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Existence Results for Fractional Integral Equations in Fréchet Spaces

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Abstract

The objective of this paper is to present results on the existence of solutions for a class of fractional integral equations in Fréchet spaces of Banach space-valued functions on the unbounded interval. Our main tool is the technique of measures of noncompactness and fixed points theorems.

1. Introduction

One of the most widely used techniques of proving that certain operator equation has a solution is to reformulate the problem as a fixed point problem and see if the latter can be solved via a fixed point argument. Measures of noncompactness play an important role in fixed point theory and have many applications in various branches of nonlinear analysis, including differential equations, integral and integro-differential equations, optimization, etc. Roughly speaking, a measure of noncompactness is a function defined on the family of all nonempty and bounded subsets of a certain metric space such that it is equal to zero on the whole family of relatively compact sets. This significant concept in mathematical science was defined by many authors in different manners [1,2]. In the last years there appeared many papers devoted to the applications of the measure noncompactness for establish some existence and stability results for various types of nonlinear integral equations [3,4]. In some recent works on this subject, authors utilize a new method of a family of measures of noncompactness and fixed point theorems for condensing operators in Fréchet spaces see [5,6]. The additional advantage of this works is the possibility of extension of the study for several problems to an unbounded domains.

Let us mention that Fréchet spaces have played an important role in functional analysis from its very beginning: Many vector spaces of holomorphic, differentiable or continuous functions which arise in connection with various problems in analysis and its applications are defined by (at most) countably many conditions, whence they carry a natural Fréchet topology (if they are, in addition, complete) [7,8].

This paper is devoted to the study of the following integral equation

$$u(x)=\varphi(x)+\frac{1}{\Gamma(r)}\int_1^x\left(\ln\frac{x}{t}\right)^{r-1}\frac{f(t,u(t))}{t}dg(t);\, x\in J, \tag{1.1}$$

where $J=[1,+\infty)$, $r>0$, $\varphi:J\to E$ is continuous function, $f:J\times E\to E$, $g:J\to\mathbb{R}$ are given functions, $(E,\|\cdot\|)$ is a Banach space and $\Gamma(\cdot)$ is the Euler gamma function. We investigate the existence of solutions of Eq. (1.1) with an application of the fixed point theorems and the technique of measure of noncompactness under some sufficient conditions.

As we know, fractional calculus have been the focus of many researchers in recent years due to their wide application in various fields of engineering, modeling of natural phenomena, optimal control, and biological mathematics [9–12]. Given the wide application of this branch of mathematics in human life, it makes sense for researchers to spend more time identifying equations that can interpret many physical phenomena and come up with newer and more powerful solutions to them. For this reason, in the last decade, many articles have been

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published in the field of ordinary and partial differential equations (see, for example, [13–15]). Let us mention that integral equations of fractional order create an interesting and important branch of the theory of integral equations. The theory of such integral equations is developed intensively in recent years together with the theory of differential equations of fractional order. On the other hand, during the last decades there has been developed the theory of functional integral equations of Stieltjes type. Nevertheless, it turns out that a lot of interesting and important problems which can be formulated inside the theory of Volterra–Stieltjes integral equations are not satisfactory solved by the results obtained up to now [16]. In the theory in question, several types of integral operators, both of linear and nonlinear types are investigated in numerous papers and monographs, we refer [17–19].

2. Preliminaries

This section is devoted to collect some definitions and auxiliary results which will be needed in further considerations.

Definition 2.1 ([20]). *A function $f : J = [a,b] \to \mathbb{R}$ is called of bounded variation if $\bigvee_J f < \infty$, where $\bigvee_J f = \sup \sum_{i=0}^{k} |f(t_{i+1}) - f(t_i)|$, and the supremum is taken over all finite subdivision of J of the forme $a = t_0 < t_1 < t_2 \cdots < t_k = b$.*

Proposition 2.2 ([20]).
- *A function f is of bounded variation on J if and only if f is the difference between two monotone increasing real-valued functions on J.*
- *If f is of bounded variation on J, then f has countable discontinuities in J.*

