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Existence results for nonlocal Cauchy problem of nonlinear $\psi-$Caputo type fractional differential equations via topological degree methods

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Abstract

This manuscript is devoted to the investigation of the existence results of fractional Cauchy problem for some nonlinear $\psi-$Caputo fractional differential equations with non local conditions. By applying fixed point theorems, some results of topological degree theory for condensing maps and some fractional analysis techniques, we establish some new existence theorems. As application, a nontrivial example is given to illustrate our theoretical results.

Keywords: $\psi-$fractional integral $\psi-$Caputo fractional derivative fixed point Kuratowski measure topological degree theory.
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1. Introduction

The subject of fractional calculus has recently evolved as an interesting and popular tool in the modelling of many phenomena in various fields of engineering, physics and economics. Indeed, fractional-order models have been found to be more adequate than integer order models for some real world problems as fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. This theory plays a very considerable role both in mathematics and in applications as material theory, transport processes, earthquakes, electrochemical processes, wave propagation, signal theory,

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biology, electromagnetic theory, fluid flow phenomena, thermodynamics, mechanics, geology, astrophysics, economics and control theory(see[1, 9, 14, 19, 25, 27]); this is the main advantage of fractional differential equations in comparison with classical integer-order models. Fractional differential equations have been of great interest recently such as boundary value problems for nonlinear fractional differential equations which can be employed in modeling and describing non-homogeneous physical phenomena that take place in their form. Almeida et al. [5] investigated the existence and uniqueness results of nonlinear fractional differential equations involving a Caputo-type fractional derivative with respect to another function by using fixed point theorems and Picard iteration metho. Zhang in [31] proved the existence and uniqueness results for nonlinear fractional boundary value problem involving Caputo type fractional derivatives by using some fixed point theorems. Many researchers have obtained some interesting results on the existence and uniqueness of solutions of boundary value problems for fractional differential equations involving different fractional derivatives such as Riemann-Liouville [22], Caputo [3], Hilfer [23], Erdelyi-Kober [24] and Hadamard [2]. For more details, the reader may also consult [6, 7, 10, 13, 17, 18, 26, 30, 32] and the references therein.
Motivated by the above works, we investigate and we generalize the results obtained in [29] involving ψ-Caputo type fractional derivatives of order $1 < \alpha < 2$. To be more precise, we establish the existence of solutions for the following nonlocal fractional Cauchy problem:

$$\begin{cases} {}^{C}D_{0^+}^{\alpha;\psi}x(t) = f(t, x(t)), \qquad t \in \Delta = [0, T], \\ \\ x(0) + \varphi(x) = x_0, \quad x'(0) = 0. \end{cases} \tag{1}$$

Where ${}^{C}D_{0^+}^{\alpha;\psi}$ is the $\psi-$Caputo fractional derivative, $T > 0$, $f \in C(\Delta \times \mathbb{R}, \mathbb{R})$ and $x_0 \in \mathbb{R}$.
The nonlocal term φ is the defined by

$$\begin{cases} \varphi : C(\Delta, \mathbb{R}) \longrightarrow \mathbb{R}, \\ \\ \qquad\qquad x \longmapsto \varphi(x). \end{cases}$$

Our paper is organized as follows: In Section 2, we give some basic definitions and properties of $\psi-$fractional integral and $\psi-$Caputo fractional derivative which will be used in the rest of this paper. In Section 3, we establish the existence of solutions for $\psi-$Caputo type fractional problem (1) by using some results of topological degree theory for condensing maps. As application, an illustrative example is presented in Section 4 followed by conclusion in Section 5.

2. Preliminaries

In this section, we give some notations, definitions and results on ψ-fractional derivatives and ψ-fractional integrals, for more details we refer the reader to [4, 8, 21].

