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Solutions Formulas for Three-dimensional Difference Equations System with Constant Coefficients

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Abstract. In this paper, we study the following three-dimensional system of difference equations

$$x_n = \frac{a x_{n-3} z_{n-2} + b}{c y_{n-1} z_{n-2} x_{n-3}},\ y_n = \frac{a y_{n-3} x_{n-2} + b}{c z_{n-1} x_{n-2} y_{n-3}},\ z_n = \frac{a z_{n-3} y_{n-2} + b}{c x_{n-1} y_{n-2} z_{n-3}},\ n \in \mathbb{N}_0,$$

where the parameters a, b, c and the initial values $x_{-j}, y_{-j}, z_{-j},\ j \in \{1, 2, 3\}$, are real numbers. We solve aforementioned system in explicit form. Then, we investigate the solutions in 3 different cases depending on whether the parameters are zero or non-zero. In addition, numerical examples are given to demonstrate the theoretical results. Finally, an application is given for solutions are related to Fibonacci numbers when $a = b = c = 1$.

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1. Introduction

Firstly, remind that $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, stand for natural, non-negative integer, integer, real and complex numbers, respectively. If $m, n \in \mathbb{Z}$, $m \le n$, the notation $i = \overline{m, n}$ stands for $\{i \in \mathbb{Z} : m \le i \le n\}$. The notation of $\lfloor i \rfloor$ means of $n \le i < n+1, n \in \mathbb{Z}$.

There are different types of difference equations. One of them is

$$x_{n+1} = \frac{a x_n + b}{c x_n + d},\ n \in \mathbb{N}_0, \tag{1.1}$$

for $c \neq 0$, $ad \neq bc$, where parameters a, b, c, d and the initial value x_0 are real numbers, which called Riccati difference equation. Indeed, equation (1.1) has the general solution can be written in the following form

$$x_n = \frac{x_0 (bc - ad) s_{n-1} + (a x_0 + b) s_n}{(c x_0 - a) s_n + s_{n+1}},\quad n \in \mathbb{N}, \tag{1.2}$$

where $(s_n)_{n \in \mathbb{N}_0}$ is the sequence satisfying

$$s_{n+1} - (a + d) s_n - (bc - ad) s_{n-1} = 0,\quad n \in \mathbb{N},$$

where $s_0 = 0$, $s_1 = 1$, in [24].

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The theory of difference equations or their systems is important. Especially, difference equation or their systems whose solutions are related to sequences of numbers has been attracted by many authors in recent years [1–14, 16–23, 25–32, 34–36].

For instance, in [33], authors obtained the solutions of the following two difference equations systems

$$x_{n+1} = \frac{x_{n-1} \pm 1}{y_n x_{n-1}},\ y_{n+1} = \frac{y_{n-1} \pm 1}{x_n y_{n-1}},\ n \in \mathbb{N}_0. \tag{1.3}$$

The solutions of systems in (1.3) are associated with Padovan numbers.

In addition, the authors of [15] dealt with the solution, stability character and asymptotic behavior of the following rational difference equation

$$x_{n+1} = \frac{\alpha x_{n-1} + \beta}{\gamma x_n x_{n-1}},\ n \in \mathbb{N}_0, \tag{1.4}$$

where $\alpha, \beta, \gamma \in \mathbb{R}^+$, the initial values x_{-1}, x_0 are non zero real numbers and they investigated the two-dimensional case of equation (1.4) given by

$$x_{n+1} = \frac{\alpha x_{n-1} + \beta}{\gamma y_n x_{n-1}},\ y_{n+1} = \frac{\alpha y_{n-1} + \beta}{\gamma x_n y_{n-1}},\ n \in \mathbb{N}_0. \tag{1.5}$$

The solutions of equation (1.4) found by the authors are related to Padovan numbers and the solution of system (1.5) are related to generalized Padovan numbers.

The motivation for above studies, we deal with the following system of difference equations

$$x_n = \frac{a x_{n-3} z_{n-2} + b}{c y_{n-1} z_{n-2} x_{n-3}},\ y_n = \frac{a y_{n-3} x_{n-2} + b}{c z_{n-1} x_{n-2} y_{n-3}},\ z_n = \frac{a z_{n-3} y_{n-2} + b}{c x_{n-1} y_{n-2} z_{n-3}},\ n \in \mathbb{N}_0, \tag{1.6}$$

where the parameters a, b, c and the initial values x_{-j}, y_{-j}, z_{-j}, $j \in \{1, 2, 3\}$, are real numbers. We solve system (1.6) in explicit form. Then, we investigate the solutions in 3 different cases depending on whether the parameters are zero or non-zero. In addition, numerical examples are given to demonstrate the theoretical results. Finally, an application is given for solutions are related to Fibonacci numbers when $a = b = c = 1$.

Definition 1.1. (Periodicity) A sequence $(x_n)_{n=-k}^{\infty}$ is said to be eventually periodic with period p if there exist $n_0 \geq -k$ such that $x_{n+p} = x_n$ for all $n \geq n_0$. If $n_0 = -k$ then the sequence $(x_n)_{n=-k}^{\infty}$ is said to be periodic with period p.

2. Explicit Solution of the System (1.6)

Let $(x_n, y_n, z_n)_{n\geq -3}$ be a solution of system (1.6). If at least one of the initial values x_{-j}, y_{-j}, z_{-j}, $j \in \{1, 2, 3\}$, is equal to zero, then the solution of system (1.6) is not defined. For example, if $x_{-3} = 0$ and so x_0, y_2 and z_4 can not be calculated. Similarly, if $y_{-3} = 0$ (or $z_{-3} = 0$) and so y_0, z_2 and x_4 (or z_0, x_2 and y_4) can not be calculated. Thus, for every well-defined solution of system (1.6), we get that $x_n y_n z_n \neq 0$, $n \geq -3$, if and only if $x_{-j} y_{-j} z_{-j} \neq 0$, $j \in \{1, 2, 3\}$. Note that the system (1.6) can be written in the form

$$x_n y_{n-1} = \frac{a}{c} + \frac{b}{c z_{n-2} x_{n-3}},\ y_n z_{n-1} = \frac{a}{c} + \frac{b}{c x_{n-2} y_{n-3}},\ z_n x_{n-1} = \frac{a}{c} + \frac{b}{c y_{n-2} z_{n-3}}, \tag{2.1}$$

for $n \in \mathbb{N}_0$. Next, by employing the change of variables

$$u_n = x_n y_{n-1},\ v_n = y_n z_{n-1},\ w_n = z_n x_{n-1},\ n \geq -2, \tag{2.2}$$

system (2.1) is transformed into the following system

$$u_n = \frac{a}{c} + \frac{b}{c w_{n-2}},\ v_n = \frac{a}{c} + \frac{b}{c u_{n-2}},\ w_n = \frac{a}{c} + \frac{b}{c v_{n-2}},\ n \in \mathbb{N}_0,$$

which can be written as

$$u_n = \frac{\left(a^3 + 2abc\right) u_{n-6} + a^2 b + b^2 c}{(a^2 c + bc^2)\, u_{n-6} + abc},\ n \geq 4, \tag{2.3}$$

$$v_n = \frac{\left(a^3 + 2abc\right) v_{n-6} + a^2 b + b^2 c}{(a^2 c + bc^2)\, v_{n-6} + abc},\ n \geq 4, \tag{2.4}$$

$$w_n = \frac{\left(a^3 + 2abc\right) w_{n-6} + a^2 b + b^2 c}{(a^2 c + bc^2)\, w_{n-6} + abc},\ n \geq 4. \tag{2.5}$$

