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Wavelet-based Numerical Approaches for Solving the Korteweg-de Vries (KdV) Equation

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Abstract. In this research work, we examine the Korteweg-de Vries equation (KdV), which is utilized to formulate the propagation of water waves and occurs in different fields such as hydrodynamics waves in cold plasma acoustic waves in harmonic crystals. This research presents two efficient computational methods based on Legendre wavelets to solve the Korteweg-de Vries. The three-step Taylor method is first applied to the Korteweg-de Vries equation for time discretization. Then, the Galerkin and collocation methods are used for spatial discretization. With these approaches, bringing the approximate solutions of the Korteweg-de Vries equation turns into getting the solution of the algebraic equation system. The solution of this system gives the Legendre wavelet coefficients. The approximate solution can be obtained by substituting the obtained coefficients into the Legendre wavelet series expansion. The presented wavelet methods are tested by studying different problems at the end of this study.

2010 AMS Classification: 65M60, 65N40

Keywords: Three-step Taylor method, KdV equation, Legendre wavelets, Galerkin method, collocation method.

1. Introduction

In this study, we present the efficient two different numerical schemes based on Legendre wavelets for solving the following Korteweg- de Vries (KdV) equation:

$$\vartheta_t + \zeta\vartheta\vartheta_x + \delta\vartheta_{xxx} = 0,\ t > 0, \tag{1.1}$$

in which ζ and δ are real constants [14]. The term $\vartheta\vartheta_x$ represents the convection effect that causes steepening of the waveform, and the term ϑ_{xxx} means the dispersion effect that propagates the waveform [37]. The KdV equation is usually used to model the waves on shallow water surfaces. Due to the competition of dispersion and convection effects, a solitary wave is formed. In solitary waves, solitons are explained as localized waves propagating without changing their shape and velocity characteristics and are stable against reciprocal collisions [8]. This equation has enormous implementations in engineering and physical sciences. This research paper suggests two efficient numerical methods for solving the KdV equation.

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Many analytic and numerical schemes have been studied and formulated to solve the KdV equation in the literature. These methods can be sorted as follows; Kumar and Mehra [4] used the wavelet Galerkin method. Oruc *et al.*presented the Haar wavelet method [25]. Desong *et al.* used a combination of finite difference and sinc collocation method [13]. A heat balance integral method was presented by Kutluay *et al.* [15]. Hepson and Dag [28] developed the exponential cubic B-spline algorithm. Soliman proposed a collocation solution of the KdV equation utilizing septic B-splines [34]. The Bubnov-Galerkin method with quadratic spline finite element Gardner was described for solving the KdV equation by Gardner *et al.* [10]. A Legendre–Petrov–Galerkin method was designed to solve the KdV equation by Ma and Sun [16]. Akdi and Sedra [1] gave the Adomian decomposition method for the numerical solution of the KdV equation. Yan and Shu presented a local discontinuous Galerkin method for solving KdV type equations [38]. A cubic B-spline Taylor–Galerkin finite element method was employed to solve the KdV equation by Canivar *et al.* [5]. Dag and Dereli used the meshless method based on the collocation with radial basis functions for solving the KdV equation [7]. Alotaibi and Ismail produced numerical schemes to examine the KdV equation via Pade approximation for space direction, trapezoidal, and implicit mid-point rule in the time direction [2], Hafiz and Andallah discussed the numerical solution of the KdV equation using Explicit finite difference schemes [11].

Wavelets presented by Daubechies have been utilized to get solutions to various problems associated with other fields of engineering and applied sciences. Since the beginning of 1990s, analytical and numerical methods based on wavelets have been applied to examine the different forms of differential equations. Because the Daubechies wavelet family has implicit expression, it is impossible to get the exact differentiation or integration of Daubechies wavelets. This situation is a drawback for Daubechies wavelet family. So, numerical wavelet methods based on orthogonal polynomials have been improved to solve many problems in the literature. Up to now, a huge of papers have focused on this topic. Some of these methods are the Gegenbauer wavelets methods [6,26,27,31], the Legendre wavelet operational matrix method [33], the Münz wavelets collocation method [3], the Jacobi wavelets method [32], wavelet-Taylor-Galerkin methods [4], Genocchi wavelet method [9] and the discontinuous Legendre wavelet Galerkin method [39].

