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ON $\mathcal{F}$-COSMALL MORPHISMS

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Abstract. In this paper, we first define the notion of $\mathcal{F}$-cosmall quotients for an additive exact substructure $\mathcal{F}$ of an exact structure $\mathcal{E}$ in an additive category $\mathcal{A}$. We show that every $\mathcal{F}$-cosmall quotient is right minimal in some cases. We also give the definition of $\mathcal{F}$-superfluous quotients and we relate it the approximation of modules. As an application, we investigate our results in a pure-exact substructure $\mathcal{F}$.

1. Introduction

In [12], Ziegler introduced the partial morphisms by using model theory of modules. Then in [9], the partial morphisms was investigated by Monari Martinez in terms of systems of linear equations. But this algebraic definition of partial morphisms was not useful in the categorical studies of purity. Then in [4] Cortés-Izurdiaga, Guil Asensio, Kaleboğaz and Srivastava studied partial morphisms by using category theory. In [4], the authors defined partial morphisms by using pushout with respect to an additive exact substructure $\mathcal{F}$ of an exact structure $\mathcal{E}$ in an additive category $\mathcal{A}$ and they call them $\mathcal{F}$-partial morphisms. They showed that the definition of $\mathcal{F}$-partial morphisms with the pure-exact substructure $\mathcal{F}$ in the category of right R-modules are coincide with the partial morphisms that defined by Ziegler in [12]. By using $\mathcal{F}$-partial morphisms they also define $\mathcal{F}$-small extension and gave an application of this definition to the pure-exact substructure $\mathcal{F}$ in the category of right modules over a ring and called it Ziegler small extension. As a dual notion of $\mathcal{F}$-partial morphisms, in [6] $\mathcal{F}$-copartial morphisms was defined by Kaleboğaz: a morphism $f : X \longrightarrow U$ is $\mathcal{F}$-copartial morphism with respect to a quotient map $p : Y \longrightarrow U$ if and only if $\mathrm{Ext}^1(f, -)$ transforms p in an $\mathcal{F}$-deflation. She studied the properties of $\mathcal{F}$-copartial morphisms and investigated the applications of $\mathcal{F}$-copartial morphisms to some exact substructures of $\mathcal{E}$ in the category of right R-modules.

Keywords. $\mathcal{F}$-cosmall quotients, right minimal morphisms, $\mathcal{F}$-superfluous quotients.

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In this paper, by using $\mathcal{F}$-copartial morphisms, we first introduce $\mathcal{F}$-cosmall quotients for any additive exact substructure $\mathcal{F}$ of an exact structure $\mathcal{E}$ in an additive category $\mathcal{A}$ (see Definition 2). We also give a new characterization of $\mathcal{F}$-cosmall quotients (see Proposition 1). As an application to a pure-exact structure $\mathcal{F}$, we give the definition of pure-cosmall quotients and we say that pure-cosmall quotients are dual of Ziegler small extensions.

A morphism $p : M \longrightarrow N$ is called *right minimal* if any endomorphism $g : M \longrightarrow M$ with $pg = p$ is an isomorphism (see [1, page 6]). In [8], right minimal morphisms are studied by Keskin Tütüncü. In [8] the author dualizes some results in [3] and gets several useful results by investigating the relationship between $\mathrm{End}_R(N)$ and $\mathrm{End}_R(M)$ when there is a right minimal epimorphism $p : M \longrightarrow N$. The author also proves that there is an isomorphism between two rings $END_R^M(N)/J(END_R^M(N))$ and $END_R^N(M)/J(END_R^N(M))$ if there exists a right minimal epimorphism $p : M \longrightarrow N$ in [8, Theorem 2.6 (1)]. As a consequence of this result the structure of the endomorphism ring of a quasi-projective module and an automorphism-coinvariant module are explained. One of the main purposes of this paper is to give an example of right minimal morphisms. In Theorem 1, we prove that every $\mathcal{F}$-cosmall quotient $f : P \longrightarrow M$ with P an $\mathcal{F}$-projective object (projective objects with respect to $\mathcal{F}$-deflations) is right minimal. An application of this theorem to the pure-exact structure gives us the dual version of [3, Proposition 1.6]. Moreover, we give the definitions of $\mathcal{F}$-superfluous quotient and weakly $\mathcal{F}$-superfluous quotient (see Definition 5). Then we investigate the relation between $\mathcal{F}$-cosmall quotient and $\mathcal{F}$-superfluous quotient (see Proposition 2). And finally we relate to the existence of approximations of modules. In Theorem 2, we show that a weakly $\mathcal{F}$-superfluous quotient $p : Y \longrightarrow U$ with $\mathcal{F}$-projective Y is an $\mathcal{F}$-Proj-cover when $\mathcal{F}$-Proj is the class of $\mathcal{F}$-projective objects of $\mathcal{A}$.

