Results in Nonlinear Analysis **5** (2022) No. 3, 325–336.
https://doi.org/10.53006/rna.1061458
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

KATUGAMPOLA KINETIC FRACTIONAL EQUATIONS WITH ITS SOLUTIONs

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Abstract

The use of fractional kinetic equations to describe the many phenomena regulated by anomalous reactions in dynamical systems with chaotic motion is examined. Several authors have used a variety of methodologies to study a number of difficulties arising from the generalised Bessel's function. In this follow-up study, the results of Katugampola kinetic fractional equations containing the first kind of generalised Bessel's function will be investigated. The result is obtained using the τ Laplace transform technique. We may use the generality of this series to deduce solutions for a fractional kinetic equation employing a different sort of Bessel's series. A graphical representation of the behaviors of the obtained solutions is also supplied.

Keywords: τ Laplace transform Katugampola fractional operator kinetic equation
2020 MSC: 26A33, 33B15, 33C05, 35A22

1. Introduction

To evaluate fractional-order differ-integral equations, fractional calculus is an effective mathematical tool. It has been established and developed in engineering and scientific fields. Fractional differential equations and their applications have made significant contributions to applied science, physics, biology, chemistry and engineering (see, e.g. [2, 7, 23]). Kinetic equations define a system of differential equations in form of reaction rates for destruction and production respectively, which examine the rate of change of a star's chemical combination for each order. The extension and generalization of kinetic fractional equations which involve many fractional operators was found in the value of fractional differential equations has grown more interest in applied science not only in mathematics but also in physics, dynamical systems, control systems and engineering in order to create many physical phenomena as

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Received :January 22, 2022; Accepted: July 08, 2022; Online: July 31, 2022

a mathematical model. Over the last decades, kinetic fractional equations in many forms have been widely used in describing and solving a variety of important astrophysics and physics (see, e.g. [1, 3, 9, 10, 11, 12, 13, 14, 15, 17, 18, 24, 25, 26]) problems. The special functions with their applications (such as Bessel's function, Hypergeometric function, Legender function, and many others) used in the solutions of fractional differential equations. They are related to every problem in a wide range of mathematical physics and mathematical fields, because kinetic fractional equations are effective and relevant in astrophysical calculations. Thoughts have prompted a number of researchers in the area of special functions to investigate applications and applications the possible extensions of the Bessel's functions. These functions are also helpful for problem solving in wave mechanics and elastic theory. Bessel's functions are an endless subject; there are much more useful properties than one is aware of.
In this paper we used the τ Laplace transform technique while other researcher used different transform techniques and already existing results are special cases of these results that is the current research work is simply generalization.

Haubold and Mathai [16] investigated the kinetic equation that describes the rate of change.

$$\frac{d\mathbb{N}}{dt} = -d(\mathbb{N}_t) + q(\mathbb{N}_t). \tag{1}$$

Where $q = q(\mathbb{N})$ denote the production rate, $d = d(\mathbb{N})$is the destruction rate, $\mathbb{N} = \mathbb{N}(t)$ indicate the reaction rate and $\mathbb{N}_t$ represent the function defined by $\mathbb{N}_t(t^*) = \mathbb{N}(t - t^*), t^* > 0$. When they obtained some appropriate cases of equation (1), when spatial variations or inhomogeneities in the quantity $\mathbb{N}(t)$ are neglected, is given by the equation

$$\frac{d\mathbb{N}}{dt} = -c_i\mathbb{N}_i(t), \tag{2}$$

with conditions $\mathbb{N}_i(t = 0) = \mathbb{N}_0$ expressing the variety of species density i at time $t = 0$ and $c_i > 0$. Ignoring index i and integrating, equation (2) becomes

$$\mathbb{N}(t) - \mathbb{N}_0 = c\ {}_0\mathbb{D}_t{}^{-1}\mathbb{N}(t), \tag{3}$$

where ${}_0\mathbb{D}_t{}^{-1}$ is the prominent case of the Riemann-Liouville integral operator ${}_0\mathbb{D}_t{}^{-n}$ defined as

$${}_0\mathbb{D}_t{}^{-n}\mathfrak{f}(t) = \frac{1}{\Gamma(n)}\int_0^t (-u + t)^{n-1}\mathfrak{f}(u)du, \qquad t > 0, \Re(n) > 0. \tag{4}$$

The fractional generalization was developed by Haubold and Mathai [16], of the classical kinetic equation (3) as

$$\mathbb{N}(t) - \mathbb{N}_0 = -c^n\ {}_0\mathbb{D}_t{}^{-n}\mathbb{N}(t), \qquad (n, c \in \Re^+), \tag{5}$$

have yield the solution of (5) as follows:

$$\mathbb{N}(t) = \mathbb{N}_0 \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(rn + 1)}(ct)^{rn}. \tag{6}$$

Equation (6) also can be rewrite in form of the Mittag-Leffler function in a combine form as

$$\mathbb{N}(t) = \mathbb{N}_0\mathbb{E}_n(-c^n t^n), \qquad n > 0. \tag{7}$$

Where $\mathbb{E}_n(z)$ is the Mittag-Leffler function, which is defined [22] as

$$\mathbb{E}_n(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(nr + 1)}, \qquad (n > 0). \tag{8}$$

The generalize form of Mittag-Leffler function [22] is given as

$$\mathbb{E}_{n,\beta}(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(nr + \beta)}, \qquad (n > 0, \beta > 0). \tag{9}$$

Furthermore, Saxena and Kalla [27] considered the kinetic fractional equation as:

$$\mathbb{N}(t) - \mathbb{N}_0 \mathfrak{f}(t) = -c^n \, {}_0\mathbb{D}_t^{-n} \mathbb{N}(t), \qquad (n, c \in \mathfrak{R}^+). \tag{10}$$