Now we consider the following equation

$$t_n = \frac{\left(a^3 + 2abc\right) t_{n-6} + a^2 b + b^2 c}{(a^2 c + bc^2)\, t_{n-6} + abc},\ n \geq 4, \tag{2.6}$$

instead of equations in (2.3)-(2.5). If we apply the decomposition of indices $n \to 6(m+1) + i$, $i = \overline{-2, 3}$ and $m \geq -1$, to (2.6), then it can be written as follows

$$t_{m+1}^{(i)} = \frac{\left(a^3 + 2abc\right) t_m^{(i)} + a^2 b + b^2 c}{(a^2 c + bc^2)\, t_m^{(i)} + abc}, \tag{2.7}$$

where $t_m^{(i)} = t_{6m+i}$, $m \in \mathbb{N}_0$, $i = \overline{-2, 3}$.
Let $A_1 := a^3 + 2abc$, $B_1 := a^2 b + b^2 c$, $C_1 := a^2 c + bc^2$, $D_1 := abc$.
From equation (1.2), the general solutions of (2.7) follows straightforwardly as

$$t_m^{(i)} = \frac{b^3 c^3 t_0^{(i)} s_{m-1} + \left(A_1 t_0^{(i)} + B_1\right) s_m}{\left(C_1 t_0^{(i)} - A_1\right) s_m + s_{m+1}},\ m \in \mathbb{N}_0, \tag{2.8}$$

for $i = \overline{-2, 3}$, where sequence of $(s_m)_{m\in\mathbb{N}_0}$ is satisfying

$$s_{m+1} - \left(a^3 + 3abc\right) s_m - b^3 c^3 s_{m-1} = 0,\ \ m \in \mathbb{N}, \tag{2.9}$$

recurrence relation with $s_0 = 0$, $s_1 = 1$. Note that by using the recurrence relation in (2.9), one can compute

$$s_{-1} = \frac{s_1 - \left(a^3 + 3abc\right) s_0}{b^3 c^3}.$$

We use (2.2) in (2.3)-(2.5) and from (2.8), equations in (2.3)-(2.5) are expressed as

$$u_{6m+i} = \frac{b^3 c^3 x_i y_{i-1} s_{m-1} + (A_1 x_i y_{i-1} + B_1)\, s_m}{(C_1 x_i y_{i-1} - A_1)\, s_m + s_{m+1}},\ m \in \mathbb{N}_0, \tag{2.10}$$

$$v_{6m+i} = \frac{b^3 c^3 y_i z_{i-1} s_{m-1} + (A_1 y_i z_{i-1} + B_1)\, s_m}{(C_1 y_i z_{i-1} - A_1)\, s_m + s_{m+1}},\ m \in \mathbb{N}_0, \tag{2.11}$$

$$w_{6m+i} = \frac{b^3 c^3 z_i x_{i-1} s_{m-1} + (A_1 z_i x_{i-1} + B_1)\, s_m}{(C_1 z_i x_{i-1} - A_1)\, s_m + s_{m+1}},\ m \in \mathbb{N}_0, \tag{2.12}$$

for $i = \overline{-2, 3}$.
From (2.2), we have that

$$x_n = \frac{u_n}{y_{n-1}} = \frac{u_n z_{n-2}}{v_{n-1}} = \frac{u_n w_{n-2}}{v_{n-1} x_{n-3}} = \frac{u_n w_{n-2} y_{n-4}}{v_{n-1} u_{n-3}} = \frac{u_n w_{n-2} v_{n-4}}{v_{n-1} u_{n-3} z_{n-5}} = \frac{u_n w_{n-2} v_{n-4}}{v_{n-1} u_{n-3} w_{n-5}} x_{n-6},\ n \geq 3, \tag{2.13}$$

$$y_n = \frac{v_n}{z_{n-1}} = \frac{v_n x_{n-2}}{w_{n-1}} = \frac{v_n u_{n-2}}{w_{n-1} y_{n-3}} = \frac{v_n u_{n-2} z_{n-4}}{w_{n-1} v_{n-3}} = \frac{v_n u_{n-2} w_{n-4}}{w_{n-1} v_{n-3} x_{n-5}} = \frac{v_n u_{n-2} w_{n-4}}{w_{n-1} v_{n-3} u_{n-5}} y_{n-6},\ n \geq 3, \tag{2.14}$$

$$z_n = \frac{w_n}{x_{n-1}} = \frac{w_n y_{n-2}}{u_{n-1}} = \frac{w_n v_{n-2}}{u_{n-1} z_{n-3}} = \frac{w_n v_{n-2} x_{n-4}}{u_{n-1} w_{n-3}} = \frac{w_n v_{n-2} u_{n-4}}{u_{n-1} w_{n-3} y_{n-5}} = \frac{w_n v_{n-2} u_{n-4}}{u_{n-1} w_{n-3} v_{n-5}} z_{n-6},\ n \geq 3. \tag{2.15}$$

From (2.13)-(2.15), we get

$$x_{6m+j} = \frac{u_{6m+j} w_{6m+j-2} v_{6m+j-4}}{v_{6m+j-1} u_{6m+j-3} w_{6m+j-5}} x_{6(m-1)+j},\, m \in \mathbb{N}_0, \tag{2.16}$$

$$y_{6m+j} = \frac{v_{6m+j} u_{6m+j-2} w_{6m+j-4}}{w_{6m+j-1} v_{6m+j-3} u_{6m+j-5}} y_{6(m-1)+j},\, m \in \mathbb{N}_0, \tag{2.17}$$

$$z_{6m+j} = \frac{w_{6m+j} v_{6m+j-2} u_{6m+j-4}}{u_{6m+j-1} w_{6m+j-3} v_{6m+j-5}} z_{6(m-1)+j},\, m \in \mathbb{N}_0, \tag{2.18}$$

for $j=\overline{3,8}$.
Multiplying the equalities which are obtained from (2.16)-(2.18), from 0 to m, it follows that

$$x_{6m+i+5}=x_{i-1}\prod_{p=0}^{m}\frac{u_{6\left(p+\lfloor\frac{i+7}{6}\rfloor\right)+i+5-6\lfloor\frac{i+7}{6}\rfloor}}{v_{6\left(p+\lfloor\frac{i+6}{6}\rfloor\right)+i+4-6\lfloor\frac{i+6}{6}\rfloor}}\frac{w_{6\left(p+\lfloor\frac{i+5}{6}\rfloor\right)+i+3-6\lfloor\frac{i+5}{6}\rfloor}}{u_{6\left(p+\lfloor\frac{i+4}{6}\rfloor\right)+i+2-6\lfloor\frac{i+4}{6}\rfloor}}\frac{v_{6\left(p+\lfloor\frac{i+3}{6}\rfloor\right)+i+1-6\lfloor\frac{i+3}{6}\rfloor}}{w_{6\left(p+\lfloor\frac{i+2}{6}\rfloor\right)+i-6\lfloor\frac{i+2}{6}\rfloor}},\tag{2.19}$$

$$y_{6m+i+5}=y_{i-1}\prod_{p=0}^{m}\frac{v_{6\left(p+\lfloor\frac{i+7}{6}\rfloor\right)+i+5-6\lfloor\frac{i+7}{6}\rfloor}}{w_{6\left(p+\lfloor\frac{i+6}{6}\rfloor\right)+i+4-6\lfloor\frac{i+6}{6}\rfloor}}\frac{u_{6\left(p+\lfloor\frac{i+5}{6}\rfloor\right)+i+3-6\lfloor\frac{i+5}{6}\rfloor}}{v_{6\left(p+\lfloor\frac{i+4}{6}\rfloor\right)+i+2-6\lfloor\frac{i+4}{6}\rfloor}}\frac{w_{6\left(p+\lfloor\frac{i+3}{6}\rfloor\right)+i+1-6\lfloor\frac{i+3}{6}\rfloor}}{u_{6\left(p+\lfloor\frac{i+2}{6}\rfloor\right)+i-6\lfloor\frac{i+2}{6}\rfloor}},\tag{2.20}$$