2. Results

Let $\mathcal{A}$ be an additive category and (i, p) be a pair of composable morphisms in $\mathcal{A}$:

$$A \xrightarrow{i} B \xrightarrow{p} C$$

If i is a kernel of p and p is a cokernel of i then (i, p) is called *kernel-cokernel pair in* $\mathcal{A}$. Let $\mathcal{E}$ be the class of kernel-cokernel pairs on $\mathcal{A}$. i is called an *admissible monomorphism* if there exists a morphism p such that $(i, p) \in \mathcal{E}$. Similarly, p is called an *admissible epimorphism* if there exists a morphism i such that $(i, p) \in \mathcal{E}$.

The class of kernel-cokernel pairs $\mathcal{E}$ is said to be an *exact structure on* $\mathcal{A}$ if it is closed under isomorphisms and satisfies the following conditions;

[E0] For every object $A \in \mathcal{A}$, the identity morphism 1_A is an admissible monomorphism.

[E0$^{\mathrm{op}}$] For every object $A \in \mathcal{A}$, the identity morphism 1_A is an admissible epimorphism.

[E1] The classes of admissible monomorphisms are closed under compositions.

[E1$^{\mathrm{op}}$] The classes of admissible epimorphisms are closed under compositions.

[E2] The pushout of an admissible monomorphism along an arbitrary morphism exists and yields an admissible monomorphism, that is, for any admissible monomorphism $i : A \longrightarrow B$ and any morphism $f : A \longrightarrow B'$, there is a pushout diagram;

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "f'"] \\ B' \arrow[r, "i'"] & P \end{tikzcd}$$

with i' an admissible monomorphism.

[E2$^{\mathrm{op}}$] The pullback of an admissible epimorphism along an arbitrary morphism exists and yields an admissible epimorphism, that is, for any admissible epimorphism $p : B \longrightarrow C$ and any morphism $g : B' \longrightarrow C$ there is a pullback diagram;

$$\begin{tikzcd} Q \arrow[r, "p'"] \arrow[d, "g'"'] & B' \arrow[d, "g"] \\ B \arrow[r, "p"] & C \end{tikzcd}$$

with p' an admissible epimorphism.

An *exact category* is a pair $(\mathcal{A}, \mathcal{E})$ with an additive category $\mathcal{A}$ and an exact structure $\mathcal{E}$ on $\mathcal{A}$. Elements of $\mathcal{E}$ are called *short exact sequences*. Keller [7] uses *conflation*, *inflation* and *deflation* for what we call short exact sequence, admissible monomorphism and admissible epimorphism, respectively. Throughout the paper we also use this terminology. Let A be an object of $\mathcal{A}$. An *admissible quotient of* A is a quotient object U of an object A such that one (and any) quotient map $p : A \longrightarrow U$ is a deflation.

An *exact substructure* $\mathcal{F}$ of $\mathcal{E}$ is just an exact structure on $\mathcal{A}$ such that each conflation in $\mathcal{F}$ (that we shall call $\mathcal{F}$-conflation) is also a conflation in $\mathcal{E}$. Inflations, deflations and admissible quotient objects with respect to $\mathcal{F}$ will be called $\mathcal{F}$-inflations, $\mathcal{F}$-deflations and $\mathcal{F}$-admissible quotient objects, respectively.

We shall start with giving the definition of $\mathcal{F}$-copartial morphisms (respectively, $\mathcal{F}$-copartial isomorphisms) for an additive substructure $\mathcal{F}$ of an exact structure $\mathcal{E}$ in an additive category $\mathcal{A}$. $\mathcal{F}$-copartial morphisms first introduced and investigated in [6] by Kaleboğaz as the dual notion of $\mathcal{F}$-partial morphism that are studied in [4].

