Solutions Formulas for Three-dimensional Difference Equations System with Constant Coefficients

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Abstract. In this paper, we study the following three-dimensional system of difference equations

\[ x_n = \frac{a x_{n-1} z_{n-2} + b}{c x_{n-1} z_{n-2} y_{n-3}}, \quad y_n = \frac{a y_{n-1} x_{n-2} + b}{c y_{n-1} x_{n-2} z_{n-3}}, \quad z_n = \frac{a z_{n-1} y_{n-2} + b}{c z_{n-1} y_{n-2} x_{n-3}}, \quad n \in \mathbb{N}_0, \]

where the parameters \(a, b, c\) and the initial values \(x_{-j}, y_{-j}, z_{-j}, \quad j \in \{1, 2, 3\}\), are real numbers. We solve aforementioned system in explicit form. Then, we investigate the solutions in 3 different cases depending on whether the parameters are zero or non-zero. In addition, numerical examples are given to demonstrate the theoretical results. Finally, an application is given for solutions are related to Fibonacci numbers when \(a = b = c = 1\).

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1. Introduction

Firstly, remind that \(\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}, \mathbb{C}\), stand for natural, non-negative integer, integer, real and complex numbers, respectively. If \(m, n \in \mathbb{Z}, \quad m \leq n\), the notation \(i = \overline{m,n}\) stands for \([i \in \mathbb{Z} : m \leq i \leq n]\). The notation of \([i]\) means of \(n \leq i < n + 1, \quad n \in \mathbb{Z}\).

There are different types of difference equations. One of them is

\[ x_{n+1} = \frac{a x_n + b}{c x_n + d}, \quad n \in \mathbb{N}_0, \quad (1.1) \]

for \(c \neq 0, ad \neq bc\), where parameters \(a, b, c, d\) and the initial value \(x_0\) are real numbers, which called Riccati difference equation. Indeed, equation (1.1) has the general solution can be written in the following form

\[ x_n = \frac{x_0 \left(bc - ad\right) s_{n-1} + \left(ax_0 + b\right) s_n}{\left(cx_0 - a\right) s_n + s_{n+1}}, \quad n \in \mathbb{N}, \quad (1.2) \]

where \((s_n)_{n \in \mathbb{N}_0}\) is the sequence satisfying

\[ s_{n+1} - \left(a + d\right) s_n - \left(bc - ad\right) s_{n-1} = 0, \quad n \in \mathbb{N}, \]

where \(s_0 = 0, s_1 = 1\), in [24].

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The theory of difference equations or their systems is important. Especially, difference equation or their systems whose solutions are related to sequences of numbers has been attracted by many authors in recent years [1–14, 16–23, 25–32, 34–36].

For instance, in [33], authors obtained the solutions of the following two difference equations systems

\[
x_{n+1} = x_{n-1} \pm \frac{1}{y_n x_{n-1}}, \quad y_{n+1} = y_{n-1} \pm \frac{1}{x_n y_{n-1}}, \quad n \in \mathbb{N}_0.
\]

(1.3)

The solutions of systems in (1.3) are associated with Padovan numbers. In addition, the authors of [15] dealt with the solution, stability character and asymptotic behavior of the following rational difference equation

\[
x_{n+1} = \frac{\alpha x_{n-1} + \beta}{\gamma y_n x_{n-1}}, \quad n \in \mathbb{N}_0,
\]

where \(\alpha, \beta, \gamma \in \mathbb{R}^+\), the initial values \(x_{-1}, x_0\) are non-zero real numbers and they investigated the two-dimensional case of equation (1.4) given by

\[
x_{n+1} = \frac{\alpha x_{n-1} + \beta}{\gamma y_n x_{n-1}}, \quad y_{n+1} = \frac{\alpha y_{n-1} + \beta}{\gamma y_n y_{n-1}}, \quad n \in \mathbb{N}_0.
\]

(1.5)

The solutions of equation (1.4) found by the authors are related to Padovan numbers and the solution of system (1.5) are related to generalized Padovan numbers.

