac{t^{-\alpha q}}{g^q(t)} t^{-p+\alpha q+1} dt \\
&= \frac{p}{p - \alpha q - 1} \int_0^{\infty} \frac{f^p(t)}{g^q(t)} dt.
\end{aligned}$$

To proof that $\dfrac{p}{p-\alpha q-1}$ is the best possible, we put $g(x)=x^{-\alpha}$ and

$$f(x)=\begin{cases}1 & \text{if } x\in(0,\xi),\\ \\ 0 & \text{if } x\in(\xi,+\infty).\end{cases}$$

Let RHS and LHS respectively be the right hand side and the left hand side of the inequality (8), then

$$\begin{aligned} RHS &= \int_0^\infty x^{\alpha q-p}\left(\int_0^x f(t)dt\right)^p dx \\ &= \frac{\xi^{\alpha q+1}}{\alpha q+1}, \end{aligned}$$

and

$$\begin{aligned} LHS &= \frac{p}{p-\alpha\, q-1}\int_0^\xi x^{\alpha q}dx \\ &= \frac{p}{p-\alpha\, q-1}\frac{\xi^{\alpha q+1}}{\alpha q+1}. \end{aligned}$$

Using $q=p$ in the Theorem 4, we get the following Corollary.

Corollary 1. *Let $0<p<1$, $g>0$ and the function $x^\alpha g(x)$ is non-decreasing for $-\frac{1}{p}<\alpha<\frac{p-1}{p}$, then for all non-negative non-increasing function f we have*

$$\left\|\frac{(Hf)(x)}{g(x)}\right\|_{L^p(0,+\infty)} \le \left(1-\alpha-\frac{1}{p}\right)^{-\frac{1}{p}} \left\|\frac{f(x)}{g(x)}\right\|_{L^p(0,+\infty)}. \tag{9}$$

The constant $\left(1-\alpha-\frac{1}{p}\right)^{-\frac{1}{p}}$ is sharp.

Remark 1. *If we take $g(x)=x^{-\alpha}$ in the inequality (9), we obtain the inequality (1).*

Theorem 5. *Let $0<p<1$, $q>0$, $g>0$ and the function $x^\alpha g(x)$ is non-decreasing for $\alpha<-\frac{1}{q}$, then for all non-negative non-decreasing function f we have*

$$\int_0^\infty \frac{(Hf)^p(x)}{g^q(x)}dx \le p\,\beta(p,\,-\alpha\, q)\int_0^\infty \frac{f^p(x)}{g^q(x)}dx, \tag{10}$$

where β is the Euler-Beta function. The constant in (10) is sharp.

Proof.
By using the Lemma 2, we get

$$\begin{aligned}\int_0^\infty \frac{(Hf)^p(x)}{g^q(x)}dx &= \int_0^\infty x^{-p}g^{-q}(x)\left(\int_0^x f(t)dt\right)^p dx\\ &\leq p\int_0^\infty x^{-p}g^{-q}(x)\left(\int_0^x f^p(t)(x-t)^{p-1}dt\right)dx\\ &= p\int_0^\infty f^p(t)\left(\int_t^{+\infty} x^{-p}g^{-q}(x)(x-t)^{p-1}dx\right)dt\\ &\leq p\int_0^\infty f^p(t)\left(\frac{t^{-\alpha}}{g(t)}\right)^q\left(\int_t^{+\infty} x^{\alpha q-p}(x-t)^{p-1}dx\right)dt.\end{aligned}$$

Using the change of variable $z=\frac{t}{x}$, then

$$\begin{aligned}\int_t^{+\infty} x^{\alpha q-p}(x-t)^{p-1}dx &= \int_0^1 \left(\frac{t}{z}\right)^{\alpha q-p}\left(\frac{t}{z}-t\right)^{p-1}\frac{t}{z^2}dz\\ &= t^{\alpha q}\int_0^1 z^{-\alpha q-1}(1-z)^{p-1}dz\\ &= t^{\alpha q}\beta(p,-\alpha q),\end{aligned}$$

therefore

$$\int_0^\infty \frac{(Hf)^p(x)}{g^q(x)}dx \leq p\beta(p,-\alpha q)\int_0^\infty \left(\frac{f^p(t)}{g^q(t)}\right)dt.$$