For the rest of the paper, we fix an exact category of $(\mathcal{A}, \mathcal{E})$ and an additive exact substructure $\mathcal{F}$ of $\mathcal{E}$.

Definition 1. Let X, Y be objects of $\mathcal{A}$ and U an admissible quotient of Y with the quotient map $p : Y \longrightarrow U$.

Let $f : X \longrightarrow U$ be a morphism and consider the pullback of f along the quotient map p:

$$\begin{tikzcd} Q \arrow[r, "\overline{p}"] \arrow[d, "\overline{f}"'] & X \arrow[d, "f"] \\ Y \arrow[r, "p"] & U \end{tikzcd}$$

Then:

(1) f is called an *$\mathcal{F}$-copartial morphism* from X to Y with codomain U if $\overline{p}$ is an $\mathcal{F}$-deflation.

(2) f is called an *$\mathcal{F}$-copartial isomorphism* from X to Y with codomain U if both $\overline{p}$ and $\overline{f}$ are $\mathcal{F}$-deflations.

Now we recall two lemmas from [6], without proofs, that we will use in the rest of the paper. The first lemma given below is a special case of the dual of Obscure Axiom in [2, Proposition 2.16] (see [6, Proposition 2.3]). The other one is one of the main properties of $\mathcal{F}$-copartial morphisms (see [6, Proposition 2.5(1)]).

Lemma 1. *Let X, Y, Z be objects of $\mathcal{A}$. If an $\mathcal{F}$-deflation $f : Z \longrightarrow Y$ factors through an deflation $p : X \longrightarrow Y$ as follows;*

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[d, "f"] \\ X \arrow[r, "p"] & Y \end{tikzcd}$$

then p is an $\mathcal{F}$-deflation too.

Lemma 2. *Let X, Y be objects of $\mathcal{A}$ and U, an admissible quotient of Y with the quotient morphism $p : Y \longrightarrow U$. Suppose that p is an $\mathcal{F}$-deflation. A morphism $f : X \longrightarrow U$ is an $\mathcal{F}$-deflation if and only if f is an $\mathcal{F}$-copartial isomorphism from X to Y with codomain U.*

As a consequence of this lemma, we can give the following corollary:

Corollary 1. *Let Y be an object of $\mathcal{A}$ and $g : Z \longrightarrow Y$ be any morphism with any object Z in $\mathcal{A}$. g is an $\mathcal{F}$-deflation if and only if g is an $\mathcal{F}$-copartial isomorphism from Z to Y with codomain Y.*

Proof. Let us take the pullback of g along 1_Y. Since 1_Y is an $\mathcal{F}$-deflation, g is an $\mathcal{F}$-deflation if and only if g is an $\mathcal{F}$-copartial isomorphism from Z to Y with codomain Y by Lemma 2. $\square$

One of the aims of this paper is to give an example of right minimal morphisms. To attain our goal we shall first give the definition of $\mathcal{F}$-cosmall quotient morphisms. These morphisms are dual of $\mathcal{F}$-small extensions that are defined in [4, Definition 3.4].

Definition 2. Let the object Y of $\mathcal{A}$ be an admissible quotient of any object X with the quotient map $p' : X \longrightarrow Y$, U be an admissible quotient of X and $p : Y \longrightarrow U$ be a deflation.

(1) We shall say that Y is *$\mathcal{F}$-cosmall in U over X* if for any $\mathcal{F}$-copartial morphism $g : Z \longrightarrow Y$ from any object Z to X with codomain Y, the following holds:
pg is an $\mathcal{F}$-copartial isomorphism from Z to X with codomain U implies that g is an $\mathcal{F}$-copartial isomorphism from Z to X with codomain Y.

(2) We shall say that Y is *$\mathcal{F}$-cosmall in U* if Y is $\mathcal{F}$-cosmall in U over Y. Namely, the deflation p' is the identity morphism of Y.

With the notion of $\mathcal{F}$-cosmall object which is defined above, now we can define $\mathcal{F}$-cosmall quotient morphisms as in the following:

Definition 3. Let $p : Y \longrightarrow U$ be a deflation. If Y is $\mathcal{F}$-cosmall in U then the deflation $p : Y \longrightarrow U$ is called an *$\mathcal{F}$-cosmall quotient*.

Namely, if Y is $\mathcal{F}$-cosmall in U over Y then p is an $\mathcal{F}$-cosmall quotient.