The motivation for above studies, we deal with the following system of difference equations

\[
x_n = \frac{a x_{n-3} - 3 x_{n-2} + b}{c y_{n-1} z_{n-2} x_{n-3}}, \quad y_n = \frac{a y_{n-3} x_{n-2} + b}{c z_{n-1} x_{n-2} y_{n-3}}, \quad z_n = \frac{a z_{n-3} y_{n-2} + b}{c x_{n-1} y_{n-2} z_{n-3}}, \quad n \in \mathbb{N}_0.
\]

(1.6)

where the parameters \(a, b, c\) and the initial values \(x_{-j}, y_{-j}, z_{-j}, j \in \{1, 2, 3\}\), are real numbers. We solve system (1.6) in explicit form. Then, we investigate the solutions in 3 different cases depending on whether the parameters are zero or non-zero. In addition, numerical examples are given to demonstrate the theoretical results. Finally, an application is given for solutions are related to Fibonacci numbers when \(a = b = c = 1\).

**Definition 1.1.** (Periodicity) A sequence \((x_n)_{n=-k}^\infty\) is said to be eventually periodic with period \(p\) if there exist \(n_0 \geq -k\) such that \(x_{n+p} = x_n\) for all \(n \geq n_0\). If \(n_0 = -k\) then the sequence \((x_n)_{n=-k}^\infty\) is said to be periodic with period \(p\).

2. Explicit Solution of the System (1.6)

Let \((x_n, y_n, z_n)_{n=-2}^{\infty}\) be a solution of system (1.6). If at least one of the initial values \(x_{-j}, y_{-j}, z_{-j}, j \in \{1, 2, 3\}\), is equal to zero, then the solution of system (1.6) is not defined. For example, if \(x_{-3} = 0\) and so \(x_0, y_2\) and \(z_4\) can not be calculated. Similarly, if \(y_{-3} = 0\) (or \(z_{-3} = 0\)) and so \(y_0, z_2\) and \(x_4\) (or \(z_0, x_2\) and \(y_4\)) can not be calculated. Thus, for every well-defined solution of system (1.6), we get that \(x_n, y_n, z_n \neq 0, n \geq -3\), if and only if \(x_{-j} y_{-j} z_{-j} \neq 0, j \in \{1, 2, 3\}\). Note that the system (1.6) can be written in the form

\[
x_n y_{n-1} = \frac{a}{c} + \frac{b}{cx_{n-2} x_{n-3}}, \quad y_n z_{n-1} = \frac{a}{c} + \frac{b}{cx_{n-2} y_{n-3}}, \quad z_n x_{n-1} = \frac{a}{c} + \frac{b}{cy_{n-2} z_{n-3}},
\]

(2.1)

for \(n \in \mathbb{N}_0\). Next, by employing the change of variables

\[
u_n = x_n y_{n-1}, \quad v_n = y_n z_{n-1}, \quad w_n = z_n x_{n-1}, \quad n \geq -2,
\]

(2.2)

system (2.1) is transformed into the following system

\[
u_n = \frac{a}{c} + \frac{b}{cw_{n-2}}, \quad v_n = \frac{a}{c} + \frac{b}{cu_{n-2}}, \quad w_n = \frac{a}{c} + \frac{b}{cv_{n-2}}, \quad n \in \mathbb{N}_0,
\]

which can be written as

\[
u_n = \frac{(a^3 + 2abc) \nu_{n-6} + a^2 b + b^2 c}{(a^2 c + bc^2) \nu_{n-6} + abc}, \quad n \geq 4,
\]

(2.3)

\[
v_n = \frac{(a^3 + 2abc) v_{n-6} + a^2 b + b^2 c}{(a^2 c + bc^2) v_{n-6} + abc}, \quad n \geq 4,
\]

(2.4)
\[ w_n = \frac{(a^3 + 2abc)w_{n-6} + a^2b + b^2c}{(a^2c + bc^2)w_{n-6} + abc}, \quad n \geq 4. \] (2.5)