The Stieltjes integral exists under several conditions, One of the most frequently used requires that f is continuous and g is of bounded variation on J, and the following inequality holds

$$\left| \int_J f(t) dg(t) \right| \leq \int_J |f(t)| \bigvee_J g.$$

Theorem 2.3 ([20]). *Suppose that g is a monotonically increasing function such that g' is Riemann integrable on J and f is continuous on J. Then*

$$\int_a^b f(t) dg(t) = \int_a^b f(t) g'(t) dt.$$

In what follows, we consider the Hadamard-Stieltjes integral of order $q > 0$ for a function u of the form

$$\left({}^{HS}I_1^q u \right)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln \frac{x}{t} \right)^{q-1} \frac{u(t)}{t} dg(t).$$

Lemma 2.4 ([21]). *Assume that the functions $\Phi, \phi_1, \phi_2 : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions such that Φ satisfies the following inequality:*

$$\Phi(t) \leq \phi_1(t) + \int_0^t \phi_2(\tau) \Phi(\tau) d\tau;\ t \geq 0,$$

then

$$\Phi(t) \leq \phi_1(t) + \int_0^t \phi_1(\tau) \phi_2(\tau) \exp\left(\int_\tau^t \phi_2(s) ds \right) d\tau;\ t \geq 0.$$

We present now some basic facts concerning measures of noncompactness. If A is a subset of a Fréchet space X then the symbols $\overline{A}$, Conv$A$ stand for the closure and convex hull of A, respectively. Moreover, for any fixed function $h : \mathbb{R}_+ \to (0,\infty)$ let us denote

$$\mathtt{M}_X = \{x \in X;\ \|x(t)\|_E \leq h(t),\ t \in J\},$$

the family of all nonempty and bounded subsets of X and by $\mathtt{N}_X$ its subfamily consisting of all relatively compact sets.
For the Fréchet space we accept the following definition of the family of measures of noncompactness.

Definition 2.5 ([6]). *A family of mappings $\mu_n : M_X \to \mathbb{R}_+$ is said to be a family of measures of noncompactness in the Fréchet space X if it satisfies the following conditions*

1. *The family $ker\{\mu_n\} = \{A \in M_X;\ \mu_n(A) = 0 \text{ for } n \in \mathbb{N}\}$ is nonempty and $ker\{\mu_n\} \subset N_X$.*
2. *$\mu_n(A) \leq \mu_n(B)$ for $A \subset B$, $n \in \mathbb{N}$.*
3. *$\mu_n(ConvA) = \mu_n(A)$ for $n \in \mathbb{N}$.*
4. *If (A_i) is a sequence of closed sets from $\mathtt{M}_X$ such that $A_{i+1} \subset A_i$ $(i = 1,2,\cdots)$ and if $\lim_{i\to\infty} \mu_n(A_i = 0$ for each $n \in \mathbb{N}$, then the intersection set $A_\infty = \bigcap_{i=1}^{\infty} A_i$ is nonempty.*
5. *$\mu_n(\lambda A) = |\lambda| \mu_n(A)$ for $\lambda \in \mathbb{R}$, $n = 1,2,\cdots$*
6. *$\mu_n(A+B) \leq \mu_n(A) + \mu_n(B)$ for $n = 1,2,\cdots$*
7. *$\mu_n(A \cup B) = \max\{\mu_n(A), \mu_n(B)\}$ for $n = 1,2,\cdots$*

We call the family $\{\mu_n\}_{n\in\mathbb{N}}$ to be *homogeneous*, *subadditive*, *sublinear*, has the *maximum property* if 5., 6., (5.6.), 7. hold respectively.

Definition 2.6. *The family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$ is said to be regular if it is full ($ker\{\mu_n\} = N_F$), sublinear and has maximum property.*

Remark 2.7. *In Fréchet space X we can also consider families of measures $\{\mu_T\}_{T\geq 0}$ indexed by nonnegative numbers instead of families $\{\mu_n\}_{n\in\mathbb{N}}$ indexed by natural numbers.*

Theorem 2.8 ([6,22]). *Let Ω be a nonempty, bounded, closed and convex subset of a Fréchet space X and let $L:\Omega\to\Omega$ be a continuous mapping. If L is a contraction with respect to a family of measures of noncompactness $\{\mu_n\}_{n\in\mathbb{N}}$ i.e for each $n\in\mathbb{N}$ and a nonempty $A\subset\Omega$ there exist a constants $k_n\in[0,1)$ such that*

$$\mu_n(L(A))\leq k_n\mu_n(A),$$

then L has at least one fixed point in the set Ω.