This article concern with the three-step Legendre wavelet collocation method and the three-step Legendre wavelet Galerkin method to estimate the solution the KdV equation in eq. (1.1). In fact, these schemes' idea is based on approximating all existed functions in the KdV equation via Legendre wavelets and using ordinary operational matrice. Until now, these methods have been few employed to acquire the numerical solution of the nonlinear partial differential equations. By using the presented methods, the mentioned problem in eq.(1.1) is transformed into a nonlinear system of equations which can be solved via Galerkin and collocation methods. The system can be easily solved with the proper numerical methods. These methods do not increase the difficulties for higher-dimensional problems and can be easily employed to solve different high- dimensions problems. The methods are easy to implement for nonlinear partial differential equations, and these techniques provide numerical solutions for nonlinear partial differential equations.

The rest parts of this paper are organized as follows: we present the Legendre wavelets, the function approximation with the Legendre wavelets, and the operational matrix of derivative in Section 2. Section 3 represents the description of the Galerkin method and the collocation method and, the application of these methods to the KdV equation. Furthermore, the produced solutions are offered in Section 4. Proper parameter values demonstrate 2D graphs in this section. Finally, the conclusion part is given in Section 5.

2. Legendre Wavelets

Wavelets form a family of functions consisting of the dilation and translation of a single function, called mother wavelet $\psi(x)$. Wavelets are expressed as:

$$\psi_{p,r}(x) = \frac{1}{\sqrt{|p|}}\psi\left(\frac{x-r}{p}\right), p, r \in \mathrm{R}, p \neq 0,$$

where p and r are the dilation and translation parameters, respectively. By limiting p and r to obtain discrete values as:$p = p_0^{-k}, r = nr_0p_0^{-k}$, where $p_0 > 1, r_0 > 0$ and $k, n \in \mathrm{N}$, we acquire the following discrete wavelets:

$$\psi_{k,n}(x) = (p_0)^{\frac{k}{2}}\,\psi\left(p_0^k x - nr_0\right),$$

in which an orthogonal basis of $L_2(\mathrm{R})$ is formed. If $p_0 = 2$ and $r_0 = 1$, then $\psi_{k,n}$ forms an orthonormal basis.

Legendre wavelets are expressed on the interval [0, 1) by:

$$\psi_{n,m}(x) = \begin{cases} \sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}} L_m\left(2^k x - \hat{n}\right), & \frac{\hat{n}-1}{2^k} \le x < \frac{\hat{n}+1}{2^k} \\ 0, & elsewhere \end{cases},$$

where $k = 1, 2, 3, ...,$ is the level of resolution, $\hat{n} = 2n - 1$ for $n = 1, 2, 3, ..., 2^{k-1}$ is the translation parameter, and $m = 0, 1, 2, ..., M - 1$ is the order of the Legendre polynomials, $M > 0$. The coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality. Legendre wavelets are orthogonal on [0, 1) with respect to the weight function $w(x) = 1$.
Here, $L_m(x)$ are Legendre polynomials of order m. These polynomials are identified on the interval $[-1, 1]$, and can be calculated from the following formulas:

$$L_0(x) = 1, \quad L_1(x) = 2x,$$
$$L_{n+1}(x) = \left(\tfrac{2n+1}{n+1}\right) x L_n(x) - \left(\tfrac{n}{n+1}\right) L_{n-1}(x), \; n = 1, 2, 3, ...$$

2.1. **Function Approximation.** A function $\vartheta(x) \in L^2\,[0, 1)$can be expressed in terms of Legendre wavelets as:

$$\vartheta(x) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} a_{lm}\psi_{lm}(x), \tag{2.1}$$

in which a_{lm}values are the coefficients of wavelets, and these wavelet coefficients are computed with the inner product

$$a_{lm} = \langle \vartheta(x), \psi_{lm}(x)\rangle .$$

If the infinite series expansion in Eq. (2.1) is truncated for an approximate solution, Eq. (2.1) can be expressed as:

$$\vartheta(x) = \sum_{l=1}^{2^{k-1}}\sum_{m=0}^{M-1} a_{lm}\psi_{lm}(x) = A^T\Psi(x), \tag{2.2}$$

in which the matrices $\Psi(x)$ and A are of order $2^{k-1}M \times 1$. Eq. (2.2) can be rewritten as:

$$v(x) = \sum_{j=1}^{\tilde{m}} a_j\psi_j(x),$$

in which $\tilde{m} = \left(2^{k-1}M\right), A = [a_1, a_2, ..., a_{\tilde{m}}]^T, \Psi(x) = [\psi_1(x), ..., \psi_{\tilde{m}}(x)]^T$, and the index j can be calculated by the aid of the relation $j = M(n-1) + m + 1$.