Now we consider the following equation
\[ t_n = \frac{(a^3 + 2abc)t_{n-6} + a^2b + b^2c}{(a^2c + bc^2)t_{n-6} + abc}, \quad n \geq 4, \] (2.6)

instead of equations in (2.3)-(2.5). If we apply the decomposition of indices \( n \to 6(m+1) + i, \ i = -2,3 \) and \( m \geq -1, \) to (2.6), then it can be written as follows
\[ t_{m+1}^{(i)} = \frac{(a^3 + 2abc)t_{m}^{(i)} + a^2b + b^2c}{(a^2c + bc^2)t_{m}^{(i)} + abc}, \] (2.7)

where \( t_{m}^{(i)} = t_{6m+i}, \ m \in \mathbb{N}_0, \ i = -2,3. \)

Let \( A_1 := a^3 + 2abc, B_1 := a^2b + b^2c, C_1 := a^2c + bc^2, D_1 := abc. \)

From equation (1.2), the general solutions of (2.7) follows straightforwardly as
\[ t_{m}^{(i)} = \frac{b^3c^3t_{0}^{(i)}s_{m-1} + (A_1t_{0}^{(i)} + B_1)s_{m}}{(C_1t_{0}^{(i)} - A_1)s_{m} + s_{m+1}}, \quad m \in \mathbb{N}_0, \] (2.8)

for \( i = -2,3, \) where sequence of \( (s_{m})_{m \in \mathbb{N}_0} \) is satisfying
\[ s_{m+1} - (a^3 + 3abc)s_{m} - b^3c^3s_{m-1} = 0, \quad m \in \mathbb{N}, \] (2.9)

recurrence relation with \( s_0 = 0, s_1 = 1. \) Note that by using the recurrence relation in (2.9), one can compute
\[ s_{1} = \frac{s_1 - (a^3 + 3abc)s_0}{b^3c^3}. \]

We use (2.2) in (2.3)-(2.5) and from (2.8), equations in (2.3)-(2.5) are expressed as
\[ u_{6m+i} = \frac{b^3c^3x_{i}y_{i-1}s_{m-1} + (A_1x_{i}y_{i-1} + B_1)s_{m}}{(C_1x_{i}y_{i-1} - A_1)s_{m} + s_{m+1}}, \quad m \in \mathbb{N}_0, \] (2.10)
\[ v_{6m+i} = \frac{b^3c^3y_{i}z_{i-1}s_{m-1} + (A_1y_{i}z_{i-1} + B_1)s_{m}}{(C_1y_{i}z_{i-1} - A_1)s_{m} + s_{m+1}}, \quad m \in \mathbb{N}_0, \] (2.11)
\[ w_{6m+i} = \frac{b^3c^3z_{i}x_{i-1}s_{m-1} + (A_1z_{i}x_{i-1} + B_1)s_{m}}{(C_1z_{i}x_{i-1} - A_1)s_{m} + s_{m+1}}, \quad m \in \mathbb{N}_0, \] (2.12)

for \( i = -2,3. \)

From (2.2), we have that
\[ x_n = \frac{u_n}{y_{n-1}} = \frac{u_nz_{n-2}}{v_{n-1}} = \frac{u_nW_{n-2}Y_{n-2}}{V_{n-1}} = \frac{u_nW_{n-2}Z_{n-3}}{V_{n-1}} = \frac{u_nW_{n-2}V_{n-4}}{V_{n-1}U_{n-3}Z_{n-5}} = \frac{u_nW_{n-2}V_{n-4}}{V_{n-1}U_{n-3}W_{n-5}}, \quad n \geq 3, \] (2.13)
\[ y_n = \frac{v_n}{z_{n-1}} = \frac{v_nX_{n-2}}{w_{n-1}} = \frac{v_nU_{n-2}Z_{n-3}}{W_{n-1}} = \frac{v_nU_{n-2}X_{n-4}}{W_{n-1}V_{n-3}X_{n-5}} = \frac{v_nU_{n-2}X_{n-4}}{W_{n-1}V_{n-3}Z_{n-5}}, \quad n \geq 3, \] (2.14)
\[ z_n = \frac{w_n}{x_{n-1}} = \frac{w_nY_{n-2}}{U_{n-1}} = \frac{w_nV_{n-2}X_{n-3}}{U_{n-1}W_{n-3}} = \frac{w_nV_{n-2}X_{n-3}}{U_{n-1}W_{n-3}Z_{n-5}}, \quad n \geq 3. \] (2.15)