Where $\mathfrak{f} \in \mathbb{L}(0,\infty)$ and $\mathbb{N}(t)$ represent the number density of a given species at time t and $\mathbb{N}_0 = \mathbb{N}(0)$ is the number density of that species at time $t = 0$, when we apply Laplace transform to equation(7), we obtain

$$\mathbb{L}\{\mathbb{N}(t)\}(p) = \mathbb{N}_0 \frac{\mathfrak{F}(s)}{1 + c^n s^{-n}} \qquad = \mathbb{N}_0 \mathfrak{F}(s) \left[\sum_{k=0}^{\infty} (-c^n)^k s^{-kn} \right] \qquad \left(k \in \mathbb{N}_0, |\frac{c}{d}| < 1 \right). \tag{11}$$

For time variable $t > 0$, $\mathfrak{f}(t)$ denote a complex valued function and s is complex or real parameter.
The Laplace transform of the function $\mathfrak{f}(t)$ is given by

$$\mathbb{L}\{\mathfrak{f}(t); s\} = \mathfrak{F}(s) = \int_0^{\infty} \exp(-st) \mathfrak{f}(t) dt, \qquad (R(s) > 0).$$

Many researchers explored solutions of equation (5) in terms of the different special functions (see, e.g., [3, 5, 6, 8, 9, 10, 11, 12, 20]) in generalized forms. Here, we recall a solution of a generalized kinetic fractional equation containing generalized Bessel's function [4] given by Dinesh et al. (see [21])

$$\mathbb{W}_{p,b,c}(x) = \mathbb{W}_p(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (c)^r}{r! \Gamma(r + p + (b+1)/2)} \left(\frac{x}{2} \right)^{(2r+p)}, \tag{12}$$

where $x \in \mathbb{C} \backslash \{0\}$ such that $\mathbb{C}$ denote the set of complex number and $b, c, p \in \mathbb{C}$, $\mathfrak{R}(p) > -1$ and $\Gamma(d)$ is known Gamma function.
Equation (12) reduces many kinds of special cases with different conditions, which can be obtained as follows

(i) If we enter $b = c = 1$ in (12), then we get the well-known first kind Bessel's function [28] of order p for $x, p \in \mathbb{C}$ with $\mathfrak{R}(p) > 0$. which is defined and represented by the following expressions (see also [4]).

.

$$\mathbb{J}_p(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \Gamma(r + p + 1)} \left(\frac{x}{2} \right)^{(2r+p)}, \tag{13}$$

(ii) By inserting $b = 1$ and $c = -1$ in equation (12), we yield the first kind modified Bessel's function of order p given by (see[4, 28]) then the expression is also convert into a known form of Galué generalized modified Bessel's function [14].
(iii)Substituting $b = 2$ and $c = 1$ in equation (12), we obtain the first kind spherical Bessel's function of order p defined by (see [4]).
In this paper, we use the τ-Laplace transform to solve the Katugampola kinetic fractional equation [1, 17], which involves generalized Bessel's function of the first kind.
Here we involve some basic definitions as follows:

2. Preliminaries

Definition 2.1. *The Katugampola fractional operator is a generalization of the Riemann- Liouville integral operator into a unique form, which was given by U. N. Katugampola [19] such as for n* $\in \mathbb{C}$ *then*

$$\left[{}^{\tau}\mathbb{I}^n_{a+} \mathfrak{f} \right](x) = \frac{\tau^{(1-n)}}{\Gamma(n)} \int_a^x \frac{u^{(\tau-1)} \mathfrak{f}(u)}{(x^{\tau} - u^{\tau})^{(1-n)}} du, \qquad (\mathfrak{R}(n) > 0, \tau > 0). \tag{14}$$

This integral is called the left-sided fractional integral. Similarly, the right-sided fractional integral of order n $\in \mathbb{C}$ *is*

$$\left[{}^{\tau}\mathbb{I}^n_{b-} \mathfrak{f} \right](x) = \frac{\tau^{(1-n)}}{\Gamma(n)} \int_x^b \frac{u^{(\tau-1)} \mathfrak{f}(u)}{(u^{\tau} - x^{\tau})^{(1-n)}} du, \qquad (\mathfrak{R}(n) > 0, \tau > 0). \tag{15}$$

Definition 2.2. *Let $\mathfrak{f}\colon [0,\infty) \to \mathfrak{R}$ be a real valued function , if $\mathfrak{f}$ is piecewise continuous function and is of τ-exponential order $\exp\left(c\frac{t^\tau}{\tau}\right)$ where c is a non negative constant, then its τ-Laplace transform exists for $s > c$ and is defined as*

$$\mathbb{L}_\tau\{\mathfrak{f}(t); s\} = \int_0^\infty \exp\left(c\frac{t^\tau}{\tau}\right)\frac{\mathfrak{f}(t)}{t^{(1-\tau)}}dt, \qquad (\tau > 0). \tag{16}$$

Convolution of the functions $\mathfrak{f}(t)$ and $\mathfrak{g}(t)$, which are expressed for $t > 0$, is important in a variety of physical applications. The τ-Laplace convolution of functions $\mathfrak{f}(t)$ and $\mathfrak{g}(t)$ is given by the following integral:

$$\{\mathfrak{f}(t) *_\tau \mathfrak{g}(t)\} = \int_0^\infty \mathfrak{f}\{(t^\tau - \rho^\tau)^{\frac{1}{\tau}}\}\mathfrak{g}(\rho)\frac{d\rho}{\rho^{(1-\tau)}}, \qquad (\tau > 0). \tag{17}$$

Which remains exists if the functions $\mathfrak{f}$ and $\mathbf{g}$ are at least piecewise continuous. One of the most important properties procured by the convolution in connection with the τ- Laplace transform is that the τ-Laplace transform of the convolution of two functions is the product of their transforms (see, e.g.[1, 17]).

