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Research Article

On matching distance between eigenvalues of unbounded operators

MICHAEL GIL*

ABSTRACT. Let A and $\tilde{A}$ be linear operators on a Banach space having compact resolvents, and let $\lambda_k(A)$ and $\lambda_k(\tilde{A})$ $(k = 1, 2, ...)$ be the eigenvalues taken with their algebraic multiplicities of A and $\tilde{A}$, respectively. Under some conditions, we derive a bound for the quantity

$$\mathrm{md}(A, \tilde{A}) := \inf_{\pi} \sup_{k=1,2,\dots} |\lambda_{\pi(k)}(\tilde{A}) - \lambda_k(A)|,$$

where π is taken over all permutations of the set of all positive integers. That quantity is called the matching optimal distance between the eigenvalues of A and $\tilde{A}$. Applications of the obtained bound to matrix differential operators are also discussed.

Keywords: Banach space, perturbations of eigenvalues, matching distance, differential operator, tensor product of Hilbert spaces.

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1. INTRODUCTION

Let $\mathcal{X}$ be a Banach space with the unit operator $I = I_{\mathcal{X}}$ and norm $\|.\|$. For a linear operator B, $\sigma(B)$ denotes the spectrum, B^{-1} is the inverse operator, $R_z(B) = (B - zI)^{-1}$ $(z \notin \sigma(B))$ is the resolvent, $\|B\|$ is the operator norm, if B is bounded; B^* is the adjoint operator, $D(B)$ is the domain and

$$d(B, z) := \inf_{s \in \sigma(B)} |s - z|, \quad z \in \mathbb{C}.$$

Throughout this paper, A and $\tilde{A}$ are linear operators on $\mathcal{X}$ having compact resolvents. So A and $\tilde{A}$ can have root vectors and all their eigenspaces are finite dimensional.

Let $\lambda_k(A)$ and $\lambda_k(\tilde{A})$ $(k = 1, 2, ...)$ be the eigenvalues of A and $\tilde{A}$, respectively, enumerated with their algebraic multiplicities taken into account. Introduce the following quantity (called the matching optimal distance between the eigenvalues of A and $\tilde{A}$):

$$\mathrm{md}(A, \tilde{A}) := \inf_{\pi} \sup_{i=1,2,\dots} |\lambda_{\pi(i)}(\tilde{A}) - \lambda_i(A)|,$$

where π is taken over all permutations of the set of all positive integers.

Our definition of $\mathrm{md}(A, \tilde{A})$ is a natural generalization of the well-known definition from the perturbation theory of finite matrices [19, p. 167].

The present paper is devoted to estimating $\mathrm{md}(A, \tilde{A})$. The perturbation theory of operators is very rich. The classical results are presented in the book [15], the recent results can be

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*Corresponding author: Michael Gil; gilmi@bezeqint.net
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found in [1]-[5], [7, 8, 9], [11], [12, 13, 14, 18] and references, which are given therein, but to the best of our knowledge, the matching optimal distance for infinite dimensional operators was not investigated in the available literature although it is important for the localization of the spectrum.

Below we suggest a bound for $\mathrm{md}(A, \tilde{A})$ assuming that

$$D(A) = D(\tilde{A}) \text{ and } q := \|A - \tilde{A}\| < \infty. \tag{1.1}$$

As a particular case, we consider a class of operators on the tensor product of a Hilbert space and a finite dimensional one. We also discuss applications of our results to matrix differential operators.

2. Operators on a Banach space

In the sequel, by $\hat{\lambda}_k(A)$ $(k = 1, 2, ...)$, we denote the distinct eigenvalues of A and assume that

$$r_0(A) := \inf_{j\neq k; j,k=1,2,...} \frac{|\hat{\lambda}_k(A) - \hat{\lambda}_j(A)|}{2} > 0.$$

Since A has a compact resolvent, if $\sigma(A)$ does not contain limit points, one can wait that this condition holds. If $\sigma(A)$ contains limit points, then $r_0(A) = 0$.