From (2.13)-(2.15), we get
\[ x_{6m+j} = \frac{u_{6m+j}W_{6m+j-2}Y_{6m+j-4}}{V_{6m+j-1}U_{6m+j-3}W_{6m+j-5}}, \quad m \in \mathbb{N}_0, \] (2.16)
\[ y_{6m+j} = \frac{v_{6m+j}U_{6m+j-2}W_{6m+j-4}}{W_{6m+j-1}V_{6m+j-3}U_{6m+j-5}}, \quad m \in \mathbb{N}_0, \] (2.17)
\[ z_{6m+j} = \frac{w_{6m+j}V_{6m+j-2}U_{6m+j-4}}{W_{6m+j-1}V_{6m+j-3}U_{6m+j-5}}, \quad m \in \mathbb{N}_0. \] (2.18)
for \( j = \frac{3}{8} \).

Multiplying the equalities which are obtained from (2.16)-(2.18), from 0 to \( m \), it follows that

\[
X_{6m+i+5} = x_{i-1} \prod_{p=0}^{m} \frac{U_6(p+\frac{i}{6}, \frac{p}{6}) + i + 5 - 6(-\frac{i}{6})}{V_6(p+\frac{i}{6}, \frac{p}{6}) + i + 3 - 6(-\frac{i}{6})} \frac{V_6(p+\frac{i}{6}, \frac{p}{6}) + i + 1 - 6(-\frac{i}{6})}{U_6(p+\frac{i}{6}, \frac{p}{6}) + i - 6(-\frac{i}{6})}, \tag{2.19}
\]

\[
Y_{6m+i+5} = y_{i-1} \prod_{p=0}^{m} \frac{V_6(p+\frac{i}{6}, \frac{p}{6}) + i + 5 - 6(-\frac{i}{6})}{W_6(p+\frac{i}{6}, \frac{p}{6}) + i + 4 - 6(-\frac{i}{6})} \frac{W_6(p+\frac{i}{6}, \frac{p}{6}) + i + 2 - 6(-\frac{i}{6})}{V_6(p+\frac{i}{6}, \frac{p}{6}) + i - 6(-\frac{i}{6})}, \tag{2.20}
\]

\[
Z_{6m+i+5} = z_{i-1} \prod_{p=0}^{m} \frac{W_6(p+\frac{i}{6}, \frac{p}{6}) + i + 5 - 6(-\frac{i}{6})}{U_6(p+\frac{i}{6}, \frac{p}{6}) + i + 4 - 6(-\frac{i}{6})} \frac{U_6(p+\frac{i}{6}, \frac{p}{6}) + i + 2 - 6(-\frac{i}{6})}{W_6(p+\frac{i}{6}, \frac{p}{6}) + i - 6(-\frac{i}{6})}, \tag{2.21}
\]

where \( m \in \mathbb{N}_0, i = -\frac{2}{3} \). By substituting the formulas in (2.10)-(2.12) into (2.19)-(2.21), we obtain

\[
X_{6m+i+5} = x_{i-1} \prod_{p=0}^{m} \frac{b^i c_i x_{i+6-6(-\frac{i}{6})}}{c_i x_{i+5-6(-\frac{i}{6})}} \left( A_{i-1} x_{i+6-6(-\frac{i}{6})} - A_i x_{i+6-6(-\frac{i}{6})} + B_i \right) y_{p+\frac{i}{6}, \frac{p}{6}} \tag{2.22}
\]

\[
Y_{6m+i+5} = y_{i-1} \prod_{p=0}^{m} \frac{b^i c_i y_{i+6-6(-\frac{i}{6})}}{c_i y_{i+5-6(-\frac{i}{6})}} \left( A_{i-1} y_{i+6-6(-\frac{i}{6})} - A_i y_{i+6-6(-\frac{i}{6})} + B_i \right) x_{p+\frac{i}{6}, \frac{p}{6}} \tag{2.23}
\]
Consider the system (1.6) with the initial values

\[ x_{0} = 0.1, \quad x_{-1} = 2.4, \quad x_{-2} = 3, \quad y_{0} = 4.06, \quad y_{-1} = 0.05, \quad y_{-2} = 0.05, \quad z_{0} = 70.54, \quad z_{-1} = 0.86, \quad z_{-2} = 9.05 \]

and the parameters \( a = 0 \), \( b = 1 \), \( c = 2 \), the solutions are represented as in the following figures.