Here we will give a characterization of $\mathcal{F}$-cosmall quotient which will be used in the rest of the paper.

Proposition 1. *Let $p : Y \longrightarrow U$ be a deflation. p is an $\mathcal{F}$-cosmall quotient if and only if for any morphism $g : Z \longrightarrow Y$ for any object Z such that pg is an $\mathcal{F}$-copartial isomorphism from Z to Y with codomain U, g is an $\mathcal{F}$-deflation.*

Proof. Let Z be an object of $\mathcal{A}$ and $g : Z \longrightarrow Y$ be a morphism such that pg is an $\mathcal{F}$-copartial isomorphism from Z to Y with codomain U. We will show that g is an $\mathcal{F}$-deflation. If we take pullback of g along 1_Y, then we get the following commutative diagram:

$$\begin{tikzcd} Q \arrow[r, "h"] \arrow[d, "\overline{g}"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "1_Y"] & Y \end{tikzcd}$$

Since 1_Y is an $\mathcal{F}$-deflation, h is an $\mathcal{F}$-deflation. Therefore, g is an $\mathcal{F}$-copartial morphism from Z to Y with codomain Y. As p is an $\mathcal{F}$-cosmall quotient, g is also an $\mathcal{F}$-copartial isomorphism from Z to Y with codomain Y. Then, by Corollary 1, g is an $\mathcal{F}$-deflation.

For the converse, to show that p is an $\mathcal{F}$-cosmall quotient, let us take an $\mathcal{F}$-copartial morphism $g : Z \longrightarrow Y$ from Z to Y with codomain Y such that pg is an $\mathcal{F}$-copartial isomorphism from Z to Y with codomain U. By assumption, g is an $\mathcal{F}$-deflation. By Corollary 1, g is an $\mathcal{F}$-copartial isomorphism from Z to Y with codomain Y. Therefore, p is an $\mathcal{F}$-cosmall quotient. □

Let R be a ring, Y and Z be right R-modules and $f : Y \longrightarrow Z$ be an epimorphism. Recall that, f is called *pure epimorphism* if $\mathrm{Hom}_R(M, f) : Hom_R(M, Y) \longrightarrow$

$\mathrm{Hom}_R(M, Z)$ is an epimorphism for all finitely presented right R-modules M. Let X be the kernel of f with the inclusion $u : X \longrightarrow Y$. Then by the theorem of Fieldhouse [5] and Warfield [10], f is pure epimorphism if and only if X is pure in Y (u is a pure monomorphism) in the sense that the natural homomorphism $X \otimes_R N \longrightarrow Y \otimes_R N$ derived from the inclusion map $u : X \longrightarrow Y$ is a monomorphism for all left R-modules N. Then, the conflation $X \longrightarrow Y \longrightarrow Z$ is said to be a *pure conflation* if f is a pure epimorphism (or u is a pure monomorphism). The class of all pure conflations is exact substructure of exact structure of the class of all conflations from [2, Exercise 5.6]. $\mathcal{F}$-copartial morphisms (respectively, $\mathcal{F}$-copartial isomorphisms) with respect to a pure-exact substructure $\mathcal{F}$ in the category of right R-modules are called *copartial morphisms (respectively, copartial isomorphisms)*, (see [6]). Here we will define pure-cosmall quotient morphisms as an application of $\mathcal{F}$-cosmall quotient with respect to a pure-exact substructure $\mathcal{F}$ in the category of right R-modules.

Definition 4. Let Y and U be right R-modules. An epimorphism $p : Y \longrightarrow U$ is called a *pure-cosmall quotient* if Y is pure-cosmall in U, that means, for any right R-module Z, any copartial morphism $g : Z \longrightarrow Y$ from Z to Y with codomain Y, the following holds:

If pg is a copartial isomorphism from Z to Y with codomain U then g is a copartial isomorphism from Z to Y with codomain Y.