τ-Laplace Convolution Theorem *If two function $\mathbf{f}$ and $\mathbf{g}$ are piecewise continuous on the interval $[0,\infty)$ and of exponential order c when $t \to \infty$, then*

$$\mathbb{L}\{\mathbf{f}(t) *_\tau \mathbf{g}(t); s\} = \mathbb{L}\{\mathbf{f}(t); s\}.\mathbb{L}\{\mathbf{g}(t); s\}, \qquad (R(s) > 0).$$

We find the τ-Laplace transform of Katugampola Fractional integral is

$$\begin{aligned}
\mathbb{L}_\tau\{{}^\tau\mathbb{I}_0^n\mathbf{f}(t); s\} &= \frac{\tau^{1-n}}{\Gamma(n)}\mathbb{L}_\tau\{t^{\tau(n-1)} *_\tau \mathbf{f}(t); s\} \\
&= \frac{\tau^{1-n}}{\Gamma(n)}\mathbb{L}_\tau\{t^{\tau(n-1)}; s\}.\mathbb{L}_\tau\{\mathbf{f}(t); s\} \\
&= s^{-n}\mathbb{L}_\tau\mathbf{f}(t),
\end{aligned} \tag{18}$$

by using the identity

$$\mathbb{L}_\tau\{t^r; s\} = \tau^{\frac{r}{\tau}}\frac{\Gamma\left(1+\frac{r}{\tau}\right)}{s^{\left(1+\frac{r}{\tau}\right)}}, \qquad (r \in R, s > 0) \tag{19}$$

$$\iff \mathbb{L}_\tau^{-1}\left(\frac{1}{s^{1+\frac{r}{\tau}}}\right) = \frac{1}{\tau^{\frac{r}{\tau}}\Gamma\left(1+\frac{r}{\tau}\right)}t^r, \tag{20}$$

in which $\mathbb{L}_\tau^{-1}$ considered as the τ-inverse Laplace transform.

3. Main Results

3.1. Solution of Katugampola kinetic fractional equations

In this section, we achieve the solution of the Katugampola kinetic fractional equation involving the generalized Bessel function of the first kind by applying the τ-Laplace transform technique.

Theorem 3.1. *For all $c, b, l, t \in \mathbb{C}, d, n > 0$ and $\mathfrak{R}(l) > -1$ then equation*

$$\mathbb{N}(t) - \mathbb{N}_0\mathbb{W}_{l,b,c}(t) = -d^n\ {}^\tau\mathbb{I}_0^n\mathbb{N}(t), \tag{21}$$

there holds the formula

$$\mathbb{N}(t) = \mathbb{N}_0\sum_{r=0}^\infty \frac{(-c)^r\Gamma\left(1+\frac{2r+l}{\tau}\right)}{r!\Gamma\left(l+r+\frac{b+1}{2}\right)}\left(\frac{t}{2}\right)^{(2r+l)} \times \mathbb{E}_{n,\frac{(\tau+2r+l)}{\tau}}\left(\frac{-d^n t^{n\tau}}{\tau^n}\right) \tag{22}$$

where $\mathbb{W}_{l,b,c}$ is the generalised Bessel's function and $\mathbb{E}_{n,\frac{(\tau+2r+l)}{\tau}}$ is the generalised Mittag-Leffler function.

Proof. Applying τ-Laplace transform both side of equation (21), we have

$$\mathbb{L}_\tau\{\mathbb{N}(t); s\} = \mathbb{N}_0 \mathbb{L}_\tau\{\mathbb{W}_{l,b,c}(t); s\} - d^n \mathbb{L}_\tau\{{}^\tau\mathbb{I}_0^n \mathbb{N}(t); s\},$$

$$\mathbb{N}_\tau(s) = \mathbb{N}_0 \sum_{r=0}^{\infty} \frac{(-1)^r (c)^r}{r!\Gamma(r+l+(b+1)/2)} \left(\frac{1}{2}\right)^{(2r+l)} \mathbb{L}_\tau\{t^{(2r+l)}; s\} - d^n s^{-n} \mathbb{N}_\tau(s), \tag{23}$$

using equation (19) in equation(23), we get

$$\begin{aligned}\mathbb{N}_\tau(s) + d^n s^{-n} \mathbb{N}_\tau(s) &= \mathbb{N}_0 \sum_{r=0}^{\infty} \frac{(-1)^r (c)^r}{r!\Gamma(r+l+(b+1)/2)} \left(\frac{1}{2}\right)^{(2r+l)} \frac{\Gamma(1+\frac{(2r+l)}{\tau})}{s^{\frac{(2r+l+\tau)}{\tau}}} \\ &\quad \times \tau^{\frac{(2r+l)}{\tau}}, \\ \mathbb{N}_\tau(s) &= \frac{1}{(1+d^n s^{-n})} \mathbb{N}_0 \sum_{r=0}^{\infty} \frac{(-1)^r (c)^r}{r!\Gamma(r+l+(b+1)/2)} \left(\frac{1}{2}\right)^{(2r+l)} \\ &\quad \times \frac{\Gamma(1+\frac{(2r+l)}{\tau})}{s^{\frac{(2r+l+\tau)}{\tau}}} \tau^{\frac{(2r+l)}{\tau}}, \\ &= \mathbb{N}_0 \sum_{r=0}^{\infty} \frac{(-1)^r (c)^r}{r!\Gamma(r+l+(b+1)/2)} \left(\frac{1}{2}\right)^{(2r+l)} \frac{\Gamma(1+\frac{(2r+l)}{\tau})}{s^{\frac{(2r+l+\tau)}{\tau}}}, \\ &\quad \times \tau^{\frac{(2r+l)}{\tau}} \sum_{k=0}^{\infty} (-1)^k \left(\frac{s}{d}\right)^{-kn}. \end{aligned} \tag{24}$$

After, some simplifications in equation (24),we may write

$$\mathbb{N}_\tau(s) = \mathbb{N}_0 \sum_{r,k=0}^{\infty} \frac{(-1)^{r+k} (c)^r (d)^{kn}\, 2^{-(2r+l)} \Gamma(1+\frac{(2r+l)}{\tau})\, \tau^{\frac{2r+l}{\tau}}}{r!\, \Gamma(r+l+(b+1)/2)} \frac{1}{s^{(1+\frac{2r+l+kn\tau}{\tau})}}. \tag{25}$$