$$z_{6m+i+5}=z_{i-1}\prod_{p=0}^{m}\frac{w_{6\left(p+\lfloor\frac{i+7}{6}\rfloor\right)+i+5-6\lfloor\frac{i+7}{6}\rfloor}}{u_{6\left(p+\lfloor\frac{i+6}{6}\rfloor\right)+i+4-6\lfloor\frac{i+6}{6}\rfloor}}\frac{v_{6\left(p+\lfloor\frac{i+5}{6}\rfloor\right)+i+3-6\lfloor\frac{i+5}{6}\rfloor}}{w_{6\left(p+\lfloor\frac{i+4}{6}\rfloor\right)+i+2-6\lfloor\frac{i+4}{6}\rfloor}}\frac{u_{6\left(p+\lfloor\frac{i+3}{6}\rfloor\right)+i+1-6\lfloor\frac{i+3}{6}\rfloor}}{v_{6\left(p+\lfloor\frac{i+2}{6}\rfloor\right)+i-6\lfloor\frac{i+2}{6}\rfloor}},\tag{2.21}$$

where $m\in\mathbb{N}_0$, $i=\overline{-2,3}$. By substituting the formulas in (2.10)-(2.12) into (2.19)-(2.21), we obtain

$$\begin{aligned}x_{6m+i+5}=&x_{i-1}\prod_{p=0}^{m}\frac{\frac{b^3c^3x_{i+5-6\lfloor\frac{i+7}{6}\rfloor}y_{i+5-6\lfloor\frac{i+7}{6}\rfloor-1}s_{p+\lfloor\frac{i+7}{6}\rfloor-1}+\left(A_1x_{i+5-6\lfloor\frac{i+7}{6}\rfloor}y_{i+5-6\lfloor\frac{i+7}{6}\rfloor-1}+B_1\right)s_{p+\lfloor\frac{i+7}{6}\rfloor}}{\left(C_1x_{i+5-6\lfloor\frac{i+7}{6}\rfloor}y_{i+5-6\lfloor\frac{i+7}{6}\rfloor-1}-A_1\right)s_{p+\lfloor\frac{i+7}{6}\rfloor}+s_{p+\lfloor\frac{i+7}{6}\rfloor+1}}}{\frac{b^3c^3y_{i+4-6\lfloor\frac{i+6}{6}\rfloor}z_{i+4-6\lfloor\frac{i+6}{6}\rfloor-1}s_{p+\lfloor\frac{i+6}{6}\rfloor-1}+\left(A_1y_{i+4-6\lfloor\frac{i+6}{6}\rfloor}z_{i+4-6\lfloor\frac{i+6}{6}\rfloor-1}+B_1\right)s_{p+\lfloor\frac{i+6}{6}\rfloor}}{\left(C_1y_{i+4-6\lfloor\frac{i+6}{6}\rfloor}z_{i+4-6\lfloor\frac{i+6}{6}\rfloor-1}-A_1\right)s_{p+\lfloor\frac{i+6}{6}\rfloor}+s_{p+\lfloor\frac{i+6}{6}\rfloor+1}}}\\&\times\frac{\frac{b^3c^3z_{i+3-6\lfloor\frac{i+5}{6}\rfloor}x_{i+3-6\lfloor\frac{i+5}{6}\rfloor-1}s_{p+\lfloor\frac{i+5}{6}\rfloor-1}+\left(A_1z_{i+3-6\lfloor\frac{i+5}{6}\rfloor}x_{i+3-6\lfloor\frac{i+5}{6}\rfloor-1}+B_1\right)s_{p+\lfloor\frac{i+5}{6}\rfloor}}{\left(C_1z_{i+3-6\lfloor\frac{i+5}{6}\rfloor}x_{i+3-6\lfloor\frac{i+5}{6}\rfloor-1}-A_1\right)s_{p+\lfloor\frac{i+5}{6}\rfloor}+s_{p+\lfloor\frac{i+5}{6}\rfloor+1}}}{\frac{b^3c^3x_{i+2-6\lfloor\frac{i+4}{6}\rfloor}y_{i+2-6\lfloor\frac{i+4}{6}\rfloor-1}s_{p+\lfloor\frac{i+4}{6}\rfloor-1}+\left(A_1x_{i+2-6\lfloor\frac{i+4}{6}\rfloor}y_{i+2-6\lfloor\frac{i+4}{6}\rfloor-1}+B_1\right)s_{p+\lfloor\frac{i+4}{6}\rfloor}}{\left(C_1x_{i+2-6\lfloor\frac{i+4}{6}\rfloor}y_{i+2-6\lfloor\frac{i+4}{6}\rfloor-1}-A_1\right)s_{p+\lfloor\frac{i+4}{6}\rfloor}+s_{p+\lfloor\frac{i+4}{6}\rfloor+1}}}\\&\times\frac{\frac{b^3c^3y_{i+1-6\lfloor\frac{i+3}{6}\rfloor}z_{i+1-6\lfloor\frac{i+3}{6}\rfloor-1}s_{p+\lfloor\frac{i+3}{6}\rfloor-1}+\left(A_1y_{i+1-6\lfloor\frac{i+3}{6}\rfloor}z_{i+1-6\lfloor\frac{i+3}{6}\rfloor-1}+B_1\right)s_{p+\lfloor\frac{i+3}{6}\rfloor}}{\left(C_1y_{i+1-6\lfloor\frac{i+3}{6}\rfloor}z_{i+1-6\lfloor\frac{i+3}{6}\rfloor-1}-A_1\right)s_{p+\lfloor\frac{i+3}{6}\rfloor}+s_{p+\lfloor\frac{i+3}{6}\rfloor+1}}}{\frac{b^3c^3z_{i-6\lfloor\frac{i+2}{6}\rfloor}x_{i-6\lfloor\frac{i+2}{6}\rfloor-1}s_{p+\lfloor\frac{i+2}{6}\rfloor-1}+\left(A_1z_{i-6\lfloor\frac{i+2}{6}\rfloor}x_{i-6\lfloor\frac{i+2}{6}\rfloor-1}+B_1\right)s_{p+\lfloor\frac{i+2}{6}\rfloor}}{\left(C_1z_{i-6\lfloor\frac{i+2}{6}\rfloor}x_{i-6\lfloor\frac{i+2}{6}\rfloor-1}-A_1\right)s_{p+\lfloor\frac{i+2}{6}\rfloor}+s_{p+\lfloor\frac{i+2}{6}\rfloor+1}}},\end{aligned}\tag{2.22}$$