To proof that $p\,\beta(p,-\alpha\,q)$ is the best possible, we put $g(x)=x^{-\alpha}$ and

$$f(x)=\begin{cases} 0 & \text{if } x\in(0,\xi),\\ 1 & \text{if } x\in(\xi,+\infty).\end{cases}$$

Let RHS and LHS respectively be the right side and the left side of the inequality (10), then

$$\begin{aligned}RHS &= \int_\xi^\infty x^{\alpha q-p}\left(\int_\xi^x f(t)dt\right)^p dx\\ &= \int_\xi^\infty x^{\alpha q-p}\,(x-\xi)^p\,dx,\end{aligned}$$

let $\mu = \frac{\xi}{x}$, then we get

$$\begin{aligned} RHS &= \xi^{\alpha q+1}\int_0^1 \mu^{-\alpha q-2}(1-\mu)^p d\mu \\ &= \xi^{\alpha q+1}\beta(p+1,\, -\alpha q-1) \\ &= \tfrac{p}{|\alpha q+1|}\xi^{\alpha q+1}\beta(p,\, -\alpha q). \end{aligned}$$

On another side

$$\begin{aligned} LHS &= p\,\beta(p,\, -\alpha\, q)\int_{\xi}^{+\infty} x^{\alpha q}dx \\ &= p\,\beta(p,\, -\alpha\, q)\tfrac{1}{|\alpha q+1|}\xi^{\alpha q+1}. \end{aligned}$$

If we set $q = p$ in the Theorem 5, we get the following Corollary.

Corollary 2. *Let $0 < p < 1$, $g > 0$ and the function $x^{\alpha} g(x)$ is non-decreasing for $\alpha < -\frac{1}{q}$, then for all non-negative non-decreasing function f we have*

$$\left\|\frac{(Hf)(x)}{g(x)}\right\|_{L^p(0,+\infty)} \leq (p\,\beta(p,\, -\alpha\, p))^{\frac{1}{p}}\left\|\frac{f(x)}{g(x)}\right\|_{L^p(0,+\infty)}. \tag{11}$$

The constant $(p\,\beta(p,\, -\alpha\, p))^{\frac{1}{p}}$ is sharp.

Remark 2. *If we take $g(x) = x^{-\alpha}$ in the inequality (11), we obtain the inequality (2).*

Theorem 6. *Let $0 < p < 1$, $q > 0$, $g > 0$ and the function $x^{\alpha} g(x)$ is non-increasing for $\alpha > \frac{p-1}{q}$, then for all non-negative non-increasing function f we have*

$$\int_0^{\infty}\frac{(\widetilde{Hf})^p(x)}{g^q(x)}dx \leq p\,\beta(p,\, \alpha\, q+1-p)\int_0^{\infty}\frac{f^p(x)}{g^q(x)}dx, \tag{12}$$

the constant in (12) is sharp.

Proof.
By applying the Lemma 1, we obtain

$$\begin{aligned}\int_0^\infty \frac{(\widetilde{Hf})^p(x)}{g^q(x)}dx &= \int_0^\infty x^{-p}g^{-q}(x)\left(\int_x^\infty f(t)dt\right)^p dx \\ &\le p\int_0^\infty x^{-p}g^{-q}(x)\left(\int_x^\infty f^p(t)(t-x)^{p-1}dt\right)dx \\ &= p\int_0^\infty f^p(t)\left(\int_0^t x^{-p}g^{-q}(x)(t-x)^{p-1}dx\right)dt \\ &\le p\int_0^\infty f^p(t)\left(\frac{t^{-\alpha}}{g(t)}\right)^q\left(\int_0^t x^{\alpha q-p}(t-x)^{p-1}dx\right)dt.\end{aligned}$$