The above Theorem is a generalization of the classical Darbo fixed point Theorem for the Fréchet space.

Theorem 2.9 ([23]). *Let Ω be a nonempty, bounded, closed and convex subset of a Hausdorff locally convex space X such that $0\in\Omega$, and let L be a continuous mapping of Ω into itself. If the implication*

$$(V=\ conv\, L(V) \text{ or } V=L(V)\cup\{0\})\Rightarrow V \text{ is relatively compact},$$

holds for every subset V of Ω, then L has a fixed point.

In the sequel we will work in the space

$$C(J,E)=\{u:J\to E;\ u \text{ is continuous}\},$$

equipped with the family of seminorms

$$\|u\|_n=\sup\{\|u(t)\|;\ t\in[1,n]\},\ n=1,2,\cdots$$

$C(J,E)$ became a Fréchet space.

Proposition 2.10 ([24]).
1. *A nonempty subset $Q\subset C(J,E)$ is said to be bounded if $\sup\{\|x\|_n;\ x\in Q\}<\infty$, $n=1,2,\cdots$*
2. *A sequence (u_n) is convergent to u in $C(J,E)$ if and only if (u_n) is uniformly convergent to u on compact subsets of J.*
3. *A family $Q\subset C(J,E)$ is relatively compact if and only if for each $T>1$, the restriction to $[1,T]$ of all functions from Q form an equicontinuous set and $Q(t)$ is relatively compact in E for each $t\in J$.*

In order to define a measure of noncompactness in the space $C(J,E)$, let us fix a nonempty bounded subset Q of the space $C(J,E)$. For $u\in Q$, $\varepsilon>0$, $t,s\in[1,n]$ such that $|t-s|\leq\varepsilon$, we denote by $\omega_0^n(u,\varepsilon)$ the modulus of continuity of the function u on the interval $[1,n]$ i.e

$$\omega_0^n(u,\varepsilon)=\sup\{\|u(t)-u(s)\|;\ t,s\in[1,n],\ |t-s|\leq\varepsilon\},$$

so

$$\begin{aligned}\omega_0^n(Q,\varepsilon) &= \sup\{\omega_0^n(u,\varepsilon);\ u\in Q\}\\ \omega_0^n(Q) &= \lim_{\varepsilon\to 0}\omega_0^n(Q,\varepsilon)\end{aligned}$$

Finally, consider the family $\{\mu_n\}_{n\geq 1}$ in $C(J,E)$ defined by the formula

$$\mu_n(Q)=\omega_0^n(Q)+\psi_n(Q);\ Q\in \mathtt{M}_{C(J,E)},\ n=1,2,\cdots, \tag{2.1}$$

where $\psi_n(Q)=\sup\limits_{t\in[1,n]}\psi(Q(t))$ and ψ is a regular measure of noncompactness in the Banach space E.

It can be shown that the family of maps $\{\mu_n\}_{n\geq 1}$ is a family of measures of noncompactness in the space $C(J,E)$. The kernel (ker μ_n) consists of nonempty and bounded sets Q such that functions from Q are equicontinuous on compact subsets of J and $Q(t)$ is relatively compact in E for each $t\in J$.

Lemma 2.11 ([24]). *Assume $Q\subset C(J,E)$ is equicontinuous on compact intervals of J and $Q(t)$ is bounded for all $t\in J$. Then*

- *The function $t\mapsto\psi(Q(t))$ is continuous on J.*
- *For each $t\in J$*

$$\psi\left(\int_1^t Q(\tau)d\tau\right)\leq\int_1^t\psi(Q(\tau))d\tau.$$

3. Main results

The equation (1.1) will be considered under the following assumptions :

(H_1) The function f is continuous and there exist two continuous functions $p,q:J\longrightarrow\mathbb{R}_+$ such that

$$\|f(x,u)\|\leq p(x)\|u\|+q(x);\ x\in J;\ u\in E.$$

(H_2) The function g is continuous and of bounded variation on J.