Corollary 2. *Let Y and U be right R-modules, $p : Y \longrightarrow U$ be a deflation. p is a pure-cosmall quotient if and only if for any right R-module Z, any morphism $g : Z \longrightarrow Y$ such that pg is a copartial isomorphism from Z to Y with codomain U is a pure epimorphism.*

Pure-cosmall quotients are the dual of Ziegler small extensions that are introduced in [4] and are studied in [3]. In [3], the authors proved that every Ziegler small extension $u : M \longrightarrow E$ with E being pure-injective is a left minimal morphism. Now we proceed to extend dual of this result to any exact substructure $\mathcal{F}$. We will show that $\mathcal{F}$-cosmall quotient morphisms are right minimal under a condition. So the following theorem gives us an example of right minimal morphisms.

Let P be an object of $\mathcal{A}$ and $p : Y \longrightarrow Z$ be a deflation. Recall that, P is said to be *p-projective* (or projective with respect to p) if for each morphism $f : P \longrightarrow Z$ there exist a morphism $g : P \longrightarrow Y$ with $pg = f$. P is said to be a *projective object in* $\mathcal{A}$ if it is projective with respect to each deflation. Projective objects with respect to $\mathcal{F}$-deflations will be called *$\mathcal{F}$-projective objects.*

Theorem 1. *Every $\mathcal{F}$-cosmall quotient $f : P \longrightarrow M$ with P being an $\mathcal{F}$-projective object is right minimal.*

Proof. Let $g : P \longrightarrow P$ be a morphism such that $fg = f$. Now we will show that g is an isomorphism. If we consider the pullback of f along fg we get the following commutative diagram;

$$\begin{tikzcd} Q \arrow[r, "h_2"] \arrow[d, "h_1"'] & P \arrow[d, "fg"] \\ P \arrow[r, "f"] & M \end{tikzcd}$$

Since $fg = f$, the identity map 1_P satisfies that $fg1_P = f1_P$. Then by the universal property of pullback, there exist $\alpha : P \longrightarrow Q$ such that $h_1\alpha = 1_P$ and $h_2\alpha = 1_P$. By Lemma 1, h_1 and h_2 are both $\mathcal{F}$-deflations. Therefore, fg is an $\mathcal{F}$-copartial isomorphism from P to P with codomain M. Since f is an $\mathcal{F}$-cosmall quotient, g is an $\mathcal{F}$-deflation by Proposition 1. So it is an epimorphism.

Now, using that P is an $\mathcal{F}$-projective, we get that there exists $h : P \longrightarrow P$ such that $gh = 1_P$. Then $f = f1_P = fgh = fh$. By using the previous argument we conclude that h is an epimorphism. Then as $hgh = h = 1_p h$, we get that $hg = 1_p$. Therefore, g is a monomorphism. So g is an isomorphism. □

Corollary 3. *Every pure-cosmall quotient $f : P \longrightarrow M$ with P being a pure-projective right R-module is right minimal.*

Now we will give the definition of $\mathcal{F}$-superfluous and weakly $\mathcal{F}$-superfluous quotients.

Definition 5. Let X and Y be objects of $\mathcal{A}$.

(1) An *$\mathcal{F}$-superfluous quotient* is an $\mathcal{F}$-deflation $p : X \longrightarrow Y$ such that for any object of Z in $\mathcal{A}$ and any morphism $\alpha : Z \longrightarrow X$ the following holds:

$$p\alpha \text{ is an } \mathcal{F}\text{-deflation implies that } \alpha \text{ is an } \mathcal{F}\text{-deflation.}$$

(2) A *weakly $\mathcal{F}$-superfluous quotient* is an $\mathcal{F}$-deflation $p : X \longrightarrow Y$ such that for any object of Z in $\mathcal{A}$ and any morphism $\alpha : Z \longrightarrow X$ the following holds:

$$p\alpha \text{ is an } \mathcal{F}\text{-deflation implies that } \alpha \text{ is a deflation.}$$

Remark 1. (1) If $\mathcal{A}$ is the category of right R-modules and $\mathcal{E}$ is the abelian exact structure, then $\mathcal{E}$-superfluous quotient morphism is coincide with the small epimorphism that is recalled in [8, Example 2.2(2)].

(2) If $\mathcal{A}$ is the category of right R-modules and $\mathcal{F}$ is the pure-exact structure, then $\mathcal{F}$-superfluous quotient morphism is coincide with the $\mathbb{S}$-superfluous epimorphism for $\mathbb{S}$ being the class of finitely presented modules that is introduced in [11].

Now we give the relation between $\mathcal{F}$-cosmall quotient and $\mathcal{F}$-superfluous quotient.