Using the change of variable $\nu = \frac{t-x}{t}$, then

$$\begin{aligned}\int_0^t x^{\alpha q-p}(t-x)^{p-1}dx &= \int_0^1 [(1-\nu)t]^{\alpha q-p}(\nu t)^{p-1}\, t d\nu \\ &= t^{\alpha q}\int_0^1 \nu^{p-1}(1-\nu)^{\alpha q-p}d\nu \\ &= t^{\alpha q}\beta(p,\, \alpha q-p+1),\end{aligned}$$

thus

$$\int_0^\infty \frac{(\widetilde{Hf})^p(x)}{g^q(x)}dx \le p\beta(p,\, \alpha q-p+1)\int_0^\infty \left(\frac{f^p(t)}{g^q(t)}\right)dt.$$

The proof that $p\,\beta(p,\, \alpha q-p+1)$ is sharp, is similar to that of Theorem 5 with the function f defined as follows

$$f(x) = \begin{cases} 1 & \text{if } x\in(0,\xi), \\ 0 & \text{if } x\in(\xi,+\infty). \end{cases}$$

If we put $q=p$ in the Theorem 6, we have the following Corollary.

Corollary 3. *Let $0<p<1$, $g>0$ and the function $x^\alpha g(x)$ is non-increasing for $\alpha < -\frac{1}{q}$, then for all non-negative non-increasing function f we have*

$$\left\|\frac{(\widetilde{Hf})(x)}{g(x)}\right\|_{L^p(0,+\infty)} \le (p\,\beta(p,\, \alpha p+1-p))^{\frac{1}{p}}\left\|\frac{f(x)}{g(x)}\right\|_{L^p(0,+\infty)}. \qquad (13)$$

The constant $(p\,\beta(p,\, \alpha p+1-p))^{\frac{1}{p}}$ is sharp.

Remark 3. *If we take $g(x)=x^{-\alpha}$ in the inequality (13), we obtain the inequality (3).*

In the second part of this work, we consider Theorems 2 and 3 for weighted Lebesgue space. Let $0 < p < \infty$, the weighted Lebesgue space $L^p_w(0,\infty)$ is the space of all Lebesgue measurable functions f such that

$$\|f\|_{L^p_w(0,\infty)} = \left(\int_0^\infty |f(t)|^p\, w(t)dt\right)^{\frac{1}{p}} < \infty, \tag{14}$$

where w is the weight function (Lebesgue measurable and positive on $(0,\infty)$).

Theorem 7. *Let $f \geq 0,\ g > 0,\ 0 < p < 1,\ 0 < \alpha < 1$. If the function $\frac{w(x)}{g^p(x)}$ is non-increasing, then*

$$\left\|\frac{(Hf)(x)}{g(x)}\right\|_{L^p_w(0,\infty)} \leq C_1 \left\|\frac{f(x)}{g(x)}\right\|_{L^p_w(0,\infty)}, \tag{15}$$

where the constant $C_1 = \frac{1}{1-\alpha}$ is sharp.

Proof.
By using Holder's inequality, we have

$$\begin{aligned}
\left\|\tfrac{(Hf)(x)}{g(x)}\right\|^p_{L^p_w(0,\infty)} &= \int_0^\infty \frac{(Hf)^p(x)}{g^p(x)} w(x)dx \\
&= \int_0^\infty \frac{g^{-p}(x)}{x^p}\left(\int_0^x f(t)t^{\alpha(1-\frac{1}{p})}t^{\alpha(\frac{1}{p}-1)}dt\right)^p w(x)dx \\
&\leq \int_0^\infty \frac{g^{-p}(x)}{x^p} w(x)\left(\int_0^x t^{\alpha(p-1)}f^p(t)dt\right)\left(\int_0^x t^{-\alpha}dt\right)^p dx \\
&= \left(\frac{1}{1-\alpha}\right)^{p-1}\int_0^\infty \frac{x^{\alpha(p-1)-1}}{g^p(x)} w(x)\left(\int_0^x t^{\alpha(p-1)}f^p(t)dt\right)dx \\
&= \left(\frac{1}{1-\alpha}\right)^{p-1}\int_0^\infty t^{\alpha(p-1)}f^p(t)\left(\int_t^\infty \frac{x^{\alpha(p-1)-1}}{g^p(x)} w(x)dx\right)dt \\
&= \left(\frac{1}{1-\alpha}\right)^{p-1}\int_0^\infty \frac{f^p(t)}{g^p(t)} w(t)K(t)dt,
\end{aligned}$$