(H_3) For each $A\in\mathtt{M}_E$ and for each $x\in J$, we have

$$\psi(f(x,A))\leq p(x)\psi(A).$$

(H_4) For each $T>1$, there exists a constant $\theta_T>0$ such that

$$\left|\int_1^T\left(\ln\frac{T}{t}\right)^{r-1}dg(t)\right|\leq\theta_T.$$

With

$$k_T=\frac{\theta_T p^*}{\Gamma(r)}<1,$$

where $p^*=\sup\{p(x);\ x\in[1,T]\}$.

Theorem 3.1. *Under the assumptions $(H_1)-(H_4)$ the integral equation (1.1) has at least one solution $u=u(x)$ in the space $C(J,E)$.*

Proof. Consider the operator L on the space $C(J,E)$ defined by

$$(Lu)(x)=\varphi(x)+\frac{1}{\Gamma(r)}\int_1^x\left(\ln\frac{x}{t}\right)^{r-1}\frac{f(t,u(t))}{t}dg(t);\ x\in J,$$

observe that in view of our assumptions, for any function $u\in C(J,E)$ the function Lu is continuous on J. For an arbitrary function $u\in C(J,E)$ and a fixed $x\in J$ we have

$$\begin{aligned}
\|Lu(x)\| &= \left\|\varphi(x)+\frac{1}{\Gamma(r)}\int_1^x\left(\ln\frac{x}{t}\right)^{r-1}\frac{f(t,u(t))}{t}dg(t)\right\|\\
&\leq \|\varphi(x)\|+\frac{1}{\Gamma(r)}\int_1^x\left(\ln\frac{x}{t}\right)^{r-1}\|f(t,u(t))\|dg(t)\\
&\leq \|\varphi(x)\|+\frac{1}{\Gamma(r)}\int_1^x\left(\ln\frac{x}{t}\right)^{r-1}[p(t)\|u(t)\|+q(t)]\,dg(t)\\
&\leq m(x)+\frac{1}{\Gamma(r)}\int_1^x\left(\ln\frac{x}{t}\right)^{r-1}p(t)\|u(t)\|dg(t),
\end{aligned}$$

where

$$m(x)=\|\varphi(x)\|+\frac{1}{\Gamma(r)}\int_1^x\left(\ln\frac{x}{t}\right)^{r-1}q(t)dg(t).$$

Next, consider the following integral inequality

$$\omega(x)\leq m(x)+\frac{1}{\Gamma(r)}\int_1^x\left(\ln\frac{x}{t}\right)^{r-1}p(t)\omega(x)dg(t).$$

In view of Lemma 2.4, we get

$$\omega(x)\leq m(x)+\frac{1}{\Gamma(r)}\int_1^x\left(\ln\frac{x}{t}\right)^{r-1}p(t)m(t)\exp\left(\int_t^x\left(\ln\frac{x}{s}\right)^{r-1}p(s)ds\right)dg(t).$$

The function

$$\Phi(x)=m(x)+\frac{1}{\Gamma(r)}\int_1^x\left(\ln\frac{x}{t}\right)^{r-1}p(t)m(t)\exp\left(\int_t^x\left(\ln\frac{x}{s}\right)^{r-1}p(s)ds\right)dg(t),$$

is continuous and nonnegative. Observe that the following implication is true :

$$\|u(x)\|\leq\Phi(x)\Rightarrow\|Lu(x)\|\leq\Phi(x);\ \text{for } x\in J.$$

We take the set

$$Q=\Big\{u\in C(J,E);\ \|u(x)\|\leq\Phi(x);\ x\in J\Big\}.$$

We see that Q is nonempty, bounded, closed and convex subset of $C(J,E)$. Moreover, the operator L transforms the set Q into itself. Further, let $T>1$, $x_1,x_2\in[1,T]$ with $x_1<x_2$ and $x_2-x_1<\varepsilon$. For a given $u\in Q$, we have