Now, we present numerical example that represent the solutions of system (1.6) when

\[ m = 0 \text{ or } i = -2, 3. \]

### 3. Particular Cases of System (1.6)

Now, we will examine the solutions in 3 different cases depending on whether the parameters are zero or non-zero.

#### 3.1. Case \( a = 0, b \neq 0 \neq c \)

In this case, system (1.6) becomes

\[
\begin{align*}
    x_{n} &= \frac{b}{c y_{n-1} z_{n-2} x_{n-3}}, \\
    y_{n} &= \frac{b}{c z_{n-1} y_{n-2} y_{n-3}}, \\
    z_{n} &= \frac{b}{c x_{n-1} y_{n-2} z_{n-3}},
\end{align*}
\]

for \( n \in \mathbb{N}_0 \).

From (3.1), we obtain

\[ x_{n} = x_{n-12}, \quad y_{n} = y_{n-12}, \quad z_{n} = z_{n-12}, \quad n \geq 9. \]  

(3.2)

We can write the solutions of equations in (3.2) as in the following form

\[ x_{12m+j} = x_{j-12}, \quad y_{12m+j} = y_{j-12}, \quad z_{12m+j} = z_{j-12}, \quad m \in \mathbb{N}_0, \quad j = 0, 1, 2, \ldots, 24. \]

Now, we present numerical example that represent the solutions of system (1.6) when \( a = 0, b \neq 0 \neq c \).

**Example 3.1.** Consider the system (1.6) with the initial values \( x_{-3} = 0.1, \ x_{-2} = 2.4, \ x_{-1} = 3, \ y_{-3} = 4.06, \ y_{-2} = 0.05, \ y_{-1} = 0.6, \ z_{-3} = 70.54, \ z_{-2} = 0.86, \ z_{-1} = 9.05 \) and the parameters \( a = 0, \ b = 1, \ c = 2 \), the solutions are represented as in the following figures.
Therefore, the solutions of system (1.6) are periodic with period 12.

3.2. **Case** $b = 0$, $a \neq 0 \neq c$. In this case, system (1.6) reduces to the following system

$$
x_n = \frac{a}{c} y_{n-1}, \quad y_n = \frac{a}{c} z_{n-1}, \quad z_n = \frac{a}{c} x_{n-1}, \quad n \in \mathbb{N}_0.
$$

(3.3)

From (3.3), we obtain

$$
x_n = x_{n-6}, \quad y_n = y_{n-6}, \quad z_n = z_{n-6}, \quad n \geq 3.
$$

(3.4)

We can write the solutions of equations in (3.4) as in the following form

$$
x_{6m+i} = x_{i-6}, \quad y_{6m+i} = y_{i-6}, \quad z_{6m+i} = z_{i-6}, \quad m \in \mathbb{N}_0, \quad i = 5, 10.
$$

Now, we present numerical example that represent the solutions of system (1.6) when $b = 0$, $a \neq 0 \neq c$.

**Example 3.2.** Consider the system (1.6) with the initial values $x_{-3} = 1$, $x_{-2} = 2$, $x_{-1} = 3$, $y_{-3} = 4$, $y_{-2} = 5$, $y_{-1} = 6$, $z_{-3} = 7$, $z_{-2} = 8$, $z_{-1} = 9$ and the parameters $a = 1$, $b = 0$, $c = 2$, the solutions are represented as in the following figures.
Therefore, the solutions of system (1.6) are eventually periodic with period 6.