Put $r_j := \inf\limits_{k\neq j} \frac{|\hat{\lambda}_k(A) - \hat{\lambda}_j(A)|}{2}$, $j = 1, 2, ...,$

$$\Omega(c, r) := \{z \in \mathbb{C} : |z - c| < r\}, \quad c \in \mathbb{C}, r > 0$$

and

$$C(c, r) := \{z \in \mathbb{C} : |z - c| = r\}, \quad c \in \mathbb{C}, r > 0.$$

By $\nu_k(A)$, we denote the algebraic multiplicity of each $\hat{\lambda}_k(A)$.

Lemma 2.1. *Let conditions (1.1) hold and for an integer j and a positive number $\hat{r}_j \leq r_j$, let*

$$q \sup_{z\in C(\hat{\lambda}_j(A),\hat{r}_j)} \|R_z(A)\| < 1. \tag{2.2}$$

Then, $\tilde{A}$ has in $\Omega(\hat{\lambda}_j(A), r_j)$ eigenvalues whose total algebraic multiplicity is equal to $\nu_j(A)$.

Proof. This result is a particular case of the well-known one [15, Theorem IV.3.18]. □

Assume that

$$\|R_\lambda(A)\| \leq \phi(1/d(A, \lambda)), \quad \text{for all } \lambda \notin \sigma(A), \tag{2.3}$$

where $\phi(x)$ is a continuous monotonically increasing non-negative function of a non-negative variable x, such that $\phi(0) = 0$ and $\phi(\infty) = \infty$.

Let conditions (1.1) and (2.3) hold, and let there be a positive number $\hat{r}_0 \leq r_0(A)$, such that

$$q\phi(1/\hat{r}_0) < 1. \tag{2.4}$$

Then, $\sigma(\tilde{A})$ lies in the set $\cup_{j=1}^{\infty}\Omega(\hat{\lambda}_j(A), \hat{r}_0)$. Indeed, assume that an eigenvalue $\tilde{\lambda}$ of $\tilde{A}$ does not belong to this set. Then for the eigenvalue $\hat{\lambda}_j(A)$ of A nearest to $\tilde{\lambda}$, we have $t = |\tilde{\lambda} - \hat{\lambda}_j(A)| \geq \hat{r}_j$. Thus

$$q\|R_{\tilde{\lambda}}(A)\| \leq q\phi(1/t) \leq q\phi(1/\hat{r}_0) < 1.$$

According to [15, Theorem IV.1.16], $\tilde{\lambda} \notin \sigma(\tilde{A})$. Hence, due to Lemma 2.1, we arrive at the following result.

Corollary 2.1. *Let conditions* (1.1) *and* (2.3) *hold, and let there be a positive number* $\hat{r}_0 \leq r_0(A)$, *such that inequality* (2.4) *is fulfilled. Then,* $\sigma(\tilde{A})$ *lies in the set* $\cup_{j=1}^{\infty}\Omega(\hat{\lambda}_j(A), \hat{r}_0)$. *Moreover, in each* $\Omega(\hat{\lambda}_j(A), \hat{r}_0)$ $(j = 1, 2, ...)$ *operator* $\tilde{A}$ *has the eigenvalues, whose total algebraic multiplicity is equal to* $\nu_j(A)$, *and therefore* $\mathrm{md}(A, \tilde{A}) \leq \hat{r}_0$.

Denote by $x(q)$ the unique positive root of the equation

$$q\phi(1/z) = 1. \tag{2.5}$$

Theorem 2.1. *Let conditions* (1.1) *and* (2.3) *hold, and let* $x(q) < r_0(A)$. *Then*

$$\sigma(\tilde{A}) \subset \cup_{j=1}^{\infty}\Omega(\hat{\lambda}_j(A), x(q)).$$

Moreover, the total algebraic multiplicity of the eigenvalues of $\tilde{A}$, *lying in each* $\Omega(\hat{\lambda}_j(A), x(q))$ $(j = 1, 2, ...)$ *is equal to the algebraic multiplicity* $\nu_j(A)$ *of* $\hat{\lambda}_j(A)$, *and consequently* $\mathrm{md}(A, \tilde{A}) \leq x(q)$.