Notations

• We denote by X a Banach space and by $\mathfrak{B}_X$ the family of all non-empty and bounded subsets of X.
• We denote by $C(\Delta, \mathbb{R})$ the space of continuous real-valued functions defined on Δ provided with the topology of the supremum norm

$$\| x \| = \sup_{t \in \Delta} | x(t) | .$$

• We denote by B_η the closed ball centered at 0 with radius $\eta > 0$.
• We denote by $L^1(\Delta, \mathbb{R})$ the space of Lebesgue integrable real-valued functions on Δ equipped with the norm

$$\| x \|_{L^1} = \int_\Delta | x(t) | \, dt.$$

Definition 2.1. *[5] Let $q > 0$, $g \in L^1([\Delta, \mathbb{R})$ and $\psi \in C^n(\Delta, \mathbb{R})$ such that $\psi'(t) > 0$ for all $t \in \Delta$. The ψ-Riemann-Liouville fractional integral at order q of the function g is given by*

$$I_{0^+}^{q,\psi} g(t) = \frac{1}{\Gamma(q)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{q-1} g(s) ds.$$

Definition 2.2. *[5] Let $q > 0$, $g \in C^{n-1}(\Delta, \mathbb{R})$ and $\psi \in C^n(\Delta, \mathbb{R})$ such that $\psi'(t) > 0$ for all $t \in \Delta$. The $\psi-$Caputo fractional derivative at order q of the function g is given by*

$${}^C D_{0^+}^{q,\psi} g(t) = \frac{1}{\Gamma(n-q)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{n-q-1} g_{\psi}^{[n]}(s) ds.$$

Where

$$g_{\psi}^{[n]}(s) = \left(\frac{1}{\psi'(s)} \frac{d}{ds} \right)^n g(s) \quad and \quad n = [q] + 1.$$

And $[q]$ denotes the integer part of the real number q.

Remark 2.3. *In particular, if $q \in]0, 1[$, then we have*

$${}^C D_{0^+}^{q,\psi} g(t) = \frac{1}{\Gamma(q)} \int_0^t (\psi(t) - \psi(s))^{q-1} g'(s) ds.$$

And

$${}^C D_{0^+}^{q,\psi} g(t) = I_{0^+}^{1-q,\psi} \left(\frac{g'(t)}{\psi'(t)} \right).$$

Proposition 2.4. *[5] Let $q > 0$, if $g \in C^{n-1}(\Delta, \mathbb{R})$, then we have*

1) ${}^C D_{0^+}^{q,\psi} I_{0^+}^{q,\psi} g(t) = g(t)$.
2) $I_{0^+}^{q,\psi}\, {}^C D_{0^+}^{q,\psi} g(t) = g(t) - \sum_{k=0}^{n-1} \frac{g_{\psi}^{[k]}(0)}{k!} (\psi(t) - \psi(0))^k$.
3) $I_{a^+}^{q,\psi}$ *is linear and bounded from $C(\Delta, \mathbb{R})$ to $C(\Delta, \mathbb{R})$.*

Proposition 2.5. *[5] Let $t > 0$ and $\alpha, \beta \geq 0$, then we have*

1) $I_{0^+}^{q,\psi} (\psi(t) - \psi(0))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\psi(t) - \psi(0))^{\alpha+\beta-1}$.
2) $D_{0^+}^{q,\psi} (\psi(t) - \psi(0))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\psi(t) - \psi(0))^{\alpha-\beta-1}$.
3) $D_{0^+}^{q,\psi} (\psi(t) - \psi(0))^n = 0, \quad$ *for all $n \in \mathbb{N}$.*

Definition 2.6. *[15] The Kuratowski measure of non-compactness is the mapping $\mu : \mathfrak{B}_X \to \mathbb{R}_+$ defined by*

$\mu(B) = \inf\{ r > 0$*: B admits a finite cover by sets of diameter $\leq r$}.*

Proposition 2.7. *[15] The Kuratowski measure of noncompactness μ satisfies the following assertions.*