where

$$K(t) = \left[\frac{t^{\alpha(p-1)}g^p(t)}{w(t)}\left(\int_t^\infty \frac{x^{\alpha(p-1)-1}}{g^p(x)} w(x)dx\right)\right].$$

Now we proof that $K(t)$ is finite for all $t > 0$. From the assumption $\frac{w(x)}{g^p(x)}$ is non-increasing, we deduce that

$$\begin{aligned}\int_t^\infty \frac{x^{\alpha(p-1)-1}}{g^p(x)} w(x)dx &\leq \frac{w(t)}{g^p(t)} \int_t^\infty x^{\alpha(p-1)-1}dx \\ &= \frac{w(t)}{g^p(t)} \frac{t^{\alpha(p-1)}}{\alpha(1-p)},\end{aligned}$$

hence

$$\text{for all } t > 0,\ K(t) < \infty.$$

Thus

$$\begin{aligned}\left\| \frac{(Hf)(x)}{g(x)} \right\|^p_{L^p_w(0,\infty)} &\leq \frac{\sup_{t>0} K(t)}{(1-\alpha)^{p-1}} \left\| \frac{f(x)}{g(x)} \right\|_{L^p_w(0,\infty)} \\ &= C^p \left\| \frac{f(x)}{g(x)} \right\|^p_{L^p(0,+\infty)}.\end{aligned}$$

To proof that $C_1 = \left(\frac{1}{1-\alpha}\right)$ is the best possible, taking $f(x) = x^{-\alpha}$, this gives us $(Hf)(x) = \frac{1}{1-\alpha}x^{-\alpha}$ and

$$\begin{aligned}\left\| \frac{(Hf)(x)}{g(x)} \right\|^p_{L^p_w(0,\infty)} &= \frac{1}{(1-\alpha)^p} \int_0^\infty \left(\frac{1}{x^\alpha g(x)}\right)^p w(x)dx, \\ \left\| \frac{f(x)}{g(x)} \right\|^p_{L^p_w(0,\infty)} &= \int_0^\infty \left(\frac{1}{x^\alpha g(x)}\right)^p w(x)dx.\end{aligned}$$

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References

[1] Benaissa, B., Budak, H., On Hardy-type integral inequalities with negative parameter, *Turkish Journal of Inequalities*, 5(2) (2021), 42-47.

[2] Benaissa, B., Sarikaya, M. Z., Senouci, A., On some new Hardy-type inequalities, *J. Math. Meth. Appl. Sci.*, 43 (2020), 8488-8495. https/doi.org/10.1002/mma.6503

[3] Yang, B., On a new Hardy type integral inequalities, *Int. Math. Forum.*, 67(2) (2007), 3317–3322.

[4] Burenkov, V. I., On the best constant in Hardy's inequality with $0 < p < 1$ for monotone functions, *Proc. Steklov Inst. Math.,* 194 (1993), 59-63.
[5] Sulaiman, W. T., Some Hardy type integral inequalities, *Appl. Math. Lett,* 25 (2012), 520-525. https://doi.org/10.1016/j.aml.2011.09.050
[6] Kufner, A., Maligranda, L., Person, L. E., The Hardy Inequality: About Its History and Some Related Results, Vydavatelsky Servis Publishing House, Pilsen, 2007.
[7] Banyat, S., More on some Hardy type integral inequalities, *J. Math. Inequal.*, 8(3) (2014), 497–501.