2.2. **Operational Matrix of Derivatives.**

Theorem 2.1. *Let $\Psi(x)$ be the Legendre wavelets vector. The derivative of $\Psi(x)$ satisfies the following equation:*

$$\frac{d}{dx}\Psi(x) = D\Psi(x),$$

where D is an operational matrix of derivative having order $2^{k-1}M$. D is defined as

$$D = \begin{bmatrix} \theta & 0 & 0 & \cdots & 0 \\ 0 & \theta & 0 & \cdots & 0 \\ 0 & 0 & \theta & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \theta \end{bmatrix},$$

in which θ is a matrix consisting of $q_{i,j}$ elements expressed as follows:

$$q_{i,j} = \begin{cases} \dfrac{2^{k+1}(j+\gamma-1)}{\sqrt{\frac{(j-1+\gamma)\Gamma(j)\Gamma(i-1+2\gamma)}{(i-1+\gamma)\Gamma(i)\Gamma(j-1+2\gamma)}}}, & j = 1, 2, \ldots, i-1, i = 2, 3, \ldots, M, \quad \text{and} \quad (i+j) \text{ odd} \\ 0, & \text{otherwise}, \end{cases}$$

where $\gamma = \frac{1}{2}$ [36].

The operational Matrix of *nth* order derivative can be expressed as:

$$\frac{d^n}{dx^n}\Psi(x) = D^n\Psi(x). \tag{2.3}$$

3. Description of the Numerical Schemes

In this method, we first examine the nonlinear partial differential equation (NPDE) of the following form

$$\vartheta_t = L\vartheta + Nf(\vartheta), \tag{3.1}$$

with the initial condition

$$\vartheta(x,0) = \vartheta_0(x),$$

in which $L\vartheta$ is the linear part of Eq. (3.1), and $Nf(\vartheta)$ that$f(\vartheta)$ is a nonlinear function is the nonlinear part of Eq. (3.1). When $\vartheta(t)$ is applied a Taylor series expansion in time, we obtain

$$\vartheta(t+\Delta t) = \vartheta(t) + \Delta t\frac{\partial\vartheta(t)}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2\vartheta(t)}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^2\vartheta(t)}{\partial t^2} + O(\Delta t^4). \tag{3.2}$$

Using the equation (3.2) as an approximation to third-order accuracy, the three-step method is expressed as [35]:

$$\begin{aligned} \vartheta\left(t+\tfrac{\Delta t}{3}\right) &= \vartheta(t) + \tfrac{\Delta t}{3}\tfrac{\partial\vartheta(t)}{\partial t}, \\ \vartheta\left(t+\tfrac{\Delta t}{2}\right) &= \vartheta(t) + \tfrac{\Delta t}{2}\tfrac{\partial\vartheta\left(t+\frac{\Delta t}{3}\right)}{\partial t}, \\ \vartheta(t,\Delta t) &= \vartheta(t) + \Delta t\tfrac{\partial\vartheta\left(t+\frac{\Delta t}{2}\right)}{\partial t}. \end{aligned} \tag{3.3}$$

In numerical schemes, we apply the Legendre wavelet Galerkin method and the Legendre wavelet collocation method for spatial discretization of equation (3.3).

3.1. Time Discretization.

Consider the KdV equation with initial condition

$$\vartheta(x,0) = \vartheta_0(x).$$

Assume that $r \geq 0$, Δt is defined the time step such that $t_r = r\Delta t$, $r = 0,1,...,R_t$ and $\vartheta(x,t_r) = \vartheta^r$. We initiate the numerical methods with the discretization of the time employing the three-step method presented in Eq. (3.3),

$$\begin{aligned} \vartheta^{r+1/3} &= \vartheta^r + \tfrac{\Delta t}{3}\left(-\zeta\vartheta^r\vartheta^r_x - \delta\vartheta^r_{xxx}\right), \\ \vartheta^{r+1/2} &= \vartheta^r + \tfrac{\Delta t}{2}\left(-\zeta\vartheta^{r+1/3}\vartheta^{r+1/3}_x - \delta\vartheta^{r+1/3}_{xxx}\right), \\ \vartheta^{r+1} &= \vartheta^r + \Delta t\left(-\zeta\vartheta^{r+1/2}\vartheta^{r+1/2}_x - \delta\vartheta^{r+1/2}_{xxx}\right). \end{aligned} \tag{3.4}$$

Here, $\vartheta^{r+1/3}, \vartheta^{r+1/2}$ and ϑ^{r+1} are the obtained solutions at time level $\left(t_r + \frac{\Delta t}{3}\right), \left(t_r + \frac{\Delta t}{2}\right)$and $(t_r + \Delta t)$, respectively.