$$\begin{aligned}
\|Lu(x_2)-Lu(x_1)\| &= \left\|\varphi(x_2)+\frac{1}{\Gamma(r)}\int_1^{x_2}\left(\ln\frac{x_2}{t}\right)^{r-1}\frac{f(t,u(t))}{t}dg(t)-\varphi(x_1)-\frac{1}{\Gamma(r)}\int_1^{x_1}\left(\ln\frac{x_1}{t}\right)^{r-1}\frac{f(t,u(t))}{t}dg(t)\right\|\\
&\leq \|\varphi(x_2)-\varphi(x_1)\|+\frac{1}{\Gamma(r)}\left\|\int_1^{x_2}\left(\ln\frac{x_2}{t}\right)^{r-1}f(t,u(t))dg(t)-\int_1^{x_2}\left(\ln\frac{x_1}{t}\right)^{r-1}f(t,u(t))d_tg(x_2,t)\right.\\
&\quad\left.+\int_1^{x_2}\left(\ln\frac{x_1}{t}\right)^{r-1}f(t,u(t))dg(t)-\int_1^{x_1}\left(\ln\frac{x_1}{t}\right)^{r-1}f(t,u(t))dg(t)\right\|\\
&\leq \|\varphi(x_2)-\varphi(x_1)\|+\frac{1}{\Gamma(r)}\int_1^{x_1}[(\ln x_2)^r-(\ln x_1)^r]\,\|f(t,u(t))\|dg(t)+\int_{x_1}^{x_2}\left(\ln\frac{x_1}{t}\right)^{r-1}\|f(t,u(t))\|dg(t)\\
&\leq \|\varphi(x_2)-\varphi(x_1)\|+\frac{p^*\Phi^*+q^*}{\Gamma(r)}\left[\int_1^{x_1}[(\ln x_2)^r-(\ln x_1)^r]\,dg(t)+\int_{x_1}^{x_2}\left(\ln\frac{x_1}{t}\right)^{r-1}dg(t)\right]\\
&= W(T,\varepsilon),
\end{aligned}$$

since φ and the logarithm function are locally uniformly continuous, so, $W(T,\varepsilon)\to 0$ when $\varepsilon\to 0$.

Remark 3.2. *In this case, the set Q is the family consisting of functions equicontinuous on compact intervals of J.*

Next, we will show that $L:Q\to Q$ is continuous. Let us fix $T>1$, $\delta>0$ and take $u_0\in Q$. Then, for $x\in[1,T]$ and any function $u\in Q$ such that $\|u(x)-u_0(x)\|<\delta$, we get

$$\begin{aligned}
\|Lu(x)-Lu_0(x)\| &= \left\|\frac{1}{\Gamma(r)}\int_1^x\left(\ln\frac{x}{t}\right)^{r-1}\frac{f(t,u(t))}{t}dg(t)-\frac{1}{\Gamma(r)}\int_1^x\left(\ln\frac{x}{t}\right)^{r-1}\frac{f(t,u_0(t))}{t}dg(t)\right\|\\
&\leq \frac{1}{\Gamma(r)}\int_1^x\left(\ln\frac{x}{t}\right)^{r-1}\|f(t,u(t))-f(t,u_0(t))\|dg(t).
\end{aligned}$$

Since f is continuous on $[1,T]\times E$, we have $\sup_{x\in[1,T]}\|f(x,u(x))-f(x,u_0(x))\|<\varepsilon(\delta)$ with $\varepsilon(\delta)\to 0$ as $\delta\to 0$. This implies

$$\sup_{x\in[1,T]}\|Lu(x)-Lu_0(x)\|\leq\frac{\theta_T}{\Gamma(r)}\varepsilon(\delta),$$

hence, the operator L is continuous on the set Q.
Further, fix arbitrarily $T>1$ and take a nonempty $\Omega\subset Q$. In view of the assumption (H_3), Remark 3.2 and by Lemma 2.11, we obtain

$$\psi(L\Omega(x)) \quad =\psi\left(\varphi(x)+\frac{1}{\Gamma(r)}\int_1^x\left(\ln\tfrac{x}{t}\right)^{r-1}\frac{f(t,\Omega(t))}{t}dg(t)\right)\leq\frac{1}{\Gamma(r)}\int_1^x\left(\ln\tfrac{x}{t}\right)^{r-1}p(t)\psi(\Omega(t))dg(t)\leq\frac{p^*\theta_T}{\Gamma(r)}\psi(\Omega(t)).$$

Thus

$$\psi_n(L\Omega)\leq k_T\psi_n(\Omega). \tag{3.1}$$

Observe, that linking (3.1) and the definition of the family of measure of noncompactness μ_n given by the formula (2.1), we obtain

$$\mu_n(L\Omega)\leq k_T\mu_n(\Omega). \tag{3.2}$$

Finally, in view of the Theorem 2.8 we deduce that L has at least one fixed point in Q which is a solution of Eq. (1.1).