Proof. Since ϕ is an increasing function, for any $\hat{r}_0 \in (x(q), r_0(A))$, we have

$$q\phi(1/\hat{r}_0) < q\phi(1/x(q)) = 1.$$

So, inequality (2.4) is fulfilled. Now, making use of Corollary 2.1, we arrive at the required result. □

3. Operators on the tensor product of a Hilbert space and a finite dimensional one

Throughout this section, $\mathcal{E}$ is a separable Hilbert space with a scalar product $\langle ., . \rangle_{\mathcal{E}}$ and the norm $\|.\|_{\mathcal{E}} = \sqrt{\langle ., . \rangle_{\mathcal{E}}}$, $\mathbb{C}^n$ is the n-dimensional complex Euclidean space with a scalar product $\langle ., . \rangle_n$ and the Euclidean norm $\|.\|_n = \sqrt{\langle ., . \rangle_n}$. Recall the definition of the tensor product $\mathcal{H} = \mathcal{E} \otimes \mathbb{C}^n$ of $\mathcal{E}$ and $\mathbb{C}^n$. To this end, consider the collection of all formal finite sums of the form

$$u = \sum_j y_j \otimes h_j \quad (y_j \in \mathcal{E}, h_j \in \mathbb{C}^n)$$

with the understanding that

$$\lambda(y \otimes h) = (\lambda y) \otimes h = y \otimes (\lambda h), \quad (y + y_1) \otimes h = y \otimes h + y_1 \otimes h,$$

$$y \otimes (h + h_1) = y \otimes h + y \otimes h_1, \quad y, y_1 \in \mathcal{E};\ h, h_1 \in \mathbb{C}^n;\ \lambda \in \mathbb{C}.$$

On that collection define the scalar product as

$$\langle h \otimes y, h_1 \otimes y_1 \rangle_{\mathcal{H}} = \langle y, y_1 \rangle_{\mathcal{E}}\ \langle h, h_1 \rangle_n, \quad y, y_1 \in \mathcal{E}; h, h_1 \in \mathbb{C}^n$$

and the cross norm is defined by $\|.\|_{\mathcal{H}} = \sqrt{\langle ., . \rangle_{\mathcal{H}}}$. Then, $\mathcal{H}$ is the completion of the considered collection in the norm $\|.\|_{\mathcal{H}}$. Besides, $I_{\mathcal{H}}, I_{\mathcal{E}}$ and I_n are the unit operators in $\mathcal{H}, \mathcal{E}$ and $\mathbb{C}^n$, respectively. From the theory of tensor products, we need only elementary facts which can be found in [6].

Note that the class of operators with compact resolvents is closed under taking the tensor product.

Everywhere below M is an $n \times n$-matrix and S is a normal operator on $\mathcal{E}$ with a compact resolvent. We will consider perturbations of the operator

$$A = S \otimes I_n + I_{\mathcal{E}} \otimes M. \tag{3.6}$$

Let $\hat{\lambda}_k(M)$ $(k = 1, ..., m \le n)$ be the distinct eigenvalues of M with the algebraic multiplicities $\nu_k(M)$: $\hat{\lambda}_k(M) \ne \hat{\lambda}_j(M)$ $(j \ne k)$ and $\hat{\lambda}_j(S)$ $(j = 1, 2, ...)$ be the distinct eigenvalues of S with multiplicities $\nu_j(S)$:

$$S = \sum_{j=1}^{\infty} \hat{\lambda}_j(S) P_j,$$

where P_j are the (mutually orthogonal and finite dimensional) eigen-projections of S. Since

$$I_{\mathcal{E}} = \sum_{k=1}^{\infty} P_k,$$

we have

$$A = \sum_{k=1}^{\infty} \hat{\lambda}_k(S) P_k \otimes I_n + M \otimes I_{\mathcal{E}} = \sum_{k=1}^{\infty} P_k \otimes (\hat{\lambda}_k(S) I_n + M).$$

Hence

$$(A - zI_{\mathcal{H}})^{-1} = \sum_{k=1}^{\infty} P_k \otimes ((\hat{\lambda}_k(S) - z) I_n + M)^{-1}$$

and therefore,

$$\|(A - zI_{\mathcal{H}})^{-1}\| = \sup_k \|((\hat{\lambda}_k(S) - z) I_n + M)^{-1}\|_n. \tag{3.7}$$

Here and below, $\|C\|_n$ means the spectral matrix norm (the operator norm with respect to the Euclidean vector norm) of a matrix C.