Proposition 2. *Let $p : Y \longrightarrow U$ be a deflation. p is an $\mathcal{F}$-superfluous quotient if and only if p is an $\mathcal{F}$-deflation and $\mathcal{F}$-cosmall quotient.*

Proof. Suppose that p is an $\mathcal{F}$-superfluous quotient. So p is an $\mathcal{F}$-deflation. Now we will show that p is an $\mathcal{F}$-cosmall quotient. Let us take an object Z and a morphism $g : Z \longrightarrow Y$ such that pg is an $\mathcal{F}$-copartial isomorphism from Z to Y with codomain U. Now if we take the pullback of pg along p we get the following commutative diagram:

$$\begin{tikzcd} Q \arrow[r, "\overline{p}"] \arrow[d, "h"'] & Z \arrow[d, "pg"] \\ Y \arrow[r, "p"] & U \end{tikzcd}$$

By Lemma 2, pg is an $\mathcal{F}$-deflation. Then g is an $\mathcal{F}$-deflation by the definition of $\mathcal{F}$-superfluous quotient. Therefore, by Proposition 1, p is an $\mathcal{F}$-cosmall quotient.

For the converse, assume that p is an $\mathcal{F}$-deflation and $\mathcal{F}$-cosmall quotient. To show that p is an $\mathcal{F}$-superfluous quotient let us take a morphism $\alpha : Z \longrightarrow Y$ such that $p\alpha$ is an $\mathcal{F}$-deflation. Now take the pullback of $p\alpha$ along p we get the following commutative diagram:

$$\begin{tikzcd} Q \arrow[r, "\overline{p}"] \arrow[d, "h"'] & Z \arrow[d, "p\alpha"] \\ Y \arrow[r, "p"] & U \end{tikzcd}$$

By Lemma 2, $p\alpha$ is an $\mathcal{F}$-copartial isomorphism from Z to Y with codomain U. Since p is an $\mathcal{F}$-cosmall quotient, α is an $\mathcal{F}$-deflation. Therefore p is an $\mathcal{F}$-superfluous quotient. □

Let $\mathcal{A}$ be any category and $\mathcal{X}$ be a class of objects in $\mathcal{A}$. Recall that, a morphism $\phi : X \longrightarrow Y$ in $\mathcal{A}$ is a $\mathcal{X}$-precover of Y if $X \in \mathcal{X}$ and for any morphism $f : Z \longrightarrow Y$ with $Z \in \mathcal{X}$, there is a morphism $g : Z \longrightarrow X$ such that $\phi g = f$. A $\mathcal{X}$-precover $\phi : X \longrightarrow Y$ is said to be a $\mathcal{X}$-cover if every morphism $g : X \longrightarrow X$ such that $\phi g = \phi$ is an isomorphism. It is clear that, an $\mathcal{X}$-cover is an $\mathcal{X}$-precover which is a right minimal morphism.

In the next result we will show that, under certain circumstances, a weakly $\mathcal{F}$-superfluous quotient $p : Y \longrightarrow U$ with Y being $\mathcal{F}$-projective is actually an $\mathcal{F}$-Proj-cover for $\mathcal{F}$-Proj being the class of $\mathcal{F}$-projective objects of $\mathcal{A}$.

Theorem 2. *Let $p : Y \longrightarrow U$ be a deflation. Consider the following assertions:*

(1) *p is an $\mathcal{F}$-superfluous quotient and Y is an $\mathcal{F}$-projective object.*
(2) *p is an $\mathcal{F}$-deflation, Y is an $\mathcal{F}$-projective and p is an $\mathcal{F}$-cosmall quotient.*
(3) *p is an $\mathcal{F}$-deflation, Y is an $\mathcal{F}$-projective and for any object X, each morphism $f : X \longrightarrow Y$ satisfying that pf is an $\mathcal{F}$-deflation, is a split epimorphism.*
(4) *p is an $\mathcal{F}$-Proj-cover for $\mathcal{F}$-Proj being the class of $\mathcal{F}$-projective objects of $\mathcal{A}$.*
(5) *p is a weakly $\mathcal{F}$-superfluous quotient with Y being $\mathcal{F}$-projective object.*

We have (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3)*,* (2) $\Rightarrow$ (4)*,* (1) $\Rightarrow$ (5)*.*

If there exists an $\mathcal{F}$-deflation $\alpha : P \longrightarrow U$ with P being an $\mathcal{F}$-projective object then (4) $\Rightarrow$ (3)*.*

If there exists an $\mathcal{F}$-superfluous quotient $\alpha : P \longrightarrow U$ with P being an $\mathcal{F}$-projective object then (5) $\Rightarrow$ (1)*.*

Proof. (1) $\Leftrightarrow$ (2) Obvious from Proposition 2.