3.2. Spatial Discretization for Legendre Wavelet Galerkin Method (LWGM).

In this section, Legendre wavelets are used to the spatial derivatives of $\vartheta(x,t)$, then the Galerkin method is employed.
The unknown solution $\vartheta(x,t)$ can be expressed by Legendre wavelets as:

$$\vartheta(x,t_r) \approx \vartheta^r = \sum_{l=0}^{2^k-1}\sum_{m=0}^{M} c^r_{lm}\psi_{lm} = \sum_{j=1}^{\widehat{m}} c^r_j\psi_j(x) = (C^r)^T\Psi(x). \tag{3.5}$$

Here, the coefficient vector $C^r$is calculated at the time t_r. Thus, the numerical solution at the time $t_{r+1/3}$ is as follows:

$$\vartheta^{r+\frac{1}{3}} = \sum_{j=1}^{\widehat{m}} c^{r+1/3}_j\psi_j(x) = \left(C^{r+\frac{1}{3}}\right)^T\Psi(x). \tag{3.6}$$

By substituting Eq. (3.6) into the first equation of Eq. (3.4), we get

$$\vartheta^{r+\frac{1}{3}} = \vartheta^r + \frac{\Delta t}{3}\left(-\zeta(\vartheta)^r\left(\frac{\partial\vartheta}{\partial x}\right)^r - \delta\left(\frac{\partial^3\vartheta}{\partial x^3}\right)^r\right). \tag{3.7}$$

Applying the operational Matrix of derivative for $\vartheta_x = \frac{\partial\vartheta}{\partial x}, \vartheta_{xxx} = \frac{\partial^3\vartheta}{\partial x^3}$ and Eq. (2.3), we get the following equations.

$$\frac{\partial\vartheta}{\partial x}(x,t_r) = \sum_{l=0}^{2^{k-1}}\sum_{m=0}^{M} c^r_{lm}\frac{\partial\psi_{lm}(x)}{\partial x} = (C^r)^T D\Psi(x) \tag{3.8}$$

and

$$\frac{\partial^3 \vartheta}{\partial x^3}(x, t_r) = \sum_{l=0}^{2^{k-1}} \sum_{m=0}^{M} c_{lm}^{r} \frac{\partial^3 \psi_{lm}(x)}{\partial x^3} = (C^r)^T D^3 \Psi(x). \tag{3.9}$$

By putting Equations (3.5), (3.6), (3.8) and (3.9) into Eq. (3.7) , we get

$$\left(C^{r+\frac{1}{3}}\right)^T \Psi(x) = (C^r)^T \Psi(x) + \frac{\Delta t}{3}\left(-\zeta\left((C^r)^T \Psi(x)\right).\left((C^r)^T D\Psi(x)\right) - \delta\,(C^r)^T D^3\Psi(x)\right). \tag{3.10}$$

When we take the initial condition firstly, we obtain

$$\vartheta_0(x) = \vartheta(x, t_0) = \vartheta^0 = \left(C^0\right)^T \Psi(x).$$

When we perform the inner product from both sides of Eq. (3.10) with $\left\{\psi_j\right\}_{j=1}^{\widehat{m}}$ as in the Galerkin method, we acquire a system of nonlinear algebraic equations with $2^{k-1}M$ unknown variables, $c_j^{r+\frac{1}{3}}$. Solving this system, we obtain $C^{r+\left(\frac{1}{3}\right)}$. To get the vectors $C^{r+\left(\frac{1}{2}\right)}$ and C^{r+1}, the resembling procedure is employed to the other equations in Eq. (3.4). The C^{r+1} is the vector coefficient in the numerical solution of $\vartheta(x, t_r)$ each step for $r = 0, 1, 2, 3, \ldots$.