□

In this section, we will give an other result using Mönch's fixed point Theorem.

The Eq. (1.1) will be considered under the following assumptions :

(C_1) The function f is continuous and there exists a continuous function $p:J\longrightarrow\mathbb{R}_+$ such that

$$\|f(x,u)\|\leq p(x);\ x\in J;\ u\in E.$$

(C_2) The function g is continuous and of bounded variation on J.
(C_3) There exists a continuous function $b:J\longrightarrow\mathbb{R}_+$ such that for each $A\in\mathtt{M}_E$ and for each $x\in J$, we have

$$\psi(f(x,A))\leq b(x)\psi(A). \tag{3.3}$$

(C_4) For each $T>1$, there exists a constant $\theta_T>0$ such that

$$\left|\int_1^T\left(\ln\frac{T}{t}\right)^{r-1}dg(t)\right|\leq\theta_T.$$

With

$$k_T=\frac{\theta_T b^*}{\Gamma(r)}<1,$$

where $b^*=\sup\{b(x);\ x\in[1,T]\}$.

Theorem 3.3. *Suppose the hypotheses $(C_1)-(C_4)$ are satisfied. Then Eq. (1.1) has at least one solution $u=u(x)$ in the space $C(J,E)$.*

Proof. Consider the operator L on the space $C(J,E)$ defined by

$$(Lu)(x)=\varphi(x)+\frac{1}{\Gamma(r)}\int_1^x\left(\ln\frac{x}{t}\right)^{r-1}\frac{f(t,u(t))}{t}dg(t);\ x\in J,$$

observe that in view of our assumptions, for any function $u\in C(J,E)$ the function Lu is continuous on J. For an arbitrary function $u\in C(J,E)$ and a fixed $x\in J$ we have

$$\begin{aligned}\|Lu(x)\| &= \left\|\varphi(x)+\frac{1}{\Gamma(r)}\int_1^x\left(\ln\frac{x}{t}\right)^{r-1}\frac{f(t,u(t))}{t}dg(t)\right\|\\ &\leq \|\varphi(x)\|+\frac{1}{\Gamma(r)}\int_1^x\left(\ln\frac{x}{t}\right)^{r-1}\|f(t,u(t))\|dg(t)\\ &\leq \|\varphi(x)\|+\frac{1}{\Gamma(r)}\int_1^x\left(\ln\frac{x}{t}\right)^{r-1}p(x)dg(t),\end{aligned}$$

hence, for $x\in[1,n]$ we infer that

$$\|Lu\|_n=\|\varphi\|_n+\frac{\theta_n p^*}{\Gamma(r)}.$$

Further, let $T>1$, $x_1,x_2\in[1,T]$ with $x_1<x_2$ and $x_2-x_1<\varepsilon$. For a given $u\in C(J,E)$, we have

$$\begin{aligned}
\|Lu(x_2)-Lu(x_1)\| &= \left\|\varphi(x_2)+\frac{1}{\Gamma(r)}\int_1^{x_2}\left(\ln\frac{x_2}{t}\right)^{r-1}\frac{f(t,u(t))}{t}dg(t)-\varphi(x_1)-\frac{1}{\Gamma(r)}\int_1^{x_1}\left(\ln\frac{x_1}{t}\right)^{r-1}\frac{f(t,u(t))}{t}dg(t)\right\| \\
&\le \|\varphi(x_2)-\varphi(x_1)\|+\frac{1}{\Gamma(r)}\left\|\int_1^{x_2}\left(\ln\frac{x_2}{t}\right)^{r-1}f(t,u(t))dg(t)-\int_1^{x_2}\left(\ln\frac{x_1}{t}\right)^{r-1}f(t,u(t))dg(t)\right. \\
&\quad \left.+\int_1^{x_2}\left(\ln\frac{x_1}{t}\right)^{r-1}f(t,u(t))dg(t)-\int_1^{x_1}\left(\ln\frac{x_1}{t}\right)^{r-1}f(t,u(t))dg(t)\right\| \\
&\le \|\varphi(x_2)-\varphi(x_1)\|+\frac{1}{\Gamma(r)}\int_1^{x_1}[(\ln x_2)^r-(\ln x_1)^r]\,p(t)dg(t)+\int_{x_1}^{x_2}\left(\ln\frac{x_1}{t}\right)^{r-1}p(t)dg(t) \\
&\le \|\varphi(x_2)-\varphi(x_1)\|+\frac{p^*}{\Gamma(r)}\left[\int_1^{x_1}[(\ln x_2)^r-(\ln x_1)^r]\,dg(t)+\int_{x_1}^{x_2}\left(\ln\frac{x_1}{t}\right)^{r-1}dg(t)\right] \\
&\le W(T,\varepsilon),
\end{aligned}$$