$$\begin{aligned}y_{6m+i+5}=&y_{i-1}\prod_{p=0}^{m}\frac{\frac{b^3c^3y_{i+5-6\lfloor\frac{i+7}{6}\rfloor}z_{i+5-6\lfloor\frac{i+7}{6}\rfloor-1}s_{p+\lfloor\frac{i+7}{6}\rfloor-1}+\left(A_1y_{i+5-6\lfloor\frac{i+7}{6}\rfloor}z_{i+5-6\lfloor\frac{i+7}{6}\rfloor-1}+B_1\right)s_{p+\lfloor\frac{i+7}{6}\rfloor}}{\left(C_1y_{i+5-6\lfloor\frac{i+7}{6}\rfloor}z_{i+5-6\lfloor\frac{i+7}{6}\rfloor-1}-A_1\right)s_{p+\lfloor\frac{i+7}{6}\rfloor}+s_{p+\lfloor\frac{i+7}{6}\rfloor+1}}}{\frac{b^3c^3z_{i+4-6\lfloor\frac{i+6}{6}\rfloor}x_{i+4-6\lfloor\frac{i+6}{6}\rfloor-1}s_{p+\lfloor\frac{i+6}{6}\rfloor-1}+\left(A_1z_{i+4-6\lfloor\frac{i+6}{6}\rfloor}x_{i+4-6\lfloor\frac{i+6}{6}\rfloor-1}+B_1\right)s_{p+\lfloor\frac{i+6}{6}\rfloor}}{\left(C_1z_{i+4-6\lfloor\frac{i+6}{6}\rfloor}x_{i+4-6\lfloor\frac{i+6}{6}\rfloor-1}-A_1\right)s_{p+\lfloor\frac{i+6}{6}\rfloor}+s_{p+\lfloor\frac{i+6}{6}\rfloor+1}}}\\&\times\frac{\frac{b^3c^3x_{i+3-6\lfloor\frac{i+5}{6}\rfloor}y_{i+3-6\lfloor\frac{i+5}{6}\rfloor-1}s_{p+\lfloor\frac{i+5}{6}\rfloor-1}+\left(A_1x_{i+3-6\lfloor\frac{i+5}{6}\rfloor}y_{i+3-6\lfloor\frac{i+5}{6}\rfloor-1}+B_1\right)s_{p+\lfloor\frac{i+5}{6}\rfloor}}{\left(C_1x_{i+3-6\lfloor\frac{i+5}{6}\rfloor}y_{i+3-6\lfloor\frac{i+5}{6}\rfloor-1}-A_1\right)s_{p+\lfloor\frac{i+5}{6}\rfloor}+s_{p+\lfloor\frac{i+5}{6}\rfloor+1}}}{\frac{b^3c^3y_{i+2-6\lfloor\frac{i+4}{6}\rfloor}z_{i+2-6\lfloor\frac{i+4}{6}\rfloor-1}s_{p+\lfloor\frac{i+4}{6}\rfloor-1}+\left(A_1y_{i+2-6\lfloor\frac{i+4}{6}\rfloor}z_{i+2-6\lfloor\frac{i+4}{6}\rfloor-1}+B_1\right)s_{p+\lfloor\frac{i+4}{6}\rfloor}}{\left(C_1y_{i+2-6\lfloor\frac{i+4}{6}\rfloor}z_{i+2-6\lfloor\frac{i+4}{6}\rfloor-1}-A_1\right)s_{p+\lfloor\frac{i+4}{6}\rfloor}+s_{p+\lfloor\frac{i+4}{6}\rfloor+1}}}\\&\times\frac{\frac{b^3c^3z_{i+1-6\lfloor\frac{i+3}{6}\rfloor}x_{i+1-6\lfloor\frac{i+3}{6}\rfloor-1}s_{p+\lfloor\frac{i+3}{6}\rfloor-1}+\left(A_1z_{i+1-6\lfloor\frac{i+3}{6}\rfloor}x_{i+1-6\lfloor\frac{i+3}{6}\rfloor-1}+B_1\right)s_{p+\lfloor\frac{i+3}{6}\rfloor}}{\left(C_1z_{i+1-6\lfloor\frac{i+3}{6}\rfloor}x_{i+1-6\lfloor\frac{i+3}{6}\rfloor-1}-A_1\right)s_{p+\lfloor\frac{i+3}{6}\rfloor}+s_{p+\lfloor\frac{i+3}{6}\rfloor+1}}}{\frac{b^3c^3x_{i-6\lfloor\frac{i+2}{6}\rfloor}y_{i-6\lfloor\frac{i+2}{6}\rfloor-1}s_{p+\lfloor\frac{i+2}{6}\rfloor-1}+\left(A_1x_{i-6\lfloor\frac{i+2}{6}\rfloor}y_{i-6\lfloor\frac{i+2}{6}\rfloor-1}+B_1\right)s_{p+\lfloor\frac{i+2}{6}\rfloor}}{\left(C_1x_{i-6\lfloor\frac{i+2}{6}\rfloor}y_{i-6\lfloor\frac{i+2}{6}\rfloor-1}-A_1\right)s_{p+\lfloor\frac{i+2}{6}\rfloor}+s_{p+\lfloor\frac{i+2}{6}\rfloor+1}}},\end{aligned}\tag{2.23}$$

$$
\begin{aligned}
z_{6m+i+5} = z_{i-1} \prod_{p=0}^{m} & \frac{\dfrac{b^3c^3 z_{i+5-6\lfloor\frac{i+7}{6}\rfloor} x_{i+5-6\lfloor\frac{i+7}{6}\rfloor-1} s_{p+\lfloor\frac{i+7}{6}\rfloor-1} + \left(A_1 z_{i+5-6\lfloor\frac{i+7}{6}\rfloor} x_{i+5-6\lfloor\frac{i+7}{6}\rfloor-1} + B_1\right) s_{p+\lfloor\frac{i+7}{6}\rfloor}}{\left(C_1 z_{i+5-6\lfloor\frac{i+7}{6}\rfloor} x_{i+5-6\lfloor\frac{i+7}{6}\rfloor-1} - A_1\right) s_{p+\lfloor\frac{i+7}{6}\rfloor} + s_{p+\lfloor\frac{i+7}{6}\rfloor+1}}}{\dfrac{b^3c^3 x_{i+4-6\lfloor\frac{i+6}{6}\rfloor} y_{i+4-6\lfloor\frac{i+6}{6}\rfloor-1} s_{p+\lfloor\frac{i+6}{6}\rfloor-1} + \left(A_1 x_{i+4-6\lfloor\frac{i+6}{6}\rfloor} y_{i+4-6\lfloor\frac{i+6}{6}\rfloor-1} + B_1\right) s_{p+\lfloor\frac{i+6}{6}\rfloor}}{\left(C_1 x_{i+4-6\lfloor\frac{i+6}{6}\rfloor} y_{i+4-6\lfloor\frac{i+6}{6}\rfloor-1} - A_1\right) s_{p+\lfloor\frac{i+6}{6}\rfloor} + s_{p+\lfloor\frac{i+6}{6}\rfloor+1}}} \\
& \times \frac{\dfrac{b^3c^3 y_{i+3-6\lfloor\frac{i+5}{6}\rfloor} z_{i+3-6\lfloor\frac{i+5}{6}\rfloor-1} s_{p+\lfloor\frac{i+5}{6}\rfloor-1} + \left(A_1 y_{i+3-6\lfloor\frac{i+5}{6}\rfloor} z_{i+3-6\lfloor\frac{i+5}{6}\rfloor-1} + B_1\right) s_{p+\lfloor\frac{i+5}{6}\rfloor}}{\left(C_1 y_{i+3-6\lfloor\frac{i+5}{6}\rfloor} z_{i+3-6\lfloor\frac{i+5}{6}\rfloor-1} - A_1\right) s_{p+\lfloor\frac{i+5}{6}\rfloor} + s_{p+\lfloor\frac{i+5}{6}\rfloor+1}}}{\dfrac{b^3c^3 z_{i+2-6\lfloor\frac{i+4}{6}\rfloor} x_{i+2-6\lfloor\frac{i+4}{6}\rfloor-1} s_{p+\lfloor\frac{i+4}{6}\rfloor-1} + \left(A_1 z_{i+2-6\lfloor\frac{i+4}{6}\rfloor} x_{i+2-6\lfloor\frac{i+4}{6}\rfloor-1} + B_1\right) s_{p+\lfloor\frac{i+4}{6}\rfloor}}{\left(C_1 z_{i+2-6\lfloor\frac{i+4}{6}\rfloor} x_{i+2-6\lfloor\frac{i+4}{6}\rfloor-1} - A_1\right) s_{p+\lfloor\frac{i+4}{6}\rfloor} + s_{p+\lfloor\frac{i+4}{6}\rfloor+1}}} \\
& \times \frac{\dfrac{b^3c^3 x_{i+1-6\lfloor\frac{i+3}{6}\rfloor} y_{i+1-6\lfloor\frac{i+3}{6}\rfloor-1} s_{p+\lfloor\frac{i+3}{6}\rfloor-1} + \left(A_1 x_{i+1-6\lfloor\frac{i+3}{6}\rfloor} y_{i+1-6\lfloor\frac{i+3}{6}\rfloor-1} + B_1\right) s_{p+\lfloor\frac{i+3}{6}\rfloor}}{\left(C_1 x_{i+1-6\lfloor\frac{i+3}{6}\rfloor} y_{i+1-6\lfloor\frac{i+3}{6}\rfloor-1} - A_1\right) s_{p+\lfloor\frac{i+3}{6}\rfloor} + s_{p+\lfloor\frac{i+3}{6}\rfloor+1}}}{\dfrac{b^3c^3 y_{i-6\lfloor\frac{i+2}{6}\rfloor} z_{i-6\lfloor\frac{i+2}{6}\rfloor-1} s_{p+\lfloor\frac{i+2}{6}\rfloor-1} + \left(A_1 y_{i-6\lfloor\frac{i+2}{6}\rfloor} z_{i-6\lfloor\frac{i+2}{6}\rfloor-1} + B_1\right) s_{p+\lfloor\frac{i+2}{6}\rfloor}}{\left(C_1 y_{i-6\lfloor\frac{i+2}{6}\rfloor} z_{i-6\lfloor\frac{i+2}{6}\rfloor-1} - A_1\right) s_{p+\lfloor\frac{i+2}{6}\rfloor} + s_{p+\lfloor\frac{i+2}{6}\rfloor+1}}},
\end{aligned} \tag{2.24}
$$