3.3. **Spatial Discretization for the Legendre Wavelet Collocation Method (LWCM).** In this section, Legendre wavelets are used to the spatial derivatives of $\vartheta(x, t)$, then the collocation method is employed.
The unknown solution can be expanded by Legendre waveletes as:

$$\vartheta(x, t_r) \approx \vartheta^r = \sum_{l=0}^{2^k-1} \sum_{m=0}^{M} c_{lm}^{r} \psi_{lm} = \sum_{j=1}^{\widehat{m}} c_j^r \psi_j(x) = (C^r)^T \Psi(x). \tag{3.11}$$

Here, the coefficient vector $C^r$is calculated at time t_r. Thus, the numerical solution at time $t_{r+1/3}$ is as follows:

$$\vartheta^{r+\frac{1}{3}} = \sum_{j=1}^{\widehat{m}} c_j^{r+1/3} \psi_j(x) = \left(C^{r+\frac{1}{3}}\right)^T \Psi(x). \tag{3.12}$$

By substituting Eq. (3.12) into the first equation of Eq. (3.4), we get

$$\vartheta^{r+\frac{1}{3}} = \vartheta^r + \frac{\Delta t}{3}\left(-\zeta\,(\vartheta)^r \left(\frac{\partial \vartheta}{\partial x}\right)^r - \delta \left(\frac{\partial^3 \vartheta}{\partial x^3}\right)^r\right). \tag{3.13}$$

Applying the operational Matrix of derivative for $\vartheta_x = \frac{\partial \vartheta}{\partial x}$, $\vartheta_{xxx} = \frac{\partial^3 \vartheta}{\partial x^3}$ and Eq. (2.3), we get

$$\frac{\partial \vartheta}{\partial x}(x, t_r) = \sum_{l=0}^{2^{k-1}} \sum_{m=0}^{M} c_{lm}^{r} \frac{\partial \psi_{lm}(x)}{\partial x} = (C^r)^T D\Psi(x) \tag{3.14}$$

and

$$\frac{\partial^3 \vartheta}{\partial x^3}(x, t_r) = \sum_{l=0}^{2^{k-1}} \sum_{m=0}^{M} c_{lm}^{r} \frac{\partial^3 \psi_{lm}(x)}{\partial x^3} = (C^r)^T D^3 \Psi(x). \tag{3.15}$$

By putting Equations (3.11), (3.12), (3.14) and (3.15) into Eq. (3.13) , we get

$$\left(C^{r+\frac{1}{3}}\right)^T \Psi(x) = (C^r)^T \Psi(x) + \frac{\Delta t}{3}\left(-\zeta\left((C^r)^T \Psi(x)\right).\left((C^r)^T D\Psi(x)\right) - \delta\,(C^r)^T D^3\Psi(x)\right). \tag{3.16}$$

When we take the initial condition firstly, we obtain

$$\vartheta_0(x) = \vartheta(x, t_0) = \vartheta^0 = \left(C^0\right)^T \Psi(x).$$

By taking the collocation points as:

$$x_j = \frac{2j-1}{2\,\widehat{m}},\ j = 1, 2, ..., \widehat{m},$$

and by putting the collocation points into Eq. (3.16), we acquire a system of algebraic equations with $2^{k-1}M$ unknown variables, $c_j^{r+\frac{1}{3}}$. Solving this system, the vector coefficient $C^{r+\left(\frac{1}{3}\right)}$is acquired. To find the vectors $C^{r+\left(\frac{1}{2}\right)}$ and C^{r+1}, the resembling procedure is employed for the other equations in Eq. (3.4), respectively. The solution C^{r+1} is the vector coefficient for $v(x, t_r)$ in each step for $r = 0, 1, 2, 3, \ldots$.

4. Illustrative Test Problems

In the present section, the numerical results on two examples are reported to confirm theapplicability, accuracy and efficiency of the presented methods. The precision of the presented methods are checked by the L_∞-error [17–24, 29, 30].

Example 4.1. As a first case, Eq. (1.1) is considered for $\zeta = -6$ and $\delta = 1$. The exact solution of the problem is [9]

$$\vartheta(x,t) = \frac{1}{6}\left(\frac{x-1}{1-t}\right).$$

We have computed the algorithms with parameters $k = 2$ and $M = 3$. Comparisons between the exact and numerical solutions at some points are tabulated in Tab. 1 for different values Δt.