(1) $\Rightarrow$ (3) Let $f : X \longrightarrow Y$ be a morphism with pf being an $\mathcal{F}$-deflation. Since p is an $\mathcal{F}$-superfluous quotient, f is an $\mathcal{F}$-deflation. As Y is an $\mathcal{F}$-projective module, f is a split epimorphism.

(3) $\Rightarrow$ (1) It is clear since split epimorphisms are $\mathcal{F}$-deflations.

(2) $\Rightarrow$ (4) Since p is an $\mathcal{F}$-deflation, it is an $\mathcal{F}$-Proj-precover for $\mathcal{F}$-Proj being the class of $\mathcal{F}$-projective objects of $\mathcal{A}$. As p is an $\mathcal{F}$-cosmall quotient, p is right minimal by Theorem 1. Therefore, p is an $\mathcal{F}$-Proj-cover for $\mathcal{F}$-Proj being the class of $\mathcal{F}$-projective objects of $\mathcal{A}$.

(1) $\Rightarrow$ (5) It is clear, since every $\mathcal{F}$-superfluous quotient is weakly $\mathcal{F}$-superfluous quotient.

(4) $\Rightarrow$ (3) Assume that there exists an $\mathcal{F}$-deflation $\alpha : P \longrightarrow U$ with P being an $\mathcal{F}$-projective object. Since p is an $\mathcal{F}$-Proj-precover for $\mathcal{F}$-Proj being the class of $\mathcal{F}$-projective objects of $\mathcal{A}$, there exists $g : P \longrightarrow Y$ such that $pg = \alpha$. Since α is an $\mathcal{F}$-deflation, then p is also an $\mathcal{F}$-deflation by Lemma 1. Now let $f : X \longrightarrow Y$ be a morphism such that pf is an $\mathcal{F}$-deflation. Since Y is an $\mathcal{F}$-projective object then there exists $h : Y \longrightarrow X$ such that $pfh = p$. As p is an $\mathcal{F}$-Proj-cover then fh is an isomorphism. Therefore, f is split.

(5) $\Rightarrow$ (1) There exists an $\mathcal{F}$-superfluous quotient $\alpha : P \longrightarrow U$ with P is an $\mathcal{F}$-projective object. Since Y is $\mathcal{F}$-projective, there exists a morphism $w : Y \longrightarrow P$ such that $\alpha w = p$. Since p is an $\mathcal{F}$-deflation and α is an $\mathcal{F}$-superfluous then w is an $\mathcal{F}$-deflation. And α is an $\mathcal{F}$-deflation too by Lemma 1. As P is an $\mathcal{F}$-projective object then there exists $h : P \longrightarrow Y$ such that $wh = 1_P$. So w is an epimorphism. We get $ph = \alpha wh = \alpha 1_P = \alpha$. Then h is an $\mathcal{F}$-deflation as p is a weakly $\mathcal{F}$-superfluous. Then $hwh = h1_P = 1_P h$. Since h is epic $hw = 1_P$. So w is a monomorphism. Therefore, w is an isomorphism. By $\alpha w = p$ and α is an $\mathcal{F}$-superfluous quotient then p is an $\mathcal{F}$-superfluous quotient. □

Remark 2. Let $p : Y \longrightarrow U$ be a deflation with Y an $\mathcal{F}$-projective object of $\mathcal{A}$. From Theorem 2 (4)$\Rightarrow$(2), we can say that if p is an $\mathcal{F}$-Proj-cover of U for $\mathcal{F}$-Proj being the class of $\mathcal{F}$-projective objects of $\mathcal{A}$, then p is an $\mathcal{F}$-cosmall quotient. But Theorem 1 shows that p can be an $\mathcal{F}$-cosmall quotient map which is not an $\mathcal{F}$-Proj-cover (since here p need not be an $\mathcal{F}$-deflation). But p is always right minimal.

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