We take the set

$$D=\left\{u\in C(J,E);\ \|u\|_n\le l_n=\|\varphi\|_n+\frac{\theta_n p^*}{\Gamma(r)};\ \text{and } \omega_0^n(u,\varepsilon)\le W(T,\varepsilon);\ n\le T\right\}.$$

Obviously D is nonempty, bounded, closed and convex subset of $C(J,E)$ and the operator L transforms the set D into itself. Moreover, the set D is the family consisting of functions equicontinuous on compact intervals of J.
Now, we show that L is continuous on the set D. Let $(u_n)_n\subset D$ be a sequence converging to u in D i.e

$$\lim_{n\to\infty}\sup_{1\le t\le T}\|u_n(t)-u(t)\|=0;\ T>1.$$

Then we get

$$\begin{aligned}
\sup_{1\le t\le T}\|(Lu_n)(x)-(Lu)(x)\| &\le \frac{1}{\Gamma(r)}\sup_{1\le x\le T}\int_1^x\left(\ln\frac{x}{t}\right)^{r-1}\|f(t,u_n(t))-f(t,u(t))\|\times dg(t) \\
&\le \frac{\theta_T}{\Gamma(r)}\sup_{1\le t\le T}\|f(t,u_n(t))-f(t,u(t))\|,
\end{aligned}$$

so

$$\lim_{n\to\infty}\sup_{1\le t\le T}\|(Lu_n)(x)-(Lu)(x)\|\le\frac{\theta_T}{\Gamma(r)}\lim_{n\to\infty}\sup_{1\le t\le T}\|f(t,u_n(t))-f(t,u(t))\|.$$

Since f is continuous on $[1,T]\times E$, we obtain

$$\lim_{n\to\infty}\sup_{1\le t\le T}\|(Lu_n)(x)-(Lu)(x)\|=0,$$

hence the operator L is continuous on the set D.

Further, let $V\subset D$ such that $V=L(V)\cup\{0\}$, fix $x\in[1,T]$ and using our assumptions we arrive at the following estimates

$$\begin{aligned}
\psi(LV(x)) &= \psi\left(\varphi(x)+\frac{1}{\Gamma(r)}\int_1^x\left(\ln\frac{x}{t}\right)^{r-1}\frac{f(t,V(t))}{t}dg(t)\right) \\
&\le \frac{1}{\Gamma(r)}\int_1^x\left(\ln\frac{x}{t}\right)^{r-1}\psi(f(t,V(t)))dg(t) \\
&\le \frac{1}{\Gamma(r)}\int_1^x\left(\ln\frac{x}{t}\right)^{r-1}b(t)\psi(V(t))dg(t) \\
&\le \frac{b^*\theta_T}{\Gamma(r)}\sup_{x\in[1,T]}\psi(V(x)),
\end{aligned}$$

thus

$$\sup_{x\in[1,T]}\psi(V(x))\le k_T\sup_{x\in[1,T]}\psi(V(x)).$$

Since for each $T>1$ we have $k_T<1$, we deduce that

$$\sup_{x\in[1,T]}\psi(V(x))=0.$$

Hence, $V(x)$ is relatively compact in E for each $x\in[1,T]$, and from the choice of the set D, we conclude that V is relatively compact in $C(J,E)$ (in view of proposition 2.10). Combining with Theorem 2.9 we complete the proof.