Table 1. Comparison of absolute errors for Example 1

Δt	Time t	LWGM	LWCM
0.005	0.1	4.9800×10^{-8}	4.7700×10^{-8}
	0.3	3.1790×10^{-7}	3.0710×10^{-7}
	0.5	1.4531×10^{-6}	1.4094×10^{-6}
0.01	0.1	1.9840×10^{-7}	1.9790×10^{-7}
	0.3	1.2710×10^{-6}	1.2611×10^{-6}
	0.5	5.8629×10^{-6}	5.8336×10^{-6}

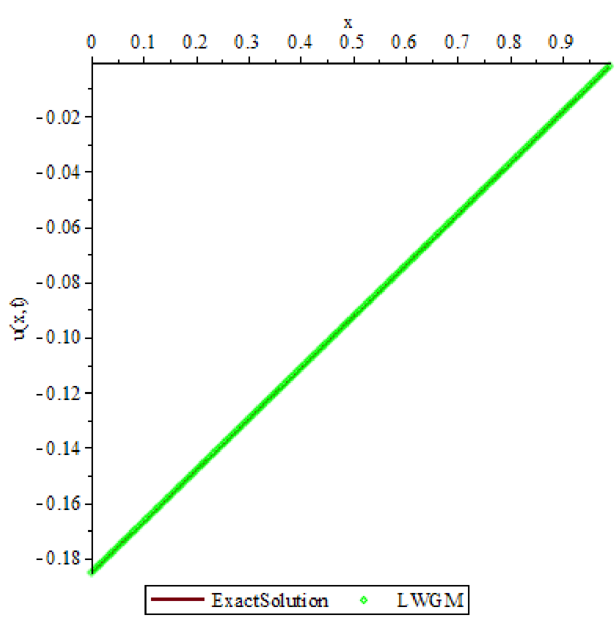

Figure 1. The exact and numerical solutions of Example 1 at $t = 0.1$ for $\Delta t = 0.001, k = 2$and $M = 9$.

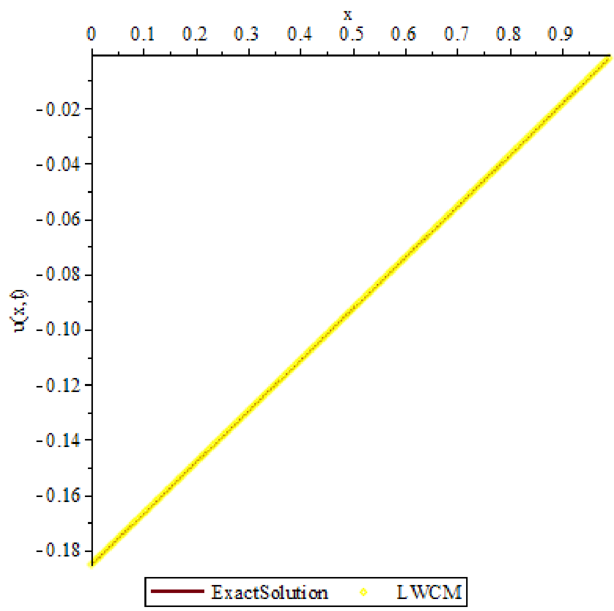

FIGURE 2. The exact and numerical solutions of Example 1 at $t = 0.1$ for $\Delta t = 0.001, k = 2$and $M = 9$.

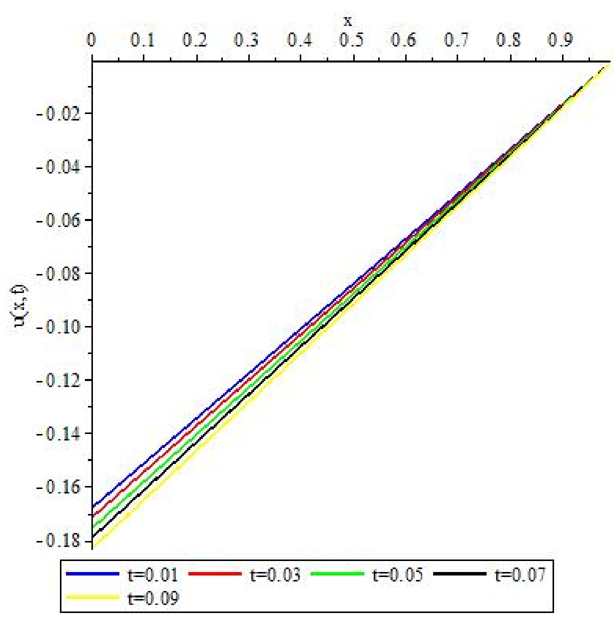

FIGURE 3. Numerical illustrations with LWGM for $\Delta t = 0.001$

Example 4.2. In this example, we take into account Eq. (1.1) with $\zeta = 1$ and $\delta = 0.0013020833$. The exact solution is [39]

$$\vartheta(x, t) = 3c \sec h^2 \left(\sqrt{\frac{c}{4\delta}}(x - ct) \right).$$

We have computed the algorithm with parameters $k = 2$ and $M = 5$. Tab. 2 and Tab.3 show comparison of the exact and numerical solutions at some points for different values Δt.