1. $\mu(B) = 0$ *if and only if B is retativety compact.*
2. $\mu(\lambda B) = |\lambda| \mu(B), \quad \lambda \in \mathbb{R}$.
3. $\mu(B_1 + B_2) \leq \mu(B_1) + \mu(B_2)$.
4. *If $B_1 \subset B_2$ then $\mu(B_1) \leq \mu(B_2)$.*
5. $\mu(B_1 \cup B_2) = \max\{\mu(B_1), \ \mu(B_2)\}$.
6. $\mu(B) = \mu(\overline{B}) = \mu(convB)$ *where $\overline{B}$ and $convB$ denote the closure and the convex hull of B respectively.*

Definition 2.8. *[15] Let and $\Phi : \Omega \subset X \to X$ be a continuous bounded map. We say that Φ is μ-Lipschitz if there exists $k \geq 0$ such that*

$$\mu(\Phi(B)) \leq k\mu(B), \quad for \quad every \quad B \subset \Omega.$$

Moreover, if $k < 1$ then we say that Φ is a strict μ-contraction.

Definition 2.9. *[15] We say that the function Φ is μ-condensing if*

$$\mu(\Phi(B)) < \mu(B),$$

for every bounded subset B of Ω with $\mu(B) > 0$.

In other words,

$$\mu(\Phi(B)) \geq \mu(B) \Rightarrow \mu(B) = 0.$$

Definition 2.10. *[15] We say that the function $\Phi : \Omega \to X$ is Lipschitz if there exists $k > 0$ such that*

$$\| \Phi(x) - \Phi(y) \| \leq k \| x - y \|, \quad for \quad all \quad x, y \in \Omega.$$

Moreover, if $k < 1$ then we say that Φ is a strict contraction.

Lemma 2.11. *[15] If $T, S : \Omega \to X$ are μ-Lipschitz mappings with constants k_1 respectively k_2, then the mapping $T + S : \Omega \to X$ is μ- Lipschitz with constants $k_1 + k_2$.*

Lemma 2.12. *[15] If $\Phi : \Omega \to X$ is compact, then Φ is $\mu-$ Lipschitz with constant $k = 0$.*

Lemma 2.13. *[15] If $\Phi : \Omega \to X$ is Lipschitz with constant k, then Φ is $\mu-$Lipschitz with the same constant k.*

Theorem 2.14. *(See Isaia [20]). Let $\Lambda : X \to X$ be μ-condensing and*

$$\mathcal{S}_\kappa = \{x \in X : x = \kappa\Lambda x \quad for \quad some \quad 0 \leq \kappa \leq 1\}.$$

If $\mathcal{S}_\kappa$ is a bounded set in X, then there exists $r > 0$ such that $\mathcal{S}_\kappa \subset B_r$ and we have

$$deg(I - \delta\Lambda, B_r, 0) = 1, \quad \forall\delta \in [0, 1].$$

Consequentty, the operator Λ has at least one fixed point and the set of the fixed points of Λ lies in B_r .

3. Main results

In this section, before we give the main result of our paper, first of all we should define what we mean by a solution for the problem (1) and prove the fundamental Lemma 3.2. For this purpose, we assume the following assumptions throughout the rest of our paper.

(A_1) There exists a constant $L_\varphi > 0$ such that

$$|\varphi(x) - \varphi(y)| \leq L_\varphi \mid x - y \mid, \quad for \quad each \quad x, y \in C(\Delta, \ \mathbb{R}).$$

(A_2) There exist two constants $K_\varphi, M_\varphi > 0$ and $q \in (0, 1)$ such that

$$|\varphi(x)| \leq K_\varphi \mid x \mid^q + M_\varphi \quad for \quad each \quad x \in C(\Delta, \ \mathbb{R}).$$

(A_3) There exist two constants $K_f, M_f > 0$ and $p \in (0, 1)$ such that

$$|f(t, x)| \leq K_f \mid x \mid^p + M_f \quad for \quad each \quad x \in C(\Delta, \ \mathbb{R}).$$

Definition 3.1. *A function $x \in C(\Delta, \mathbb{R})$ such that its $\alpha-$derivative existing on Δ is said te be a solution of the problem* (1) *if x satisfies the equation ${}^{C}D^{\alpha,\psi}_{0^+}x(t) = f(t, x(t))$ on Δ and the condition $x(0) + \varphi(x) = x_0,\ x'(0) = 0$.*