□

4. Example

Let $E = l^\infty$ be the space of all bounded sequences $(w_p)_{p\in\mathbb{N}}$ of real numbers endowed with the norm

$$\|w\|_\infty = \max_{p\in\mathbb{N}} |w_p|;\ w \in E.$$

We consider an infinite system of fractional integral equations

$$u_p(z) = \frac{z+p}{z^2+2p} + \frac{1}{\Gamma(r)} \int_1^z \left(\ln\frac{z}{t}\right)^{r-1} \frac{\sqrt{e^{-2t}u_p^2(t) + \frac{1}{pt}}}{t} d\left(\frac{1}{t} - \frac{1}{t^2}\right);\ p\in\mathbb{N};\ r > 1. \tag{4.1}$$

It is clear that equation (4.1) can be written as equation (1.1), where

$$\begin{aligned} u : J = [1,\infty) &\to l^\infty \\ z &\mapsto (u_p(z))_{p\in\mathbb{N}}, \end{aligned}$$

Set

$$\varphi(z) = (\varphi_p(z))_{p\in\mathbb{N}} = \frac{z+p}{z^2+2p};\quad g(t) = \frac{1}{t} - \frac{1}{t^2},$$

$$f(z,u(z)) = (f_p(z,u_p(z)))_{p\in\mathbb{N}} = \sqrt{e^{-2z}u_p^2(z) + \frac{1}{pz}}.$$

Remark 4.1. *We can see that for each $u(z) \in l^\infty$ and $z \in J$ we have $(f_p(z,u_p(z)))_{p\in\mathbb{N}} \in l^\infty$, so, the function $f : J \times l^\infty \to l^\infty$ is well defined.*

Let us show that conditions $(H_1)-(H_4)$ hold. The function $t \mapsto \frac{1}{t} - \frac{1}{t^2}$ is continuous on J, increasing on $[1,2]$ and decreasing on $[2,\infty)$. Moreover, we have

$$\lim_{t\to+\infty} \left(\frac{1}{t} - \frac{1}{t^2}\right) = 0.$$

So it is of bounded variation on J. It follows that

$$\begin{aligned} \left|f_p(z,u_p(z))\right| &= \sqrt{e^{-2z}u_p^2(z) + pz} \\ &\le \sqrt{e^{-2z}u_p^2(z)} + \sqrt{\frac{1}{pz}} \\ &\le e^{-z}|v_p(z)| + \sqrt{\frac{1}{pz}}, \end{aligned}$$

thus

$$\sup_{p\in\mathbb{N}} \left|f_p(z,u_p(z))\right| \le e^{-z} \sup_{p\in\mathbb{N}} |v_p(z)| + \sqrt{\frac{1}{pz}}.$$

Then

$$\|f(z,u(z))\|_\infty \le e^{-z}\|u(z)\|_\infty + \sqrt{\frac{1}{pz}}. \tag{4.2}$$

So $p(z) = e^{-z}$; $\quad p^* = \frac{1}{e}$; $\quad q(z) = \sqrt{\frac{1}{pz}}$ and for a fixed $T > 1$ we have

$$\begin{aligned} \left|\int_1^T \left(\ln\frac{T}{t}\right)^{r-1} d\left(\frac{1}{t} - \frac{1}{t^2}\right)\right| &\le (\ln T)^r \left|\int_1^T d\left(\frac{1}{t} - \frac{1}{t^2}\right)\right| \\ &\le (\ln T)^r \left(\frac{1}{T} - \frac{1}{T^2}\right) \\ &= \theta_T. \end{aligned}$$

Observe that

$$k = \frac{\theta_T p^*}{\Gamma(r)} = \frac{(\ln T)^r (T-1)}{eT^2\Gamma(r)} < 1;\ \text{for each } T > 1.$$

In view of (4.2), we deduce that

$$\psi(f(t,A)) \le e^{-t}\psi(A);\ \text{for each } A \in \mathtt{M}_E.$$

Consequently from Theorem 3.1 the Eq. (4.1) has at least solution in $C(J,E)$.

5. Conclusion

In this work, we have presented an existence result for a type of integral equation by application of MNCs and the fixed point theorems. The interest of this work is the possibility of dealing with several nonlinear problems on unbounded domains, on the other hand, we have given an illustrative example which indicates the applicability of this study to deal with an infinite system of integral equations. Some of the results in this direction are our future plan especially the choice of MNCs which allows us to characterize the qualitative aspect of the solutions.

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