Taking τ-inverse Laplace transform both side of equation (25)and using equation (20) in it, we have

$$\begin{aligned}\mathbb{N}_\tau(t) &= \mathbb{N}_0 \sum_{r,k=0}^{\infty} \frac{(-1)^{r+k} (c)^r (d)^{kn}\, 2^{-(2r+l)} \Gamma(1+\frac{(2r+l)}{\tau})\, \tau^{\frac{2r+l}{\tau}}}{r!\, \Gamma(r+l+(b+1)/2)} \\ &\quad \times \frac{t^{(2r+l+kn\tau)}}{\tau^{\frac{2r+l}{\tau}+nk} \Gamma\left(1+\frac{(2r+l+kn\tau)}{\tau}\right)}, \\ &= \mathbb{N}_0 \sum_{r=0}^{\infty} \frac{(-1)^r (c)^r \Gamma(1+\frac{(2r+l)}{\tau})}{r!\, \Gamma(r+l+(b+1)/2)} \left(\frac{t}{2}\right)^{(2r+l)} \sum_{k=0}^{\infty} \left(\frac{(-1)^k d^n t^{n\tau}}{\tau^n}\right)^k \\ &\quad \times \frac{1}{\Gamma\left(\frac{(2r+l+\tau)}{\tau}+kn\right)}, \\ &= \mathbb{N}_0 \sum_{r=0}^{\infty} \frac{(-1)^r (c)^r \Gamma(1+\frac{(2r+l)}{\tau})}{r!\, \Gamma(r+l+(b+1)/2)} \left(\frac{t}{2}\right)^{(2r+l)} \mathbb{E}_{n,\frac{\tau+2r+l}{\tau}} \left(\frac{-d^n t^{n\tau}}{\tau^n}\right). \end{aligned} \tag{26}$$

□

This is a proposition This is an example.

Corollary 3.2. *If $d>0, n, \tau>0, l \in \mathbb{C}$ and $\Re(l) > -1$ then, the solution of the equation*

$$\mathbb{N}(t) - \mathbb{N}_0 \mathbb{J}_l(t) = -d^n\ {}^\tau\mathbb{I}_0^n \mathbb{N}(t), \tag{27}$$

there holds the formula

$$\mathbb{N}(t) = \mathbb{N}_0 \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma\left(1 + \frac{2r+l}{\tau}\right)}{r!\Gamma\left(l + r + 1\right)} \left(\frac{t}{2}\right)^{(2r+l)} \times \mathbb{E}_{n,\frac{(\tau+2r+l)}{\tau}} \left(\frac{-d^n t^{n\tau}}{\tau^n}\right) \tag{28}$$

where $\mathbb{J}_l$ is Bessel's function of first kind and $\mathbb{E}_{n,\frac{(\tau+2r+l)}{\tau}}$ is the generalized Mittag-Leffler function.

Corollary 3.3. *If $d > 0, n, \tau > 0, l \in \mathbb{C}$ and $\Re(l) > -1$ then, the solution of the equation*

$$\mathbb{N}(t) - \mathbb{N}_0 \mathbb{I}_l(t) = -d^n \ {}^{\tau}\mathbb{I}_0^n \mathbb{N}(t), \tag{29}$$

there holds the formula

$$\mathbb{N}(t) = \mathbb{N}_0 \sum_{r=0}^{\infty} \frac{\Gamma\left(1 + \frac{2r+l}{\tau}\right)}{r!\Gamma\left(l + r + 1\right)} \left(\frac{t}{2}\right)^{(2r+l)} \times \mathbb{E}_{n,\frac{(\tau+2r+l)}{\tau}} \left(\frac{-d^n t^{n\tau}}{\tau^n}\right) \tag{30}$$

where $\mathbb{I}_l$ is modified Bessel's function of first kind and $\mathbb{E}_{n,\frac{(\tau+2r+l)}{\tau}}$ is the generalized Mittag-Leffler function.

Theorem 3.4. *If $d > 0, n, \tau > 0, c, b, l, \in \mathbb{C}$ and $\Re(l) > -1$ then, the solution of the equation*

$$\mathbb{N}(t) - \mathbb{N}_0 \mathbb{W}_{l,b,c}(d^n t^n) = -d^n \ {}^{\tau}\mathbb{I}_0^n \mathbb{N}(t), \tag{31}$$

there holds the formula

$$\mathbb{N}(t) = \mathbb{N}_0 \sum_{r=0}^{\infty} \frac{(-c)^r \Gamma\left(1 + \frac{n(2r+l)}{\tau}\right)}{r!\Gamma\left(l + r + \frac{b+1}{2}\right)} \left(\frac{d^n t^n}{2}\right)^{(2r+l)} \times \mathbb{E}_{n,\frac{(\tau+n(2r+l))}{\tau}} \left(\frac{-d^n t^{n\tau}}{\tau^n}\right) \tag{32}$$