for $m \in \mathbb{N}_0$, $i = \overline{-2, 3}$.

3. Particular Cases of System (1.6)

Now, we will examine the solutions in 3 different cases depending on whether the parameters are zero or non-zero.

3.1. **Case $a = 0, b \neq 0 \neq c$.** In this case, system (1.6) becomes

$$
x_n = \frac{b}{c y_{n-1} z_{n-2} x_{n-3}}, \ y_n = \frac{b}{c z_{n-1} x_{n-2} y_{n-3}}, \ z_n = \frac{b}{c x_{n-1} y_{n-2} z_{n-3}}, \ n \in \mathbb{N}_0. \tag{3.1}
$$

From (3.1), we obtain

$$
x_n = x_{n-12}, \ y_n = y_{n-12}, \ z_n = z_{n-12}, \ n \geq 9. \tag{3.2}
$$

We can write the solutions of equations in (3.2) as in the following form

$$
x_{12m+j} = x_{j-12}, \ y_{12m+j} = y_{j-12}, \ z_{12m+j} = z_{j-12}, \ m \in \mathbb{N}_0, \ j = \overline{9, 20}.
$$

Now, we present numerical example that represent the solutions of system (1.6) when $a = 0$, $b \neq 0 \neq c$.

Example 3.1. Consider the system (1.6) with the initial values $x_{-3} = 0.1$, $x_{-2} = 2.4$, $x_{-1} = 3$, $y_{-3} = 4.06$, $y_{-2} = 0.05$, $y_{-1} = 0.6$, $z_{-3} = 70.54$, $z_{-2} = 0.86$, $z_{-1} = 9.05$ and the parameters $a = 0$, $b = 1$, $c = 2$, the solutions are represented as in the following figures.

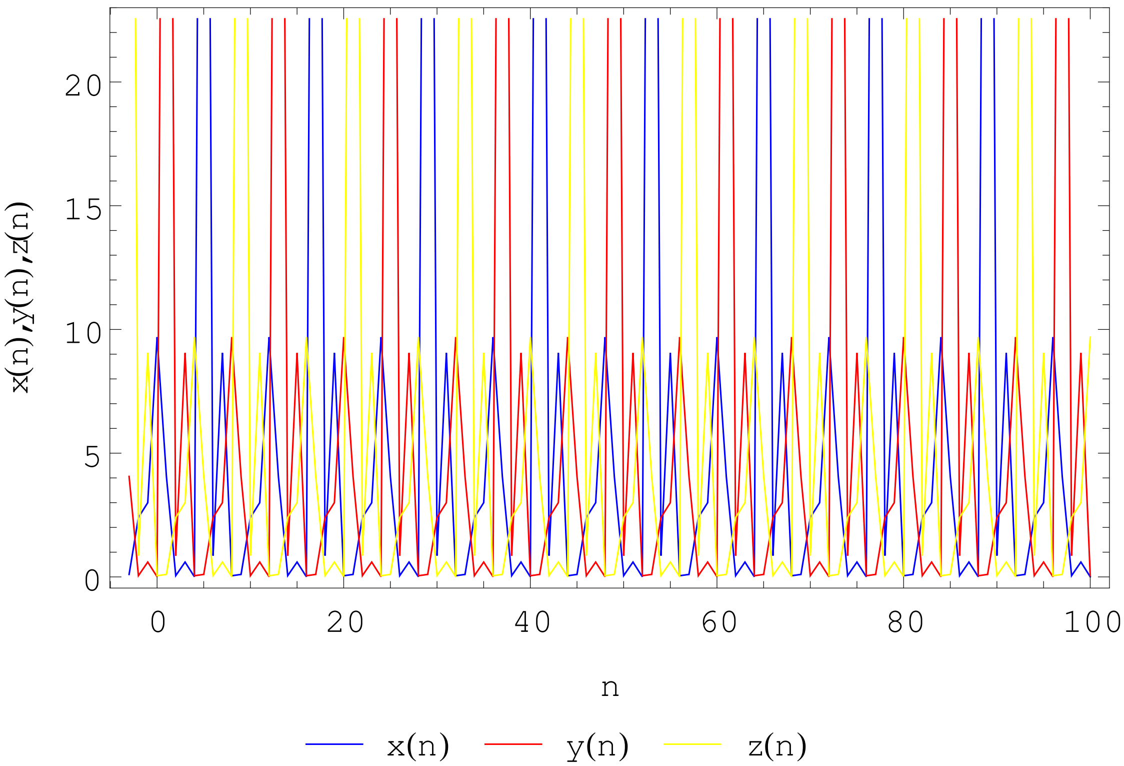

FIGURE 1. Plots of x_n, y_n, z_n in case $a = 0$, $b \neq 0 \neq c$

Therefore, the solutions of system (1.6) are periodic with period 12.