Table 2. The absolute errors for Example 2

Δt	c	t	LWGM
0.005	$\frac{1}{9}$	0.02	2.3832858×10^{-3}
		0.04	$3.81444275 \times 10^{-3}$
		0.06	$4.98929151 \times 10^{-3}$
		0.08	$6.18301202 \times 10^{-3}$
		0.1	$7.39568769 \times 10^{-3}$
0.005	$\frac{1}{5}$	0.02	$2.92528788 \times 10^{-2}$
		0.04	$3.44036171 \times 10^{-2}$
		0.06	$4.01002000 \times 10^{-2}$
		0.08	$4.63078563 \times 10^{-2}$
		0.1	$5.29861363 \times 10^{-2}$

Table 3. The absolute errors for Example 2

Δt	c	t	LWGM
0.001	$\frac{1}{9}$	0.02	$6.78696573 \times 10^{-3}$
		0.04	$3.81443942 \times 10^{-3}$
		0.06	$4.98928615 \times 10^{-3}$
		0.08	$6.18300455 \times 10^{-3}$
		0.1	$7.39567695 \times 10^{-3}$
0.001	$\frac{1}{5}$	0.02	$2.92528622 \times 10^{-2}$
		0.04	$3.44035845 \times 10^{-2}$
		0.06	$4.01001489 \times 10^{-2}$
		0.08	$4.63077843 \times 10^{-2}$
		0.1	$5.29861820 \times 10^{-2}$

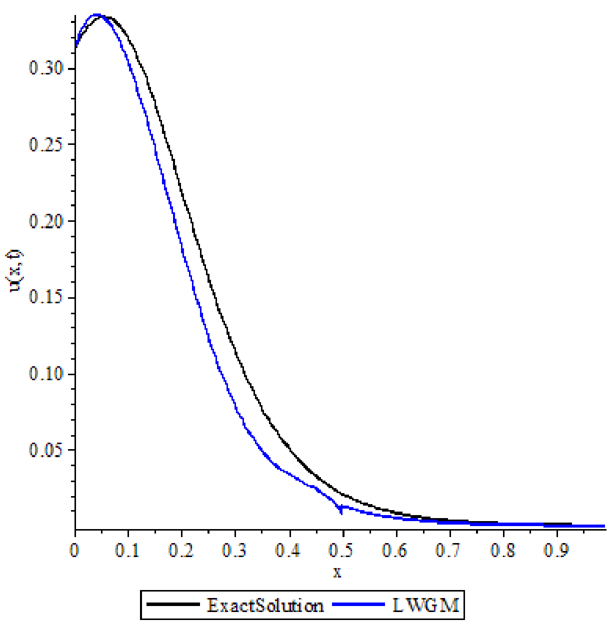

Figure 4. The exact and numerical solutions of Example 2 at $t = 0.5$ for $\Delta t = 0.005$and $c = \frac{1}{9}$

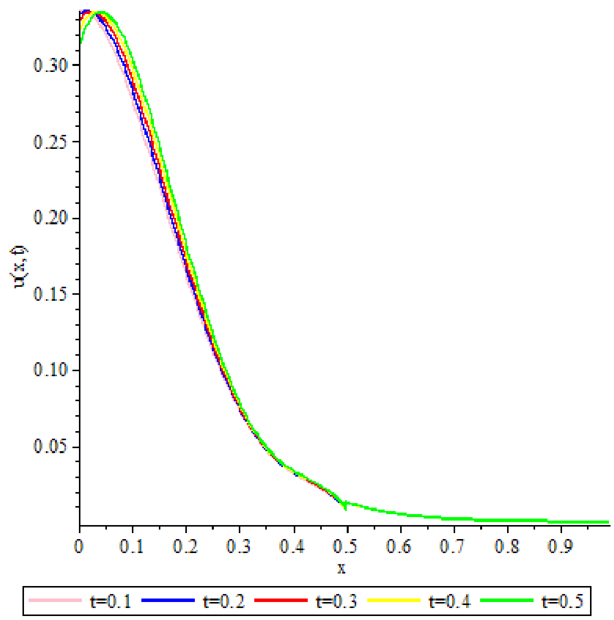

Figure 5. Numerical illustrations with LWGM for $c = \frac{1}{9}$ and $\Delta t = 0.005$