Proof. Applying τ-Laplace transform both side of equation (31), we have

$$\mathbb{L}_\tau\{\mathbb{N}(t); s\} = \mathbb{N}_0 \mathbb{L}_\tau\{\mathbb{W}_{l,b,c}(d^n t^n); s\} - d^n \mathbb{L}_\tau\{{}^{\tau}\mathbb{I}_0^n \mathbb{N}(t); s\},$$

$$\mathbb{N}_\tau(s) = \mathbb{N}_0 \sum_{r=0}^{\infty} \frac{(-c)^r}{r!\Gamma(r + l + (b+1)/2)} \left(\frac{d^n}{2}\right)^{(2r+l)} \mathbb{L}_\tau\{t^{n(2r+l)}; s\} - d^n s^{-n} \mathbb{N}_\tau(s), \tag{33}$$

using equation (19) in equation (33), we obtain

$$\begin{aligned}
\mathbb{N}_\tau(s) + d^n s^{-n} \mathbb{N}_\tau(s) &= \mathbb{N}_0 \sum_{r=0}^{\infty} \frac{(-c)^r}{r!\Gamma(r + l + (b+1)/2)} \left(\frac{d^n}{2}\right)^{(2r+l)} \\
&\quad \times \frac{\Gamma(1 + \frac{n(2r+l)}{\tau})}{s^{\left(1+\frac{n(2r+l)}{\tau}\right)}} \tau^{\frac{n(2r+l)}{\tau}}, \\
\mathbb{N}_\tau(s) &= \frac{1}{(1 + d^n s^{-n})} \mathbb{N}_0 \sum_{r=0}^{\infty} \frac{(-c)^r}{r!\Gamma(r + l + (b+1)/2)} \left(\frac{d^n}{2}\right)^{(2r+l)} \\
&\quad \times \frac{\Gamma(1 + \frac{n(2r+l)}{\tau})}{s^{\left(1+\frac{n(2r+l)}{\tau}\right)}} \tau^{\frac{n(2r+l)}{\tau}}, \\
&= \mathbb{N}_0 \sum_{r=0}^{\infty} \frac{(-c)^r}{r!\Gamma(r + l + (b+1)/2)} \left(\frac{d^n}{2}\right)^{(2r+l)} \frac{\Gamma(1 + \frac{n(2r+l)}{\tau})}{s^{\left(1+\frac{n(2r+l)}{\tau}\right)}} \\
&\quad \times \tau^{\frac{n(2r+l)}{\tau}} \sum_{k=0}^{\infty} (-1)^k \left(\frac{s}{d}\right)^{-kn}.
\end{aligned} \tag{34}$$

After, some simplifications in equation (34),we may write

$$\mathbb{N}_\tau(s) = \mathbb{N}_0 \sum_{r,k=0}^{\infty} \frac{(-1)^{r+k}(c)^r(d)^{n(k+2r+l)}\, 2^{-(2r+l)}\Gamma(1+\frac{n(2r+l)}{\tau})\, \tau^{\frac{n(2r+l)}{\tau}}}{r!\,\Gamma(r+l+(b+1)/2)} \times \frac{1}{s^{(1+\frac{n(2r+l+k\tau)}{\tau})}}. \tag{35}$$

Taking τ-inverse Laplace transform both side of equation (35) and using equation (20) in it, we have

$$\begin{aligned}
\mathbb{N}_\tau(t) &= \mathbb{N}_0 \sum_{r,k=0}^{\infty} \frac{(-1)^{r+k}(c)^r(d)^{n(k+2r+l)}\, 2^{-(2r+l)}\Gamma(1+\frac{n(2r+l)}{\tau})\, \tau^{\frac{n(2r+l)}{\tau}}}{r!\,\Gamma(r+l+(b+1)/2)} \\
&\quad \times \frac{t^{n(2r+l+k\tau)}}{\tau^{\frac{n(2r+l+k\tau)}{\tau}}\Gamma\left(1+\frac{n(2r+l+k\tau)}{\tau}\right)}, \\
&= \mathbb{N}_0 \sum_{r=0}^{\infty} \frac{(-c)^r\Gamma(1+\frac{n(2r+l)}{\tau})}{r!\,\Gamma(r+l+(b+1)/2)} \left(\frac{d^n t^n}{2}\right)^{(2r+l)} \sum_{k=0}^{\infty}\left(\frac{(-1)^k d^n t^{n\tau}}{\tau^n}\right)^k \\
&\quad \times \frac{1}{\Gamma\left(\frac{n(2r+l)+\tau}{\tau}+kn\right)}, \\
&= \mathbb{N}_0 \sum_{r=0}^{\infty} \frac{(-c)^r\Gamma(1+\frac{n(2r+l)}{\tau})}{r!\,\Gamma(r+l+(b+1)/2)} \left(\frac{d^n t^n}{2}\right)^{(2r+l)} \mathbb{E}_{n,\frac{\tau+n(2r+l)}{\tau}}\left(\frac{-d^n t^{n\tau}}{\tau^n}\right).
\end{aligned} \tag{36}$$

□

Corollary 3.5. *If $d > 0, n, \tau > 0, l \in \mathbb{C}$ and $\Re(l) > -1$ then, the solution of the equation*

$$\mathbb{N}(t) - \mathbb{N}_0\mathbb{J}_l(d^n t^n) = -d^n\ {}^{\tau}\mathbb{I}_0^n\mathbb{N}(t), \tag{37}$$

there holds the formula

$$\mathbb{N}(t) = \mathbb{N}_0 \sum_{r=0}^{\infty} \frac{(-1)^r\Gamma\left(1+\frac{2r+l}{\tau}\right)}{r!\Gamma(l+r+1)} \left(\frac{d^n t^n}{2}\right)^{(2r+l)} \times \mathbb{E}_{n,\frac{(\tau+2r+l)}{\tau}}\left(\frac{-d^n t^{n\tau}}{\tau^n}\right) \tag{38}$$

where $\mathbb{J}_l$ is Bessel's function of the first kind and $\mathbb{E}_{n,\frac{(\tau+2r+l)}{\tau}}$ is the generalized Mittag-Leffler function.

Corollary 3.6. *If $d > 0, n, \tau > 0, l \in \mathbb{C}$ and $\Re(l) > -1$ then, the solution of the equation*

$$\mathbb{N}(t) - \mathbb{N}_0\mathbb{I}_l(d^n t^n) = -d^n\ {}^{\tau}\mathbb{I}_0^n\mathbb{N}(t), \tag{39}$$

there holds the formula

$$\mathbb{N}(t) = \mathbb{N}_0 \sum_{r=0}^{\infty} \frac{\Gamma\left(1+\frac{2r+l}{\tau}\right)}{r!\Gamma(l+r+1)} \left(\frac{d^n t^n}{2}\right)^{(2r+l)} \times \mathbb{E}_{n,\frac{(\tau+2r+l)}{\tau}}\left(\frac{-d^n t^{n\tau}}{\tau^n}\right) \tag{40}$$

where $\mathbb{I}_l$ is modified Bessel's function of first kind and $\mathbb{E}_{n,\frac{(\tau+2r+l)}{\tau}}$ is the generalized Mittag-Leffler function.