3.2. **Case $b = 0$, $a \neq 0 \neq c$.** In this case, system (1.6) reduces to the following system

$$x_n = \frac{a}{cy_{n-1}},\ y_n = \frac{a}{cz_{n-1}},\ z_n = \frac{a}{cx_{n-1}},\ n \in \mathbb{N}_0. \tag{3.3}$$

From (3.3), we obtain

$$x_n = x_{n-6},\ y_n = y_{n-6},\ z_n = z_{n-6},\ n \geq 3. \tag{3.4}$$

We can write the solutions of equations in (3.4) as in the following form

$$x_{6m+i} = x_{i-6},\ y_{6m+i} = y_{i-6},\ z_{6m+i} = z_{i-6},\ m \in \mathbb{N}_0,\ i = \overline{5, 10}.$$

Now, we present numerical example that represent the solutions of system (1.6) when $b = 0$, $a \neq 0 \neq c$.

Example 3.2. Consider the system (1.6) with the initial values $x_{-3} = 1$, $x_{-2} = 2$, $x_{-1} = 3$, $y_{-3} = 4$, $y_{-2} = 5$, $y_{-1} = 6$, $z_{-3} = 7$, $z_{-2} = 8$, $z_{-1} = 9$ and the parameters $a = 1$, $b = 0$, $c = 2$, the solutions are represented as in the following figures.

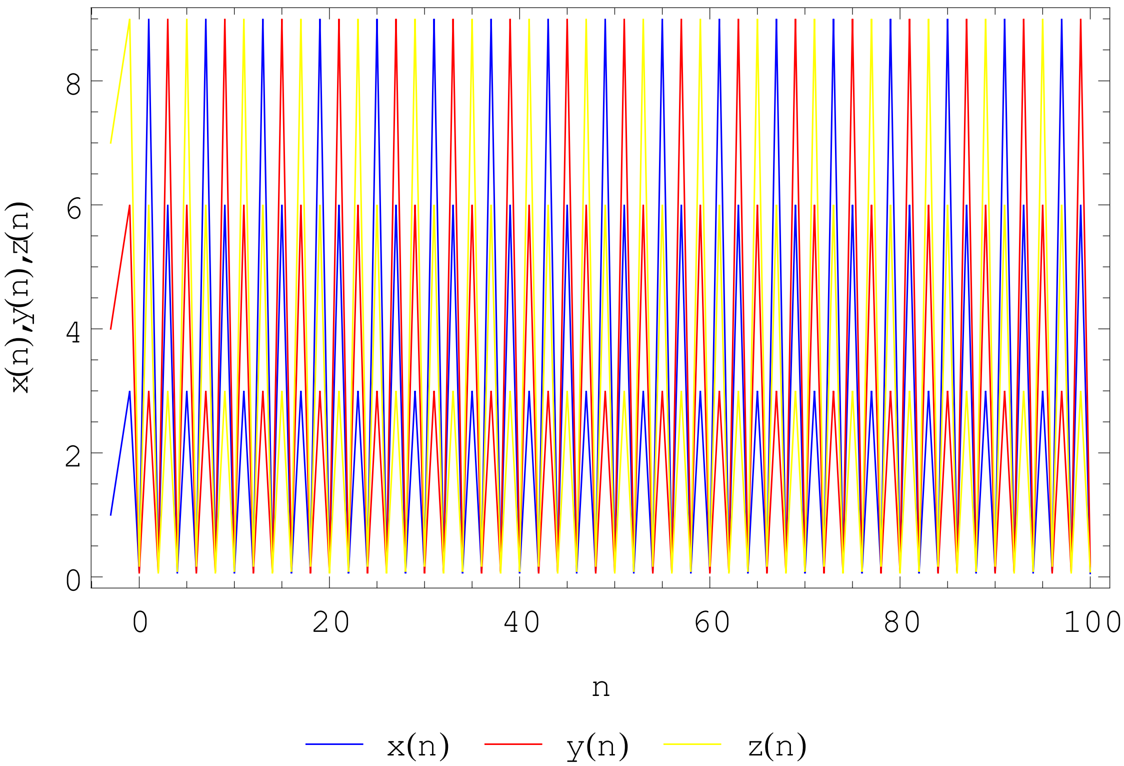

FIGURE 2. Plots of x_n, y_n, z_n in case $b = 0$, $a \neq 0 \neq c$

Therefore, the solutions of system (1.6) are eventually periodic with period 6.

3.3. **Case $c = 0, a \neq 0 \neq b$.** In this case, system (1.6) does not have well-defined solutions.

4. AN APPLICATION

Firstly, we will derive the solution forms of the system (1.6) with $a = b = c = 1$, that is, the system

$$x_n = \frac{x_{n-3}z_{n-2} + 1}{y_{n-1}z_{n-2}x_{n-3}},\ y_n = \frac{y_{n-3}x_{n-2} + 1}{z_{n-1}x_{n-2}y_{n-3}},\ z_n = \frac{z_{n-3}y_{n-2} + 1}{x_{n-1}y_{n-2}z_{n-3}},\ n \in \mathbb{N}_0.$$

From (2.9), we obtain

$$s_{m+1} - 4s_m - s_{m-1} = 0, \quad m \in \mathbb{N}, \tag{4.1}$$

where $s_0 = 0$, $s_1 = 1$.
Binet Formula for (4.1) is

$$s_m = \frac{\left(2 + \sqrt{5}\right)^m - \left(2 - \sqrt{5}\right)^m}{\left(2 + \sqrt{5}\right) - \left(2 - \sqrt{5}\right)}, \quad m \in \mathbb{N}_0. \tag{4.2}$$

Note that

$$\left(\frac{1 \pm \sqrt{5}}{2}\right)^3 = 2 \pm \sqrt{5}.$$

Using this in (4.2), we obtain

$$s_m = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{3m} - \left(\frac{1-\sqrt{5}}{2}\right)^{3m}}{\left(\frac{1+\sqrt{5}}{2}\right)^3 - \left(\frac{1-\sqrt{5}}{2}\right)^3} = \frac{f_{3m}}{2}, \quad m \in \mathbb{N}_0. \tag{4.3}$$

Using (4.3) into (2.22)-(2.24), we have

$$x_{6m+i+5}=x_{i-1}\prod_{p=0}^{m}\frac{\frac{x_{i+5-6\lfloor\frac{i+7}{6}\rfloor}y_{i+5-6\lfloor\frac{i+7}{6}\rfloor-1}F_{3(p+\lfloor\frac{i+7}{6}\rfloor-1)}+\left(3x_{i+5-6\lfloor\frac{i+7}{6}\rfloor}y_{i+5-6\lfloor\frac{i+7}{6}\rfloor-1}+2\right)F_{3(p+\lfloor\frac{i+7}{6}\rfloor)}}{\left(2x_{i+5-6\lfloor\frac{i+7}{6}\rfloor}y_{i+5-6\lfloor\frac{i+7}{6}\rfloor-1}-3\right)F_{3(p+\lfloor\frac{i+7}{6}\rfloor)}+F_{3(p+\lfloor\frac{i+7}{6}\rfloor+1)}}}{\frac{y_{i+4-6\lfloor\frac{i+6}{6}\rfloor}z_{i+4-6\lfloor\frac{i+6}{6}\rfloor-1}F_{3(p+\lfloor\frac{i+6}{6}\rfloor-1)}+\left(3y_{i+4-6\lfloor\frac{i+6}{6}\rfloor}z_{i+4-6\lfloor\frac{i+6}{6}\rfloor-1}+2\right)F_{3(p+\lfloor\frac{i+6}{6}\rfloor)}}{\left(2y_{i+4-6\lfloor\frac{i+6}{6}\rfloor}z_{i+4-6\lfloor\frac{i+6}{6}\rfloor-1}-3\right)F_{3(p+\lfloor\frac{i+6}{6}\rfloor)}+F_{3(p+\lfloor\frac{i+6}{6}\rfloor+1)}}}$$