3.3. Case \( c = 0, a \neq 0 \neq b \). In this case, system (1.6) does not have well-defined solutions.

4. An Application

Firstly, we will derive the solution forms of the system (1.6) with \( a = b = c = 1 \), that is, the system

\[
\begin{align*}
x_n &= x_{n-3}z_{n-2} + 1, \\
y_n &= y_{n-3}x_{n-2} + 1, \\
z_n &= z_{n-3}y_{n-2} + 1,
\end{align*}
\]

\( n \in \mathbb{N}_0 \).

From (2.9), we obtain

\[
s_{m+1} - 4s_m - s_{m-1} = 0, \quad m \in \mathbb{N},
\]

where \( s_0 = 0, s_1 = 1 \).

Binet Formula for (4.1) is

\[
s_m = \frac{(2 + \sqrt{5})^m - (2 - \sqrt{5})^m}{(2 + \sqrt{5}) - (2 - \sqrt{5})}, \quad m \in \mathbb{N}_0.
\]

Note that

\[
\left( \frac{1 \pm \sqrt{5}}{2} \right)^3 = 2 \pm \sqrt{5}.
\]

Using this in (4.2), we obtain

\[
s_m = \frac{(1 + \sqrt{5})^{3m} - (1 - \sqrt{5})^{3m}}{2} = f_{3m}, \quad m \in \mathbb{N}_0.
\]

Using (4.3) into (2.22)-(2.24), we have
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\[ X_{m+i+5} = X_{i-1} \prod_{p=0}^{m} \left( 5_{i+6-0+4}^{i+6-1} F_{i+p+1}^{(i+6-1)} \right)^{\left(3_{i+6-0+4}^{i+6-1} Y_{i+6-0+4}^{i+6-1} + 3^{i+6-0+4} F_{i+p+1}^{(i+6-1)} \right) + F_{i+p+1}^{(i+6-1)}} \]

\[ Y_{m+i+5} = Y_{i-1} \prod_{p=0}^{m} \left( 5_{i+6-0+4}^{i+6-1} F_{i+p+1}^{(i+6-1)} \right)^{\left(3_{i+6-0+4}^{i+6-1} Y_{i+6-0+4}^{i+6-1} + 3^{i+6-0+4} F_{i+p+1}^{(i+6-1)} \right) + F_{i+p+1}^{(i+6-1)}} \]

\[ Z_{m+i+5} = Z_{i-1} \prod_{p=0}^{m} \left( 5_{i+6-0+4}^{i+6-1} F_{i+p+1}^{(i+6-1)} \right)^{\left(3_{i+6-0+4}^{i+6-1} Y_{i+6-0+4}^{i+6-1} + 3^{i+6-0+4} F_{i+p+1}^{(i+6-1)} \right) + F_{i+p+1}^{(i+6-1)}} \]
All authors have read and agreed to the published version of manuscript. Moreover, numerical examples are given to demonstrate the theoretical results. In addition, the solutions of Riccati type. Also, we have solved the solutions in 3 dimensions for \( M. \) Kara, Y. Yazlik, Turk. J. Math. Comput. Sci., 14(1)(2022), 107–116

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5. Conclusion

In this paper, we have considered the following three-dimensional system of difference equations

\[
\begin{align*}
x_n &= \frac{ax_{n-3} + b}{cy_{n-2} + z_{n-3}}, \quad y_n = \frac{ay_{n-3} + b}{cz_{n-2} + y_{n-3}}, \quad z_n = \frac{az_{n-3} + b}{cx_{n-2} + y_{n-3}}, \quad n \in \mathbb{N}_0,
\end{align*}
\]

where the parameters \( a, b, c \) and the initial values \( x_{-j}, y_{-j}, z_{-j}, \ j \in \{1, 2, 3\} \), are real numbers. Firstly, we have obtained the explicit form of solutions of the aforementioned system using suitable transformation reducing to the equations in Riccati type. Also, we have solved the solutions in 3 different cases depending on whether the parameters are zero or non-zero. Moreover, numerical examples are given to demonstrate the theoretical results. In addition, the solutions of this system are related to Fibonacci numbers for the case \( a = b = c = 1. \)

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have contributed sufficiently in the planning, execution or analysis of this study to be included as authors.

All authors have read and agreed to the published version of manuscript.

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