Theorem 3.7. *If $a, d > 0, n, \tau > 0, c, b, l, \in \mathbb{C}$ and $\Re(l) > -1$ then, the solution of the equation*

$$\mathbb{N}(t) - \mathbb{N}_0\mathbb{W}_{l,b,c}(d^n t^n) = -a^n\ {}^{\tau}\mathbb{I}_0^n\mathbb{N}(t), \tag{41}$$

there holds the formula

$$\mathbb{N}(t) = \mathbb{N}_0 \sum_{r=0}^{\infty} \frac{(-c)^r\Gamma\left(1+\frac{n(2r+l)}{\tau}\right)}{r!\Gamma\left(l+r+\frac{b+1}{2}\right)} \left(\frac{d^n t^n}{2}\right)^{(2r+l)} \times \mathbb{E}_{n,\frac{(\tau+n(2r+l))}{\tau}}\left(\frac{-a^n t^{n\tau}}{\tau^n}\right) \tag{42}$$

Proof. This result can be verified by the similar procedure given in the proof of Theorem 1. So we omit all details. □

3.2. Graphical representations

Figures 1–6 describe the graphical form of the equation (22) by using some fractional values of *n*, and we can see that if we use small values of *n*, the graph increases and then decreases in nature. Aside from that, if we choose larger values for *n*, it gradually decreases. The graphical results show that the region of convergence of solutions depends on the fractional parameter *n* in a continuous manner. As a result, by watching the behavior of the solutions for different parameters and time intervals, it is deduced that $\mathbb{N}(t)$ can be both negative and positive. A similar observation can be made for solutions (32) and (42).