$$\times\frac{\frac{z_{i+3-6\lfloor\frac{i+5}{6}\rfloor}x_{i+3-6\lfloor\frac{i+5}{6}\rfloor-1}F_{3(p+\lfloor\frac{i+5}{6}\rfloor-1)}+\left(3z_{i+3-6\lfloor\frac{i+5}{6}\rfloor}x_{i+3-6\lfloor\frac{i+5}{6}\rfloor-1}+2\right)F_{3(p+\lfloor\frac{i+5}{6}\rfloor)}}{\left(2z_{i+3-6\lfloor\frac{i+5}{6}\rfloor}x_{i+3-6\lfloor\frac{i+5}{6}\rfloor-1}-3\right)F_{3(p+\lfloor\frac{i+5}{6}\rfloor)}+F_{3(p+\lfloor\frac{i+5}{6}\rfloor+1)}}}{\frac{x_{i+2-6\lfloor\frac{i+4}{6}\rfloor}y_{i+2-6\lfloor\frac{i+4}{6}\rfloor-1}F_{3(p+\lfloor\frac{i+4}{6}\rfloor-1)}+\left(3x_{i+2-6\lfloor\frac{i+4}{6}\rfloor}y_{i+2-6\lfloor\frac{i+4}{6}\rfloor-1}+2\right)F_{3(p+\lfloor\frac{i+4}{6}\rfloor)}}{\left(2x_{i+2-6\lfloor\frac{i+4}{6}\rfloor}y_{i+2-6\lfloor\frac{i+4}{6}\rfloor-1}-3\right)F_{3(p+\lfloor\frac{i+4}{6}\rfloor)}+F_{3(p+\lfloor\frac{i+4}{6}\rfloor+1)}}}$$

$$\times\frac{\frac{y_{i+1-6\lfloor\frac{i+3}{6}\rfloor}z_{i+1-6\lfloor\frac{i+3}{6}\rfloor-1}F_{3(p+\lfloor\frac{i+3}{6}\rfloor-1)}+\left(3y_{i+1-6\lfloor\frac{i+3}{6}\rfloor}z_{i+1-6\lfloor\frac{i+3}{6}\rfloor-1}+2\right)F_{3(p+\lfloor\frac{i+3}{6}\rfloor)}}{\left(2y_{i+1-6\lfloor\frac{i+3}{6}\rfloor}z_{i+1-6\lfloor\frac{i+3}{6}\rfloor-1}-3\right)F_{3(p+\lfloor\frac{i+3}{6}\rfloor)}+F_{3(p+\lfloor\frac{i+3}{6}\rfloor+1)}}}{\frac{z_{i-6\lfloor\frac{i+2}{6}\rfloor}x_{i-6\lfloor\frac{i+2}{6}\rfloor-1}F_{3(p+\lfloor\frac{i+2}{6}\rfloor-1)}+\left(3z_{i-6\lfloor\frac{i+2}{6}\rfloor}x_{i-6\lfloor\frac{i+2}{6}\rfloor-1}+2\right)F_{3(p+\lfloor\frac{i+2}{6}\rfloor)}}{\left(2z_{i-6\lfloor\frac{i+2}{6}\rfloor}x_{i-6\lfloor\frac{i+2}{6}\rfloor-1}-3\right)F_{3(p+\lfloor\frac{i+2}{6}\rfloor)}+F_{3(p+\lfloor\frac{i+2}{6}\rfloor+1)}}},$$

$$y_{6m+i+5}=y_{i-1}\prod_{p=0}^{m}\frac{\frac{y_{i+5-6\lfloor\frac{i+7}{6}\rfloor}z_{i+5-6\lfloor\frac{i+7}{6}\rfloor-1}F_{3(p+\lfloor\frac{i+7}{6}\rfloor-1)}+\left(3y_{i+5-6\lfloor\frac{i+7}{6}\rfloor}z_{i+5-6\lfloor\frac{i+7}{6}\rfloor-1}+2\right)F_{3(p+\lfloor\frac{i+7}{6}\rfloor)}}{\left(2y_{i+5-6\lfloor\frac{i+7}{6}\rfloor}z_{i+5-6\lfloor\frac{i+7}{6}\rfloor-1}-3\right)F_{3(p+\lfloor\frac{i+7}{6}\rfloor)}+F_{3(p+\lfloor\frac{i+7}{6}\rfloor+1)}}}{\frac{z_{i+4-6\lfloor\frac{i+6}{6}\rfloor}x_{i+4-6\lfloor\frac{i+6}{6}\rfloor-1}F_{3(p+\lfloor\frac{i+6}{6}\rfloor-1)}+\left(3z_{i+4-6\lfloor\frac{i+6}{6}\rfloor}x_{i+4-6\lfloor\frac{i+6}{6}\rfloor-1}+2\right)F_{3(p+\lfloor\frac{i+6}{6}\rfloor)}}{\left(2z_{i+4-6\lfloor\frac{i+6}{6}\rfloor}x_{i+4-6\lfloor\frac{i+6}{6}\rfloor-1}-3\right)F_{3(p+\lfloor\frac{i+6}{6}\rfloor)}+F_{3(p+\lfloor\frac{i+6}{6}\rfloor+1)}}}$$

$$\times\frac{\frac{x_{i+3-6\lfloor\frac{i+5}{6}\rfloor}y_{i+3-6\lfloor\frac{i+5}{6}\rfloor-1}F_{3(p+\lfloor\frac{i+5}{6}\rfloor-1)}+\left(3x_{i+3-6\lfloor\frac{i+5}{6}\rfloor}y_{i+3-6\lfloor\frac{i+5}{6}\rfloor-1}+2\right)F_{3(p+\lfloor\frac{i+5}{6}\rfloor)}}{\left(2x_{i+3-6\lfloor\frac{i+5}{6}\rfloor}y_{i+3-6\lfloor\frac{i+5}{6}\rfloor-1}-3\right)F_{3(p+\lfloor\frac{i+5}{6}\rfloor)}+F_{3(p+\lfloor\frac{i+5}{6}\rfloor+1)}}}{\frac{y_{i+2-6\lfloor\frac{i+4}{6}\rfloor}z_{i+2-6\lfloor\frac{i+4}{6}\rfloor-1}F_{3(p+\lfloor\frac{i+4}{6}\rfloor-1)}+\left(3y_{i+2-6\lfloor\frac{i+4}{6}\rfloor}z_{i+2-6\lfloor\frac{i+4}{6}\rfloor-1}+2\right)F_{3(p+\lfloor\frac{i+4}{6}\rfloor)}}{\left(2y_{i+2-6\lfloor\frac{i+4}{6}\rfloor}z_{i+2-6\lfloor\frac{i+4}{6}\rfloor-1}-3\right)F_{3(p+\lfloor\frac{i+4}{6}\rfloor)}+F_{3(p+\lfloor\frac{i+4}{6}\rfloor+1)}}}$$