Any eigenvalue of A can be written as

$$\hat{\lambda}_{jk}(A) = \hat{\lambda}_j(S) + \hat{\lambda}_k(M), \quad j = 1, 2...; k = 1, ..., m.$$

Assume that

$$r_0(A) = \inf\{|\hat{\lambda}_j(S) + \hat{\lambda}_k(M) - \hat{\lambda}_{j_1}(S) - \hat{\lambda}_{k_1}(M)| :$$

$$j \ne j_1, k \ne k_1; j, j_1 = 1, 2, ...; k_1, k = 1, ..., m\} > 0. \tag{3.8}$$

Denote by $\|M\|_F$ the Frobenius norm of M : $\|M\|_F := (\text{trace } M^*M)^{1/2}$. The following quantity plays an essential role hereafter:

$$g(M) := [\|M\|_F^2 - \sum_{k=1}^{m} \nu_k(M) |\hat{\lambda}_k(M)|^2]^{1/2}.$$

The following properties of $g(M)$ are checked in [10, Section 3.1]. If M is normal, then $g(M) = 0$. In addition,

$$g(e^{it} M + zI_n) = g(M), \quad t \in \mathbb{R}; z \in \mathbb{C} \tag{3.9}$$

and

$$g^2(M) \le 2\|M_I\|_F^2 \;\; (M_I = (M - M^*)/2i), \text{ and } g^2(M) \le \|M\|_F^2 - |\text{trace } M^2|.$$

Due to [10, Theorem 3.2], for any $n \times n$-matrix M, one has

$$\|R_\lambda(M)\|_n \le \sum_{k=0}^{n-1} \frac{g^k(M)}{\sqrt{k!}\, d^{k+1}(M, \lambda)}, \quad \lambda \notin \sigma(M). \tag{3.10}$$

This inequality is sharp: if M is normal, then $g(M) = 0$ and with $0^0 = 1$ (3.10) is attained: $\|R_\lambda(M)\|_n = \frac{1}{d(M,\lambda)}$.

According to (3.7) and (3.10),

$$\|(A - zI_{\mathcal{H}})^{-1}\|_{\mathcal{H}} \le \sup_j \|((\hat{\lambda}_j(S) - z)I_n + M)^{-1}\|_n$$

$$\le \sup_j \sum_{k=0}^{n-1} \frac{g^k(M)}{\sqrt{k!}d^{k+1}(M, z - \hat{\lambda}_j(S))}, \quad z - \hat{\lambda}_j(S) \notin \sigma(M).$$

But

$$d(M, z - \hat{\lambda}_j(S)) \ge \inf_{j,k} |z - \hat{\lambda}_{jk}(A)| = d(z, A).$$

Thus

$$\|(A - zI_{\mathcal{H}})^{-1}\|_{\mathcal{H}} \le \sum_{k=0}^{n-1} \frac{g^k(M)}{\sqrt{k!}d^{k+1}(A, z)}, \quad z \notin \sigma(A). \tag{3.11}$$

So, we can take

$$\phi(x) = \sum_{k=0}^{n-1} \frac{g^k(M)x^{k+1}}{\sqrt{k!}}.$$

Besides, equation (2.5) has the form

$$q \sum_{k=0}^{n-1} \frac{g^k(M)}{z^{k+1}\sqrt{k!}} = 1.$$

This equation is equivalent to the equation

$$z^n = q \sum_{k=0}^{n-1} \frac{g^k(A)}{\sqrt{k!}} z^{n-k-1}. \tag{3.12}$$

Now, Theorem 2.1 implies

Theorem 3.2. *Let A be defined by (3.6), condition (3.8) hold and $\tilde{A}$ be a closed operator on $\mathcal{H}$ satisfying conditions (1.1). Let the unique positive root $y(M, q)$ of (3.12) satisfy the inequality $y(M, q) < r_0(A)$, where $r_0(A)$ is defined by (3.8). Then, $\mathrm{md}(A, \tilde{A}) \le y(A, q)$.*

If M is normal, then $g(M) = 0$ and with $0^0 = 1$, we have $y(M, q) = q$. Theorem 2.1 gives us the inequality $\mathrm{md}(A, \tilde{A}) \le q$, provided $q < r_0(A)$.

