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Research Article

A family of Newton-type methods with seventh and eighth-order of convergence for solving systems of nonlinear equations

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Abstract

In this work, we first develop a new family of three-step seventh- and eighth-order Newton-type iterative methods for solving systems of nonlinear equations. We also propose some different choices of parameter matrices that ensure the convergence order. The proposed family includes some known methods as special cases. The computational cost and efficiency index of our methods are discussed. Numerical experiments are conducted to support the theoretical results.

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1. Introduction

Nonlinear systems arise in many applications in civil engineering, theoretical physics, finance, chemistry, biology, nanotechnology etc. Finding the solution of systems of nonlinear equations $\mathcal{F}(x) = 0$ is an important problem with a wide variety of applications in numerical analysis, where $\mathcal{F} : D \subseteq R^n \to R^n$ and D is an open convex domain in R^n. For this purpose, the most widely used iterative scheme is Newtons method [22]

$$x_{k+1} = x_k - \mathcal{F}'(x_k)^{-1}\mathcal{F}(x_k), \quad k = 0, 1, \dots,$$

where x_0 is the initial approximation and $\mathcal{F}'(x_k)^{-1}$ is the inverse of Fréchet derivative $\mathcal{F}'(x_k)$ of the function $\mathcal{F}(x)$. On the other hand, high-order methods achieve sufficiently high accuracy as well as the desired order of convergence with fewer iterations, see [6,9,13]. In [10,14,18] the usefulness of higher order iterative methods is explained. Generally, multi-step methods with high order have overcome the theoretical limit of one-step methods concerning the convergence order; see for instance [1–5,11,12,15–17,21–27] and references.

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This paper aims to develop a new family of high-order iterative methods with high-efficiency indexes and low computational cost. In section 2, we present a family of three-step seventh-order iterative methods. As per our knowledge, existing seventh-order three-step methods were designed only for special cases $a = 1$ (see Eq. (2.3)), while our scheme was designed for any real parameter $a \neq 0$. We suggest concrete choices of parameters in this family. It should be pointed out that the results of Theorems 2.3 and 2.5 (see section 2) are novel contributions to construct new seventh-order iterations. The obtained sufficient convergence condition allows us not only to prove the convergence of known methods analytically but also to design new iterations. In section 3, we propose a family of eighth-order iterative methods. We also suggest five different choices for parameters that ensure the eighth-order convergence of methods. The proposed family includes some existing methods as particular cases. Additionally, a derivative-free variant of the proposed method is also presented in this section. The computational cost and efficiency index of the proposed iterative methods are discussed in section 4. Numerical examples are given in section 5 to illustrate the convergence behavior of the proposed methods.

2. Seventh-order iterations

We recall the following result of Taylor's expansion on vector functions (see [16]). Letting $\mathcal{F} : D \subseteq R^n \to R^n$ be $p-$times Fréchet differentiable in a convex set $D \subseteq R^n$, then for any $x, \hat{h} \in D$ the following expression holds:

$$\mathcal{F}(x+\hat{h}) = \mathcal{F}(x) + \mathcal{F}'(x)\hat{h} + \frac{1}{2!}\mathcal{F}''(x)\hat{h}^2 + \frac{1}{3!}\mathcal{F}'''(x)\hat{h}^3 + \cdots + \frac{1}{p!}\mathcal{F}^{(p)}(x)\hat{h}^p + R_p, \tag{2.1}$$

where

$$\|R_p\| \leq \frac{1}{p!}\sup_{0<t<1}\|\mathcal{F}^{(p)}(x+t\hat{h})\|\|\hat{h}\|^p \quad and \quad \hat{h}^p = (\overbrace{\hat{h}, \hat{h}, \cdots, \hat{h}}^{p}), \tag{2.2}$$

and $\|\cdot\|$ denotes any norm in R^n, or a corresponding operator norm. We also will use the following definition [7]:

Definition 2.1. Let $\{x_k\}_{k\geq 0}$ be a sequence in R^n which converges to x^*. Then, the convergence is called of order p, $p > 1$, if

$$\frac{\|\mathcal{F}(x_{k+1})\|}{\|\mathcal{F}(x_k)\|^p} = O(1)$$

or

$$\mathcal{F}(x_{k+1}) = O(h^p), \quad h = \mathcal{F}(x_k), \quad h^p = (\overbrace{h, h, \cdots, h}^{p}).$$

We consider the following three-point iterative method:

$$\begin{aligned} y_k &= x_k - a\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(x_k),\ a \in R \setminus \{0\}, \\ z_k &= \psi_p(x_k, y_k) = y_k - \bar{\tau}_k\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(y_k), \\ x_{k+1} &= z_k - \alpha_k\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(z_k), \end{aligned} \tag{2.3}$$

where $\bar{\tau}_k$ and α_k are matrices to be determined for seventh and eight-order convergence and $\psi_p(x_k, y_k)$ is an iteration with order $p \geq 3$ and we assume that $\mathcal{F}'(x_k)$ is invertible for all k. Let τ_k be the matrix depending on $\mathcal{F}(x_k)$ and derivatives of $\mathcal{F}(x_k)$ at x_k satisfying

$$z_k = x_k - \tau_k\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(x_k), \tag{2.4}$$

where x_k and z_k are as in (2.3). In [25], it was proven that $\mathcal{F}(z_k) = \mathcal{O}(h^4)$ if τ_k satisfies

$$\tau_k = I + Q_k + 2Q_k^2 + \bar{d}_k + \mathcal{O}(h^3), \tag{2.5}$$

where

$$Q_k = \frac{1}{2}\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}''(x_k)\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(x_k), \tag{2.6}$$

and

$$\bar{d}_k = -\frac{1}{6}\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}'''(x_k)\left(\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(x_k)\right)^2. \tag{2.7}$$

To obtain high order iterations of type (2.3), we need the following assumption

$$\frac{1}{2}\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}''(x_k)Q_k\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(x_k) = Q_k^2 + \mathcal{O}(h^3) \tag{2.8}$$

which holds for some methods.

To proceed with the convergence analysis of iteration (2.3), first, we will establish the relation of $\bar{\tau}_k$ and τ_k which allows ensuring the fourth-order convergence of $\mathcal{F}(z_k)$.

Theorem 2.2. *The fourth-order convergence condition* (2.5) *for* τ_k *is equivalent to*

$$\bar{\tau}_k = I + (a+1)Q_k + (a^2+2a+2)Q_k^2 + (a^2+a+1)\bar{d}_k + \mathcal{O}(h^3). \tag{2.9}$$

Proof. From (2.3), (2.4) we can get

$$x_k - \tau_k\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(x_k) = x_k - a\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(x_k) - \bar{\tau}_k\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(y_k).$$

Then

$$\bar{\tau}_k\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(y_k) = (\tau_k - aI)\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(x_k). \tag{2.10}$$

On the other hand, using the Taylor expansion of $\mathcal{F}(y_k)$ around x_k gives

$$\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(y_k) = \Big((1-a)I + a^2Q_k + a^3\bar{d}_k\Big)\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(x_k) + \mathcal{O}(h^4). \tag{2.11}$$

Using (2.11) in (2.10) we get

$$\bar{\tau}_k(1-a)\Big(I + \frac{a^2Q_k + a^3\bar{d}_k}{1-a}\Big) = \tau_k - aI + \mathcal{O}(h^3). \tag{2.12}$$