Lemma 3.2. *A function $x(t) \in C(\Delta, \mathbb{R})$ is a solution of the fractional differential equation (1) if and only if x satisfies the following fractional integral equation*

$$x(t) = x_0 - \varphi(x) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, x(s))ds. \tag{2}$$

Proof. Let x be a solution of the problem (1), then we apply the $\psi-$fractional integral $I^{\alpha,\psi}_{0^+}$ on both sides of (1) we get

$$I^{\alpha,\psi}_{0^+}\ {}^{C}D^{\alpha,\psi}_{0^+}x(t) = I^{\alpha,\psi}_{0^+}f(t, x(t)),$$

and by using Proposition 2.4 we obtain

$$x(t) = c_0 + (\psi(t) - \psi(0))c_1 + I^{\alpha,\psi}_{0^+}f(t, x(t)),$$

where $c_0, c_1 \in \mathbb{R}$.
It follows that

$$x'(t) = c_1\psi'(t) + \frac{1}{\Gamma(\alpha)} \int_0^t \left(\psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, x(s))\right)' ds.$$

Since $x(0) + \varphi(x) = x_0$ and $x'(0) = 0$, then $c_0 = x_0 - \varphi(x)$ and $c_1 = 0$.
Hence the integral equation (2) holds.
Conversely, by direct computation, it is clear that if x satisfies the integral equation (2), then the equation (1) holds which completes the proof. □

To show that the fractional integral equation (2) has at least one solution $x \in C(\Delta, \mathbb{R})$, we consider two operators $\mathcal{L}, \mathcal{F} : C(\Delta, \mathbb{R}) \to C(\Delta,\ \mathbb{R})$ defined as follow:

$$\mathcal{F}x(t) = x_0 - \varphi(x), \quad t \in \Delta. \tag{3}$$

And

$$\mathcal{L}x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, x(s))ds, \quad t \in \Delta. \tag{4}$$

Then, the fractional integral equation (2) can be written as follow:

$$\mathcal{T}x(t) = \mathcal{F}x(t) + \mathcal{L}x(t), \quad t \in \Delta. \tag{5}$$

Theorem 3.3. *Assume that the hypotheses $(A_1) - (A_3)$ hold, then the nonlocal fractional Cauchy problem (1) has at least one solution $x \in C(\Delta, \mathbb{R})$. In addition, the set of the solutions of (1) is bounded in $C(\Delta, \mathbb{R})$.*

In order to prove the Theorem 3.3, we will need to show some lemmas and preliminary results.

Lemma 3.4. *The operator $\mathcal{F}$ is μ- Lipschitz with the constant L_φ. Moreover, $\mathcal{F}$ satisfies the following inequality:*

$$\|\mathcal{F}x\|_C \leq |x_0| + K_\varphi \|x\|^q + M_\varphi, \quad for \quad every \quad x \in C(J,\ \mathbb{R}). \tag{6}$$

Proof. To prove that the operator $\mathcal{F}$ is Lipschitz with constant $L\varphi$.
Let $x, y \in C(\Delta, \mathbb{R})$, then we have

$$|\mathcal{F}x(t) - \mathcal{F}y(t)| \leq |\varphi(x) - \varphi(y)|,$$

by using the hypothesis (A_1) we get

$$|\mathcal{F}x(t) - \mathcal{F}y(t)| \le L_\varphi \|x - y\|,$$

taking supremum over t, we obtain

$$\|\mathcal{F}x - \mathcal{F}y\| \le L_\varphi \|x - y\|.$$

It follows that $\mathcal{F}$ is is Lipschitz with constant L_φ. By using the Lemma 2.13, we deduce that the operator $\mathcal{F}$ is μ-Lipschitz with the same constant $_\varphi$.
To show the inequality (6), let $x \in C(\Delta, \mathbb{R})$, then we have

$$|\mathcal{F}x(t)| = |x_0 - \varphi(x)| \le |x_0| + |\varphi(x)|,$$

by using the assumption (A_2) we get

$$\|\mathcal{F}x\| \le |x_0| + K_\varphi \|x\|^q + M_\varphi.$$

□

Lemma 3.5. *The operator $\mathcal{L}$ is continuous and satisfies the following inequality:*

$$\|\mathcal{L}x\| \le \frac{(K_\varphi \|x\|^p + M_\varphi)(\psi(T) - \psi(0))^\alpha}{\Gamma(\alpha+1)}, \quad for \quad every \quad x \in C(J,\ \mathbb{R}). \tag{7}$$