$$\times\frac{\frac{z_{i+1-6\lfloor\frac{i+3}{6}\rfloor}x_{i+1-6\lfloor\frac{i+3}{6}\rfloor-1}F_{3(p+\lfloor\frac{i+3}{6}\rfloor-1)}+\left(3z_{i+1-6\lfloor\frac{i+3}{6}\rfloor}x_{i+1-6\lfloor\frac{i+3}{6}\rfloor-1}+2\right)F_{3(p+\lfloor\frac{i+3}{6}\rfloor)}}{\left(2z_{i+1-6\lfloor\frac{i+3}{6}\rfloor}x_{i+1-6\lfloor\frac{i+3}{6}\rfloor-1}-3\right)F_{3(p+\lfloor\frac{i+3}{6}\rfloor)}+F_{3(p+\lfloor\frac{i+3}{6}\rfloor+1)}}}{\frac{x_{i-6\lfloor\frac{i+2}{6}\rfloor}y_{i-6\lfloor\frac{i+2}{6}\rfloor-1}F_{3(p+\lfloor\frac{i+2}{6}\rfloor-1)}+\left(3x_{i-6\lfloor\frac{i+2}{6}\rfloor}y_{i-6\lfloor\frac{i+2}{6}\rfloor-1}+2\right)F_{3(p+\lfloor\frac{i+2}{6}\rfloor)}}{\left(2x_{i-6\lfloor\frac{i+2}{6}\rfloor}y_{i-6\lfloor\frac{i+2}{6}\rfloor-1}-3\right)F_{3(p+\lfloor\frac{i+2}{6}\rfloor)}+F_{3(p+\lfloor\frac{i+2}{6}\rfloor+1)}}},$$

$$z_{6m+i+5}=z_{i-1}\prod_{p=0}^{m}\frac{\frac{z_{i+5-6\lfloor\frac{i+7}{6}\rfloor}x_{i+5-6\lfloor\frac{i+7}{6}\rfloor-1}F_{3(p+\lfloor\frac{i+7}{6}\rfloor-1)}+\left(3z_{i+5-6\lfloor\frac{i+7}{6}\rfloor}x_{i+5-6\lfloor\frac{i+7}{6}\rfloor-1}+2\right)F_{3(p+\lfloor\frac{i+7}{6}\rfloor)}}{\left(2z_{i+5-6\lfloor\frac{i+7}{6}\rfloor}x_{i+5-6\lfloor\frac{i+7}{6}\rfloor-1}-3\right)F_{3(p+\lfloor\frac{i+7}{6}\rfloor)}+F_{3(p+\lfloor\frac{i+7}{6}\rfloor+1)}}}{\frac{x_{i+4-6\lfloor\frac{i+6}{6}\rfloor}y_{i+4-6\lfloor\frac{i+6}{6}\rfloor-1}F_{3(p+\lfloor\frac{i+6}{6}\rfloor-1)}+\left(3x_{i+4-6\lfloor\frac{i+6}{6}\rfloor}y_{i+4-6\lfloor\frac{i+6}{6}\rfloor-1}+2\right)F_{3(p+\lfloor\frac{i+6}{6}\rfloor)}}{\left(2x_{i+4-6\lfloor\frac{i+6}{6}\rfloor}y_{i+4-6\lfloor\frac{i+6}{6}\rfloor-1}-3\right)F_{3(p+\lfloor\frac{i+6}{6}\rfloor)}+F_{3(p+\lfloor\frac{i+6}{6}\rfloor+1)}}}$$

$$\times\frac{\frac{y_{i+3-6\lfloor\frac{i+5}{6}\rfloor}z_{i+3-6\lfloor\frac{i+5}{6}\rfloor-1}F_{3(p+\lfloor\frac{i+5}{6}\rfloor-1)}+\left(3y_{i+3-6\lfloor\frac{i+5}{6}\rfloor}z_{i+3-6\lfloor\frac{i+5}{6}\rfloor-1}+2\right)F_{3(p+\lfloor\frac{i+5}{6}\rfloor)}}{\left(2y_{i+3-6\lfloor\frac{i+5}{6}\rfloor}z_{i+3-6\lfloor\frac{i+5}{6}\rfloor-1}-3\right)F_{3(p+\lfloor\frac{i+5}{6}\rfloor)}+F_{3(p+\lfloor\frac{i+5}{6}\rfloor+1)}}}{\frac{z_{i+2-6\lfloor\frac{i+4}{6}\rfloor}x_{i+2-6\lfloor\frac{i+4}{6}\rfloor-1}F_{3(p+\lfloor\frac{i+4}{6}\rfloor-1)}+\left(3z_{i+2-6\lfloor\frac{i+4}{6}\rfloor}x_{i+2-6\lfloor\frac{i+4}{6}\rfloor-1}+2\right)F_{3(p+\lfloor\frac{i+4}{6}\rfloor)}}{\left(2z_{i+2-6\lfloor\frac{i+4}{6}\rfloor}x_{i+2-6\lfloor\frac{i+4}{6}\rfloor-1}-3\right)F_{3(p+\lfloor\frac{i+4}{6}\rfloor)}+F_{3(p+\lfloor\frac{i+4}{6}\rfloor+1)}}}$$

$$\times \frac{\dfrac{x_{i+1-6\lfloor \frac{i+3}{6} \rfloor} y_{i+1-6\lfloor \frac{i+3}{6} \rfloor -1} F_{3\left(p+\lfloor \frac{i+3}{6} \rfloor -1\right)} + \left(3x_{i+1-6\lfloor \frac{i+3}{6} \rfloor} y_{i+1-6\lfloor \frac{i+3}{6} \rfloor -1} + 2\right) F_{3\left(p+\lfloor \frac{i+3}{6} \rfloor\right)}}{\left(2x_{i+1-6\lfloor \frac{i+3}{6} \rfloor} y_{i+1-6\lfloor \frac{i+3}{6} \rfloor -1} - 3\right) F_{3\left(p+\lfloor \frac{i+3}{6} \rfloor\right)} + F_{3\left(p+\lfloor \frac{i+3}{6} \rfloor +1\right)}}}{\dfrac{y_{i-6\lfloor \frac{i+2}{6} \rfloor} z_{i-6\lfloor \frac{i+2}{6} \rfloor -1} F_{3\left(p+\lfloor \frac{i+2}{6} \rfloor -1\right)} + \left(3y_{i-6\lfloor \frac{i+2}{6} \rfloor} z_{i-6\lfloor \frac{i+2}{6} \rfloor -1} + 2\right) F_{3\left(p+\lfloor \frac{i+2}{6} \rfloor\right)}}{\left(2y_{i-6\lfloor \frac{i+2}{6} \rfloor} z_{i-6\lfloor \frac{i+2}{6} \rfloor -1} - 3\right) F_{3\left(p+\lfloor \frac{i+2}{6} \rfloor\right)} + F_{3\left(p+\lfloor \frac{i+2}{6} \rfloor +1\right)}}},$$

for $m \in \mathbb{N}_0$, $i = \overline{-2, 3}$, where F_n is n-th Fibonacci number.

5. Conclusion

In this paper, we have considered the following three-dimensional system of difference equations

$$x_n = \frac{ax_{n-3}z_{n-2} + b}{cy_{n-1}z_{n-2}x_{n-3}}, \ y_n = \frac{ay_{n-3}x_{n-2} + b}{cz_{n-1}x_{n-2}y_{n-3}}, \ z_n = \frac{az_{n-3}y_{n-2} + b}{cx_{n-1}y_{n-2}z_{n-3}}, \ n \in \mathbb{N}_0,$$

where the parameters a, b, c and the initial values x_{-j}, y_{-j}, z_{-j}, $j \in \{1, 2, 3\}$, are real numbers. Firstly, we have obtained the explicit form of solutions of the aforementioned system using suitable transformation reducing to the equations in Riccati type. Also, we have solved the solutions in 3 different cases depending on whether the parameters are zero or non-zero. Moreover, numerical examples are given to demonstrate the theoretical results. In addition, the solutions of this system are related to Fibanacci numbers for the case $a = b = c = 1$.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have contributed sufficiently in the planning, execution or analysis of this study to be included as authors. All authors have read and agreed to the published version of manuscript.

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