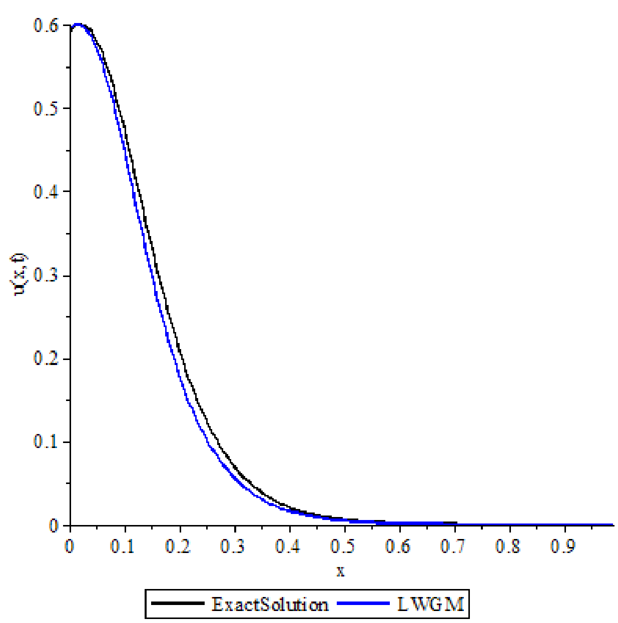

Figure 6. The exact and numerical solutions of Example 2 at $t = 0.1$ for $k = 2, M = 9, \Delta t = 0.001$and $c = \frac{1}{5}$

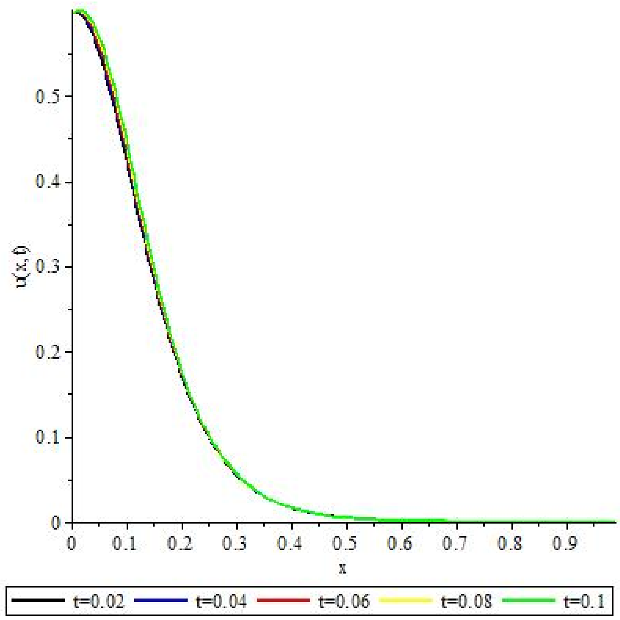

Figure 7. Numerical illustrations with LWGM for $c = \frac{1}{5}$ and $\Delta t = 0.001$

For diverse values of c and Δt, the physical behaviors of exact and numerical solutions obtained using the presented numerical algorithms at different times are depicted in fig. 4, fig.5, fig. 6 and fig. 7, respectively.

5. Conclusion

In this paper, the Kdv equation was numerically solved by using the presented Legendre wavelet methods. The time discretization is employed in the solution procedures before the spatial discretization. Then, the spatial discretization is performed by Legendre wavelets. Therefore, a fully discrete scheme is acquired. Solving by the obtained discrete scheme, an approximate solution is obtained. To exhibit the accuracy and efficienct of the presented methods, two test problems were examined. It is seen that the numerical results are in good agreement with the exact solutions. All of the calculations of the presented methods were executed successfully by MAPLE. Consequently, it is manifestly seen that the LWCM is more fast and effective method than the LWGM and, the presented methods construct the acceptable results for the numerical solution of the KdV equation. As the next step, the presented methods can be employed to acquire the approximation solution of a partial differential equation with different nonlinearity, partial differential equations, and fractional partial differential equations.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have contributed sufficiently to the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

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