Proof. To prove that the operator $\mathcal{L}$ is continuous, let $x_n \in C(\Delta, \mathbb{R})$ converging to x in $C(\Delta, \mathbb{R})$, it follows that there exists $\delta > 0$ such that $\|x_n\| \le \delta$ and $\|x\| \le \delta$. Now let $t \in \Delta$, then we have

$$|\mathcal{L}x_n(t) - \mathcal{L}x(t)| \le \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |f(s, x_n(s)) - f(s, x(s))| \, ds,$$

by using the continuity of the function f, it is easy to see that

$$\lim_{n \to \infty} f(s, x_n(s)) = f(s, x(s)).$$

On the one other hand, we use the assumpsion (A_3) we get the following inequality

$$\frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \|f(s, x_n(s)) - f(s, x(s))\| \le (K_\varphi \delta^p + M_\varphi) \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)},$$

and since the function $s \mapsto \dfrac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)}$ is integrable over $[0, t]$, then by using the Lebesgue dominated convergence theorem we get

$$\lim_{n \mapsto +\infty} \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \|f(s, x_n(s)) - f(s, x(s))\| \, ds = 0,$$

it follows that

$$\lim_{n \mapsto +\infty} \| \mathcal{L}x_n - \mathcal{L}x \| = 0.$$

Wich shows that $\mathcal{L}$ is a continuous operator on $C(\Delta, \mathbb{R})$.
Let us show the inequality (7), for this purpose let $x(t) \in C(\Delta, \mathbb{R})$, then we have

$$|\mathcal{L}x(t)| \le \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |f(s, x(s))| \, ds,$$

by using (A_3) we obtain

$$|\mathcal{L}x(t)| \leq \frac{(K_\varphi \|x\|^p + M_\varphi)}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} ds,$$

Finally we get

$$\| \mathcal{L}x \| \leq \frac{(K_\varphi \|x\|^p + M_\varphi)(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)}.$$

□

Lemma 3.6. *The operator $\mathcal{L} : C(\Delta, \mathbb{R}) \to C(\Delta, \mathbb{R})$ is compact.*

Proof. In order to show the compactness of $\mathcal{L}$ we need to show that $\mathcal{L}B_\eta$ is relatively compact in $C(\Delta, \mathbb{R})$ and we usse the Arzela-Ascoli Theorem [16].
For this purpose let $x \in B_\eta$, then by using the inequality (7) we have

$$\| \mathcal{L}x \| \leq \frac{(K_\varphi \eta^p + M_\varphi)(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)} := \xi.$$

It follows that $\mathcal{L}B_\eta \subset B_\xi$. This shows that $\mathcal{L}B_\eta$ is bounded.
Now, let us also show that $\mathcal{L}B_\eta$ is equicontinuous on Δ.

Let $x \in \mathcal{L}B_\eta$ and $t_1, t_2 \in \Delta$ such that $t_1 < t_2$, then we have

$$|\mathcal{L}x(t_2) - \mathcal{L}x(t_1)| \leq \frac{K_\varphi \mid x \mid^p + M_\varphi}{\Gamma(\alpha)} \int_{t_1}^{t_2} \psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1} ds,$$

$$|\mathcal{L}x(t_2) - \mathcal{L}x(t_1)| \leq \frac{K_\varphi \eta^p + M_\varphi}{\Gamma(\alpha)} \int_{t_1}^{t_2} \psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1} ds,$$

$$|\mathcal{L}x(t_2) - \mathcal{L}x(t_1)| \leq \frac{K_\varphi \eta^p + M_\varphi}{\Gamma(\alpha+1)} (\psi(t_2)-\psi(t_1))^\alpha,$$

Since ψ is a continuous function, then we obatin

$$\lim_{t_1 \to t_2} |\mathcal{L}x(t_1) - \mathcal{L}x(t_2)| = 0.$$

which shows that $\mathcal{L}B_\eta$ is equicontinuous.