Now, let M be non-normal: $g(M) \ne 0$. Substitute $z = g(M)w$ into (3.12). We obtain the equation

$$w^n = \frac{q}{g(A)} \sum_{k=0}^{n-1} \frac{1}{\sqrt{k!}} w^{n-k-1}. \tag{3.13}$$

Put

$$p_n = \sum_{j=0}^{n-1} \frac{1}{\sqrt{k!}}.$$

Due to [10, Lemma 3.17], the unique positive root w_0 of equation (3.13) satisfies the inequality

$$w_0 \le \begin{cases} \frac{qp_n}{g(A)} & \text{if } qp_n > g(A), \\ (qp_n/g(A))^{1/n} & \text{if } qp_n \le g(A). \end{cases}$$

But $y(A, q) = w_0 g(A)$. This implies $y(M, q) \le \eta(M, q)$, where

$$\eta(M, q) = \begin{cases} qp_n & \text{if } qp_n > g(M), \\ g^{1-1/n}(M)(qp_n)^{1/n} & \text{if } qp_n \le g(M). \end{cases}$$

Now, Theorem 3.2 yields

Corollary 3.2. *Let A be defined by (3.6), condition (3.8) hold and $\tilde{A}$ be a closed operator on $\mathcal{H}$, such that (1.1) holds. If, in addition, $\eta(M,q) < r_0(A)$, then $\mathrm{md}(A,\tilde{A}) \leq \eta(M,q)$.*

Theorem 3.2 is based on the estimate (3.10). If M is diagonalizable, i.e. there is a nonsingular matrix W, such that $W^{-1}MW$ is a normal matrix, then

$$\|R_\lambda(M)\| \leq \frac{\kappa}{d(M,\lambda)},$$

where

$$\kappa = \|W^{-1}\|_n \|W\|_n, \quad \lambda \notin \sigma(M).$$

According to (3.7), we obtain

$$\|R_\lambda(A)\| \leq \frac{\kappa}{d(A,\lambda)}, \quad \lambda \notin \sigma(A).$$

Equation (2.5) in the considered case takes the form $q\kappa/z = 1$ and thus $x(q) = q\kappa$. So, if M is diagonalizable, then Theorem 2.1 implies

$$\mathrm{md}(A,\tilde{A}) \leq q\kappa \text{ provided } q\kappa < r_0(A). \tag{3.14}$$

Some bounds for κ can be found, in particular, in [10, p.105].

4. Differential operators with matrix coefficients

Let $L^2_n = L^2([0,1],\mathbb{C}^n)$ be the space of functions defined on $[0,1]$, with values in $\mathbb{C}^n$ and the scalar product

$$\langle f,h\rangle_{L^2_n} = \int_0^1 \langle f(x),h(x)\rangle_n dx, \quad f,h \in L^2_n.$$

Let $C(x)$ be an $n\times n$-matrix continuously dependent on x. Consider the operators

$$\tilde{A} = -\frac{d^2}{dx^2} + C(x) \tag{4.15}$$

and

$$A = -\frac{d^2}{dx^2} + M, \quad x \in (0,1) \tag{4.16}$$

with a constant $n\times n$-matrix M and the domain

$$D(A) = D(\tilde{A}) = \{u \in L^2_n : u'' \in L^2_n : \ u(0) = u(1) = 0\}.$$

For instance, one can take $M = C(0)$ or $M = \int_0^1 C(x)dx$. Clearly,

$$q = \|A - \tilde{A}\|_{L^2_n} \leq \sup_x \|C(x) - M\|_n.$$

Here, $\|A-\tilde{A}\|_{L^2_n}$ is the operator norm in L^2_n of $A-\tilde{A}$.

We have $L^2_n = L^2(0,1)\otimes\mathbb{C}^n$, where $L^2(0,1)$ is the standard complex space of scalar functions. On $D(S) = H^2_0(0,1)$, i.e. on

$$D(S) = \{u \in L^2(0,1) : u'' \in L^2(0,1) : \ u(0) = u(1) = 0\},$$

put $S := -\frac{d^2}{dx^2}$. Since $\hat{\lambda}_j(S) = \pi^2 j^2$ ($j = 1,2,...$) with $\nu_j(S) = 1$, $\sigma(A)$ consists of the eigenvalues $\lambda_{jk}(A) = \pi^2 j^2 + \hat{\lambda}_k(M)$ ($j = 1,2,...\,; k = 1,...,m$), and the algebraic multiplicity of $\hat{\lambda}_{jk}(A)$ is equal to $\nu_k(M)$. Let