Since $Q_k = \mathcal{O}(h)$, $\bar{d}_k = \mathcal{O}(h^2)$ under (2.6), (2.7) and using well-known expansion

$$(I-P)^{-1} = I + P + P^2 + \cdots, \; \|P\| < 1, \tag{2.13}$$

from (2.12) we obtain

$$\begin{aligned}
\bar{\tau}_k =& \frac{aI - \tau_k}{a-1}\Big(I + \frac{a^2Q_k + a^3\bar{d}_k}{1-a}\Big)^{-1} + \mathcal{O}(h^3)\\
=& \Big(I + \frac{Q_k + 2Q_k^2 + \bar{d}_k}{1-a}\Big)\Big(I - \frac{a^2Q_k + a^3\bar{d}_k}{1-a} + \left(\frac{a^2Q_k + a^3\bar{d}_k}{1-a}\right)^2 + \cdots\Big) + \mathcal{O}(h^3)\\
=& I + (a+1)Q_k + (a^2+2a+2)Q_k^2 + (a^2+a+1)\bar{d}_k + \mathcal{O}(h^3),
\end{aligned}$$

in which we used (2.5). Thus, we arrive at (2.9). The converse is obvious from (2.9) and (2.10). □

In [25] it was confirmed that $\mathcal{F}(z_k) = O(h^4)$ under condition

$$\bar{\tau}_k = I + 2Q_k + \mathcal{O}(h^2). \tag{2.14}$$

Theorem 2.3. *Let the function* $\mathcal{F} : D \subseteq R^n \to R^n$ *be sufficiently differentiable in an open neighborhood* D *of* x^**, that is a solution of* $\mathcal{F}(x) = 0$*. Moreover, assume the Jacobian matrix of* $\mathcal{F}$ *is continuous and nonsingular in* D*. Suppose that the assumption (2.8) is also fulfilled. Then, for an initial value* x_0 *sufficiently close to* x^**, the iteration (2.3) has the order of convergence* $p+3$ *when the matrix* α_k *in (2.3) satisfies*

$$\alpha_k = I + 2Q_k + 3\bar{d}_k + 6Q_k^2 + \mathcal{O}(h^3). \tag{2.15}$$

Proof. Using Taylor's formula of $\mathcal{F}(x_{k+1})$ in a neighborhood z_k, one gets:

$$\mathcal{F}(x_{k+1}) = \Big(I - \mathcal{F}'(z_k)\alpha_k \left[\mathcal{F}'(x_k)\right]^{-1}\Big)\mathcal{F}(z_k) + \mathcal{O}(\|\mathcal{F}(z_k)\|^2). \tag{2.16}$$

Due to $\mathcal{F}(z_k) = \mathcal{O}(h^p)$, then from (2.16) it is obvious that

$$\mathcal{F}(x_{k+1}) = \mathcal{O}(h^{p+3}),$$

under the condition

$$\alpha_k = \left[\mathcal{F}'(z_k)\right]^{-1} \mathcal{F}'(x_k) + \mathcal{O}(h^3). \tag{2.17}$$

Hence, we have to find the inverse of $\mathcal{F}'(z_k)$ with accuracy $\mathcal{O}(h^3)$. To make it, we apply Taylor's formula of $\mathcal{F}'(z_k)$ at point x_k

$$\mathcal{F}'(z_k) = \mathcal{F}'(x_k) + \mathcal{F}''(x_k)(z_k - x_k) + \frac{\mathcal{F}'''(x_k)}{2}(z_k - x_k)^2 + \mathcal{O}((z_k - x_k)^3). \tag{2.18}$$

From the first two steps in (2.3), we can obtain

$$z_k - x_k = -a\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(x_k) - \bar{\tau}_k\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(y_k). \tag{2.19}$$

Using (2.11) in (2.19) we get

$$z_k - x_k = -(aI + (1-a)\bar{\tau}_k + a^2\bar{\tau}_k Q_k)\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(x_k) + \mathcal{O}(h^3). \tag{2.20}$$

Using (2.9) in (2.20) we obtain

$$z_k - x_k = -(I + Q_k)\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(x_k) + \mathcal{O}(h^3). \tag{2.21}$$