Now the set $\mathcal{L}B_\eta$ is uniformly bounded and equicontinuous and by using Arzelà–Ascoli Theorem [16] we deduce that $\mathcal{L}B_\eta$ is relatively compact, wich implies that the operator $\mathcal{L}$ is compact.

□

Corollary 3.7. *The operator $\mathcal{L} : C(\Delta, \mathbb{R}) \to C(\Delta, \mathbb{R})$ is μ-Lipschitz with zero constant.*

Proof. Since $\mathcal{L}$ is compact and by using Lemma 2.12 we deduce that $\mathcal{L}$ is μ-Lipschitz with zero constant. □

Now, we have all the ingredients to give the proof of our main result; Theorem 3.3.

Proof of Theorem 3.3.
Let $\mathcal{F}, \mathcal{L}, \mathcal{T}$: $C(J, \mathbb{R}) \to C(J, \mathbb{R})$ be the operators given by the equations (3),(4) and (5) respectively.
They are continuous and bounded. Moreover, by using Lemma 3.4 we have $\mathcal{F}$ is μ-Lipschitz with constant $L_\varphi \in [0, 1)$ and by using Corollary 3.7 we have $\mathcal{F}$ is μ-Lipschitz with zero constant. It follows from Lemma

2.11 that $\mathcal{T}$ is a strict μ-contraction with constant L_φ.
We consider the following set

$$\mathcal{S}_\gamma = \{x \in C(\Delta,\ \mathbb{R}) : x = \gamma \mathcal{T} x \quad for \quad some \quad \gamma \in [0,1]\}.$$

Let us show that $\mathcal{S}_\gamma$ is bounded in $C(\Delta, \mathbb{R})$. For this purpose let $x \in \mathcal{S}_\gamma$, then $x = \gamma\mathcal{T}x = \gamma(\mathcal{F}x + \mathcal{L}x)$. It follows that

$$\|x\| = \gamma\|\mathcal{T}x\| \leq \gamma(\|\mathcal{F}x\| + \|\mathcal{L}x\|),$$

by using Lemmas 3.4 and 3.5 we get

$$\|x\| \leq \left(|x_0| + K_\varphi\|x\|^q + M_\varphi + \frac{(K_f\|x\|^p + M_f)(\psi(T) - \psi(0))^\alpha}{\Gamma(\alpha+1)}\right). \tag{8}$$

From the inequality (8) we deduce that $\mathcal{S}_\gamma$ is bounded in $C(\Delta,\ \mathbb{R})$ with $p < 1$ and $q < 1$.

if it's not the case, we suppose that $\sigma := \|x\| \longrightarrow \infty$. Dividing both sides of (8) by σ, and taking $\sigma \to \infty$, it follows that

$$1 \leq \lim_{\sigma\to\infty} \frac{\left(|x_0| + K_\varphi\sigma^q + M_\varphi + \frac{(K_\varphi\sigma^p + M_\varphi)(\psi(T) - \psi(0))^\alpha}{\Gamma(\alpha+1)}\right)}{\sigma} = 0,$$

which is a contradiction. Hence , as a consequence of the Theorem 2.14 we conclude that $\mathcal{T}$ has at least one fixed point which is the solution of the fractional problem (1) and the set of the fixed points of $\mathcal{T}$ is bounded in $C(\Delta,\ \mathbb{R})$. □