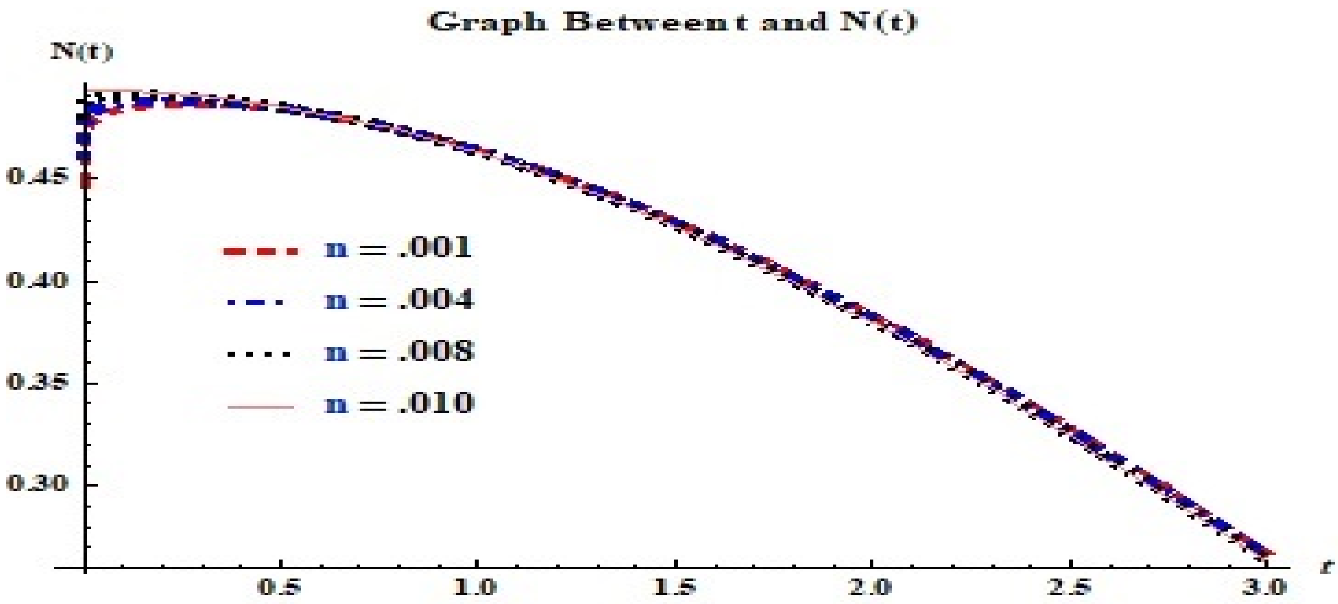

Figure 1: Curve between t and $\mathbb{N}(t)$ with different value of n

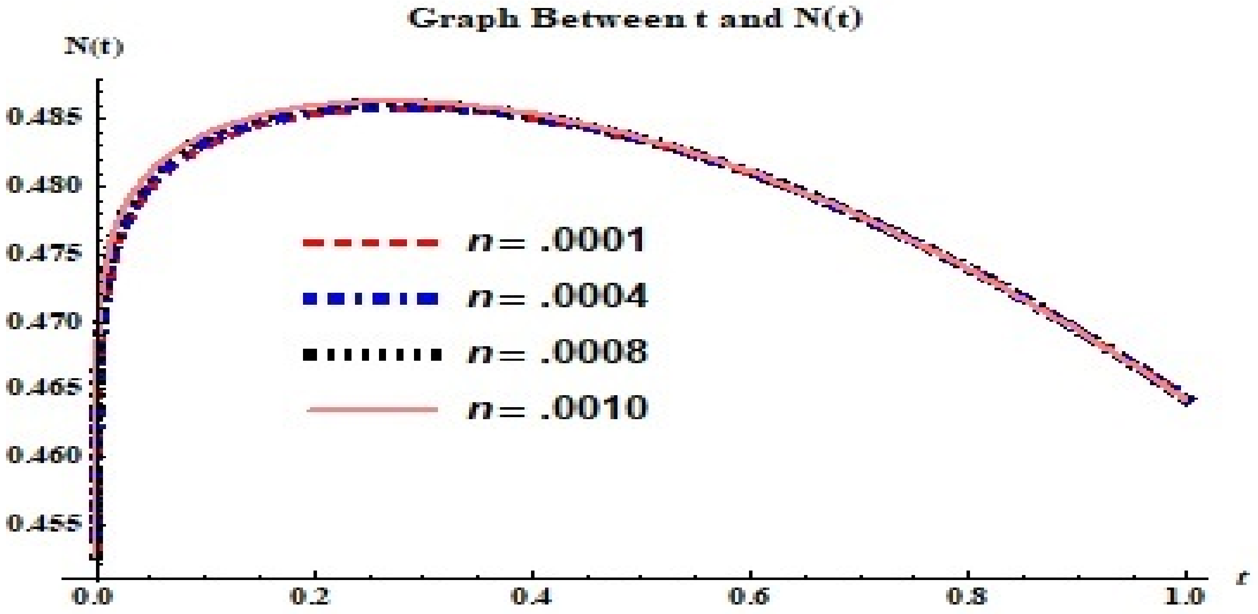

Figure 2: Curve between t and $\mathbb{N}(t)$ with different value of n

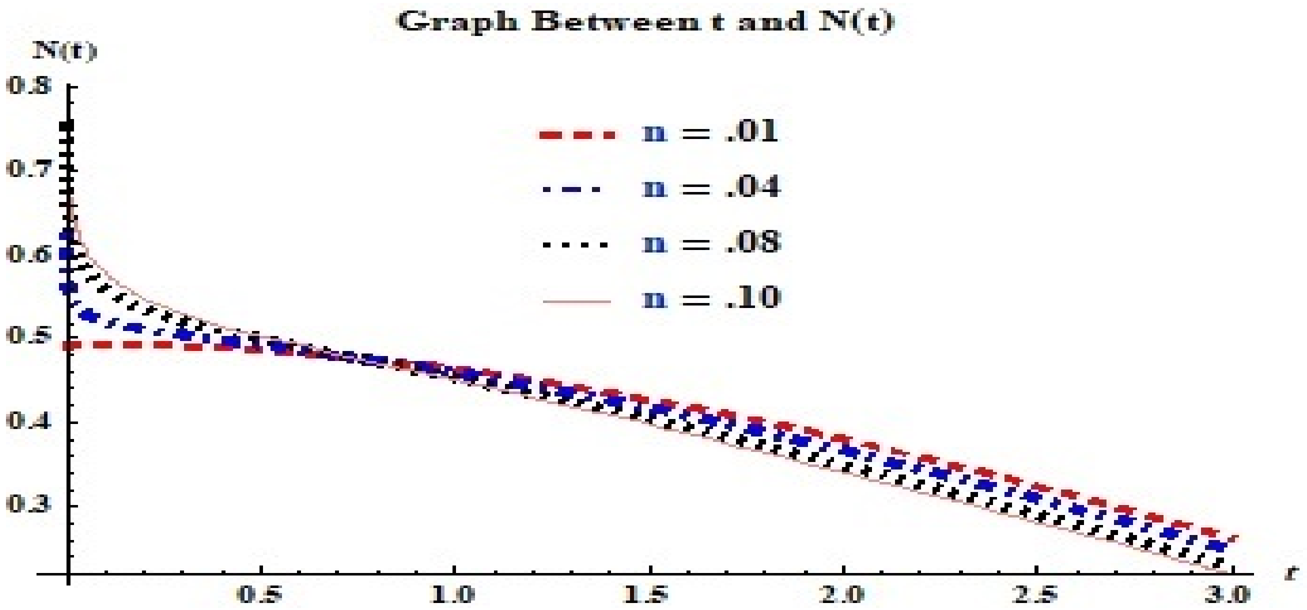

Figure 3: Curve between t and $\mathbb{N}(t)$ with different value of n

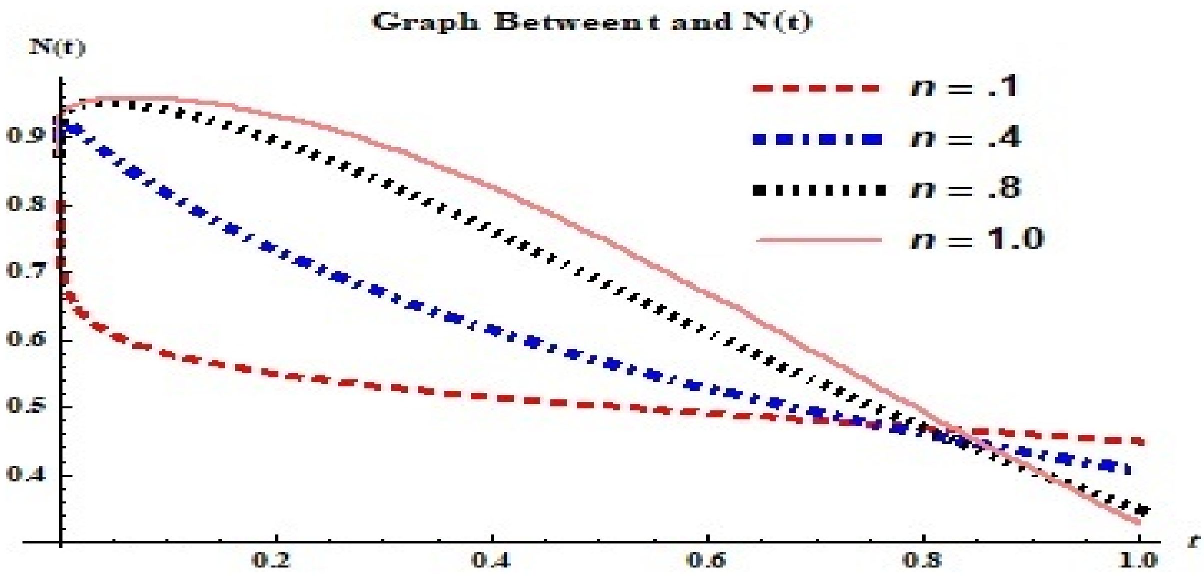

Figure 4: Curve between t and $\mathbb{N}(t)$ with different value of n