$$\delta(M) := \inf\{|\pi^2(j^2 - j_1^2) + \hat{\lambda}_k(M) - \hat{\lambda}_{k_1}(M)| :$$

$$j \neq j_1, k_1 \neq k; j, j_1 = 1, 2, ...; k, k_1 = 1, ..., m\} > 0.$$

Then, $r_0(A) = \delta(M) > 0$. Now, Corollary 3.2 yields

Corollary 4.3. *Let $\tilde{A}$ and A be defined by* (4.16) *and* (4.15), $\delta(M) > 0$ *and* $\eta(M, q) < \delta(M)$. *Then,* $\mathrm{md}(A, \tilde{A}) \leq \eta(M, q)$.

In particular, from this corollary, it follows that

$$\sigma(\tilde{A}) \subset \bigcup_{j=1,2,...;\ k=1,...,m} \Omega(\pi^2 j^2 + \hat{\lambda}_k(M), \eta(M, q)),$$

provided $\eta(M, q) < \delta(M)$. If M is diagonalizable, then one can apply inequality (3.14).

For the recent results on the spectra of differential operators see, for instance, the works [16, 17, 20] and the references given therein.

5. Elliptic operators

Let $\omega = [0, 1]^2$ and $L^2(\omega)$ be the space of complex-valued functions defined on ω, with the scalar product

$$\langle f, h \rangle_{L^2(\omega)} = \int_0^1 \int_0^1 f(x, y)\overline{h}(x, y) dx dy, \;\; f, h \in L^2(\omega).$$

Let $c(x, y)$ be a complex continuous function and

$$R := \frac{\partial^2}{\partial x^2} + a\frac{\partial^2}{\partial y^2}, \;\; 0 \leq x, y \leq 1, a \in \mathbb{C}.$$

Consider the operators A and $\tilde{A}$ defined by

$$(\tilde{A} f)(x, y) = (Rf)(x, y) + c(x, y)f(x, y) \tag{5.17}$$

and

$$(Af)(x, y) = (Rf)(x, y) + c_0 f(x, y), \;\; x, y \in (0, 1), f \in D(A) \tag{5.18}$$

with a constant $c_0 \in \mathbb{C}$ and the domain

$$D(A) = \{u \in L^2(\omega) : Ru \in L^2(\omega) : \;\; u(0, y) = u(1, y) = u(x, 0) = u(x, 1) = 0; \;\; 0 \leq x, y \leq 1\}.$$

Clearly,

$$q = \|A - \tilde{A}\|_{L^2(\omega)} \leq \sup_{x,y} |c(x, y) - c_0|.$$

Here, $\|A - \tilde{A}\|_{L^2(\omega)}$ is the operator norm in $L^2(\omega)$ of $A - \tilde{A}$. The eigenfunctions of A are $\sin(\pi j x)\sin(\pi k y)$ and $\sigma(A)$ consists of the simple eigenvalues $\lambda_{jk}(A) = \pi^2(j^2 + ak^2) + c_0$ $(j, k = 1, 2, ...)$. Assume that

$$\delta(R) := \inf\{|\pi^2(j^2 + ak^2 - j_1^2 - ak_1^2)| : j_1 \neq j, k_1 \neq k; j, j_1, k, k_1 = 1, 2, ...\} > 0.$$

Then, $r_0(A) = \delta(R) > 0$. For example, if a is imaginary, then $\delta(R) \geq 3\pi^2(1 + |a|)$. Omitting simple calculations, under consideration, we obtain

$$\|(A - \lambda I)^{-1}\|_{L^2(\omega)} \leq \frac{1}{d(A, \lambda)}.$$

Now, Theorem 2.1 yields

Corollary 5.4. *Let $\tilde{A}$ and A be defined by* (5.17) *and* (5.18), *and* $\delta(R) > q$. *Then,* $\mathrm{md}(A, \tilde{A}) \leq q$.

Similarly, making use of Corollary 3.2, one can consider elliptic operators with matrix coefficients.

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MICHAEL GIL
BEN GURION UNIVERSITY OF THE NEGEV
DEPARTMENT OF MATHEMATICS
P.0. BOX 653, BEER-SHEVA 84105, ISRAEL
ORCID: 0000-0002-6404-9618
E-mail address: gilmi@bezeqint.net