Remark 3.8. *Note that if the assumptions (A_2) and (A_3) are formulated for $q = 1$ and $p = 1$, then the conclusions of Theorem 3.3 remain valid provided that*

$$K_\varphi + \frac{K_f(\psi(T) - \psi(0))^\alpha}{\Gamma(\alpha+1)} < 1.$$

4. An illustrative example

In this section, we give a nontrivial example to illustrate our main result.
Consider the following fractional Cauchy problem:

$$\begin{cases} {}^C D_{0^+}^{\frac{3}{2},t} x(t) = \dfrac{e^{-t}}{9+e^t}\left(\dfrac{|x(t)|}{1+|x(t)|}\right), \quad t \in \Delta = [0,1], \\ \\ x'(0) = 0, \quad x(0) = \displaystyle\sum_{i=1}^{10} \beta_i |x(t_i)|, \quad \beta_i > 0, \quad 0 < t_i < 1, \quad i = 1,2,..,10. \end{cases} \tag{9}$$

In this example we set $\alpha = \frac{3}{2}$, $T = 1$, $\psi(t) = t$, $f(t, x(t)) = \dfrac{e^{-t}}{9+e^t}\left(\dfrac{|x(t)|}{1+|x(t)|}\right)$ and $\varphi(x) = \displaystyle\sum_{i=1}^{10} \beta_i|x(t_i)|$ with $\sum_{i=1}^{10} \beta_i < 1$.

It is clear that the assumptions (A_1) and (A_2) are satisfied with $K_\varphi = L_\varphi = \sum_{i=1}^{10} \beta_i$, $M_\varphi = 0$ and $q = 1$.
Indeed, we have

$$|\varphi(x(t))| = \left|\sum_{i=1}^{10} \beta_i |x(t_i)|\right|,$$

it follows that

$$|\varphi(x)| \leq \sum_{i=1}^{10} \beta_i \|x\|,$$

hence $K_\varphi = \sum\limits_{i=1}^{10} \beta_i, \quad M_\varphi = 0$ and $q = 1$.
On the other hand, we have

$$|\varphi(x(t)) - \varphi(y(t))| = \left| \sum_{i=1}^{10} \beta_i |x(t_i) - \sum_{i=1}^{10} \beta_i |y(t_i) \right|,$$

from which, we have

$$|\varphi(x) - \varphi(y)| \leq \sum_{i=1}^{10} \beta_i |x - y|,$$

thus $L_\varphi = \sum\limits_{i=1}^{10} \beta_i$.
To prove the assumption (A_3), let $t \in \Delta$ and $x \in \mathbb{R}$, then we have

$$|f(t, x(t))| = \left| \frac{e^{-t}}{9 + e^t} \left(\frac{|x(t)|}{1 + |x(t)|} \right) \right|,$$

$$|f(t, x(t))| \leq \left| \frac{e^{-t}}{9 + e^t} \right| \left| \left(\frac{|x(t)|}{1 + |x(t)|} \right) \right|,$$

$$|f(t, x(t))| \leq \frac{1}{10} (|x| + 1).$$

Thus, the assumption (H_3) holds true with $K_f = M_f = \dfrac{1}{10}$ and $p = 1$.
Finally, all the conditions of Theorem 3.3 are satisfied, thus it is easy to see that the fractional nonlinear Cauchy problem (9) has at least one solution defined on $[0, 1]$. Moreover, the set of its solutions is bounded in $C(\Delta, \mathbb{R})$. Indeed, from the inequality (8) we have

$$\|x\| \leq \frac{\frac{M_f(\psi(1)-\psi(0))^\alpha}{\Gamma(\alpha+1)}}{1 - \frac{K_f(\psi(1)-\psi(0))^\alpha}{\Gamma(\alpha+1)}} = \frac{1}{10\Gamma(5/2) - 1} = 0.082.$$

5. Conclusion

In this paper, we studied the existence of solutions for nonlocal fractional Cauchy problem of nonlinear fractional differential equations involving Caputo type fractional derivative with respect to another function ψ. The existence theorems are proved by using some fixed point theorems based on topological degree theory for condensing maps. As application, an example is given to illustrate the obtained result.

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Conflict of interest

The authors declare that they have no conflict of interest.

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