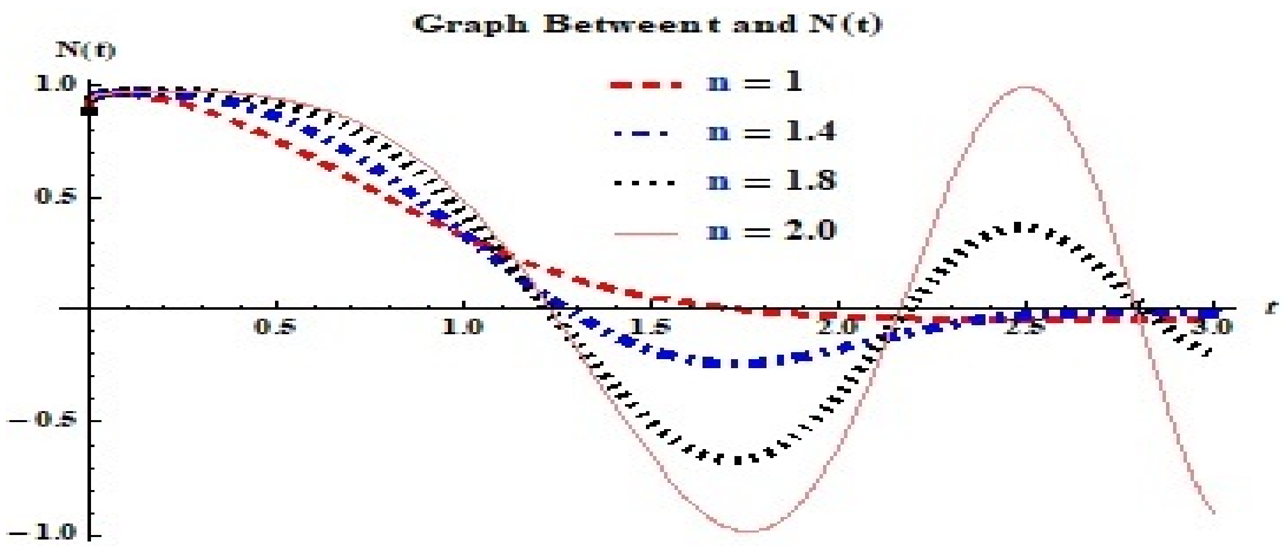

Figure 5: Curve between t and $\mathbb{N}(t)$ with different value of n

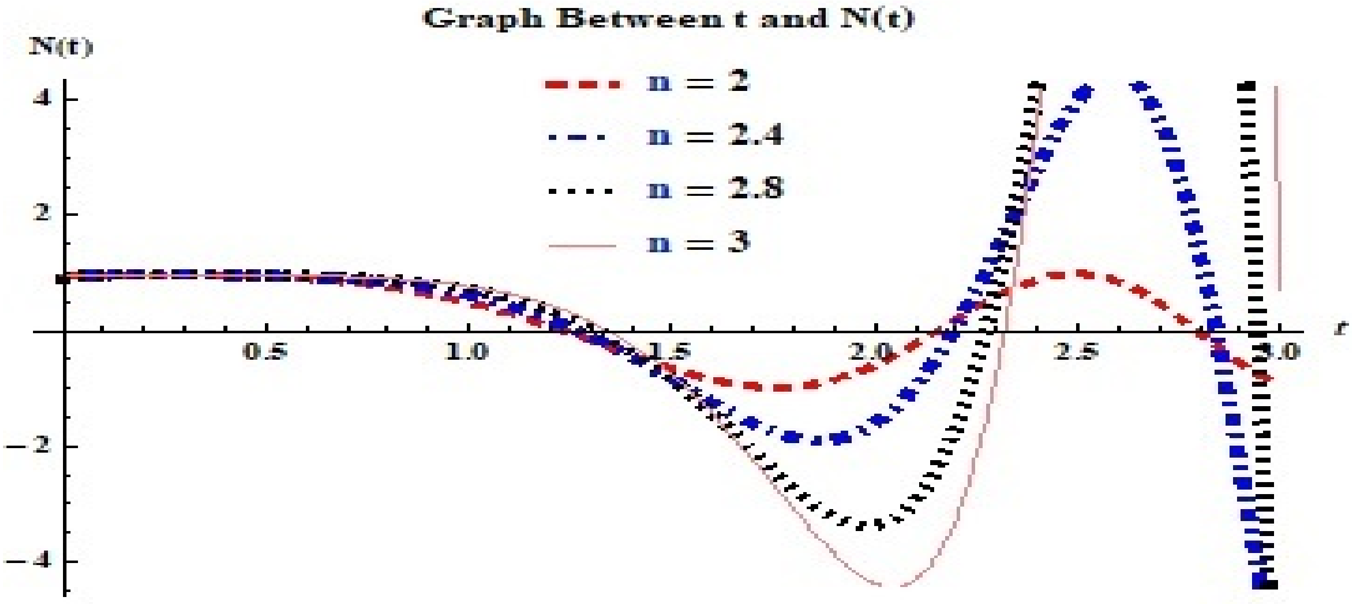

Figure 6: Curve between t and $\mathbb{N}(t)$ with different value of n

Acknowledgement

The authors express their sincere thanks to the reviewers for their careful reading and suggestions that helped to improve this paper.

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