Substituting (2.21) into (2.18) we get

$$\begin{aligned}\mathcal{F}'(z_k) &= \mathcal{F}'(x_k)(I - \left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}''(x_k)(I + Q_k)\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(x_k)\\ &\quad + \frac{1}{2}\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}'''(x_k)\Big(\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(x_k)\Big)^2) + \mathcal{O}(h^3)\\ &= \mathcal{F}'(x_k)(I - 2Q_k - 2Q_k^2 - 3\bar{d}_k) + \mathcal{O}(h^3),\end{aligned}$$

in which (2.6), (2.7), and (2.8) are used. Using (2.13), we get

$$\begin{aligned}\left[\mathcal{F}'(z_k)\right]^{-1} &= (I - 2Q_k - 2Q_k^2 - 3\bar{d}_k)^{-1}\left[\mathcal{F}'(x_k)\right]^{-1} + \mathcal{O}(h^3)\\ &= (I + 2Q_k + 2Q_k^2 + 3\bar{d}_k + 4Q_k^2)\left[\mathcal{F}'(x_k)\right]^{-1} + \mathcal{O}(h^3).\end{aligned} \tag{2.22}$$

Substituting (2.22) into (2.17) we arrive at (2.15). □

Thus, the choice (2.15) ensures $\mathcal{F}(x_{k+1}) = \mathcal{O}(h^{p+3})$.

Remark 2.4. The parameter a in (2.3) belongs to the set $R\setminus\{0\}$. But there is a suitable choice for a from computational point of view.

From (2.3) we see that the first step in (2.3) can be considered independent Newton method with damping parameter a. According to [22], it is desirable to choose

$$0 < a < 2. \tag{2.23}$$

Indeed, from (2.11) we have

$$\|\mathcal{F}(y_k)\| < |1-a|\|\mathcal{F}(x_k)\| + \mathcal{O}(h^2).$$

That is, such choice (2.23) allows us to decrease the residual $\|\mathcal{F}(y_k)\|$ and thereby speed up the convergence of iteration (2.3).

The direct consequence of Theorem 2.2 and 2.3 is:

Theorem 2.5. *Let assumptions of Theorem 2.3 be satisfied. Then the convergence order of iteration* (2.3) *equal to seven, when* $\bar{\tau}_k$ *and* α_k *satisfy* (2.9) *(*(2.14) *when* $a = 1$*) and* (2.15) *respectively.*

Proof. By Theorem 2.2, $\mathcal{F}(z_k) = \mathcal{O}(h^4)$ provided (2.9). Then under Theorem 2.3, $\mathcal{F}(x_{k+1}) = \mathcal{O}(h^7)$ under choice (2.15). $\square$

Note that Theorems 2.3 and 2.5 hold for any value of parameter a. Our next goal is to express Q_k and $\bar{d}_k$ in (2.9), (2.15) in terms of $\mathcal{F}(x_k)$, $\mathcal{F}'(x_k)$ and $\mathcal{F}'(y_k)$ for computational convenience. To do this, we introduce

$$w_k = x_k + a\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(x_k). \tag{2.24}$$

From the first equation in (2.3) and (2.24), we find that

$$w_k = 2x_k - y_k.$$

Further, we will consider Taylor's formula of $\mathcal{F}'(y_k)$ around the point x_k:

$$\mathcal{F}'(y_k) = \mathcal{F}'(x_k)\Big(I - 2aQ_k - 3a^2\bar{d}_k\Big) + \mathcal{O}(h^3). \tag{2.25}$$

Analogously, the following expression holds:

$$\mathcal{F}'(w_k) = \mathcal{F}'(x_k)\Big(I + 2aQ_k - 3a^2\bar{d}_k\Big) + \mathcal{O}(h^3). \tag{2.26}$$

Let us also use the notations

$$t_k = \left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}'(y_k),\ s_k = \left[\mathcal{F}'(y_k)\right]^{-1}\mathcal{F}'(x_k),\ q_k = \left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}'(w_k).$$

Then from (2.25) and (2.26), it follows that

$$t_k = I - 2aQ_k - 3a^2\bar{d}_k + \mathcal{O}(h^3), \tag{2.27}$$

and

$$q_k = I + 2aQ_k - 3a^2\bar{d}_k + \mathcal{O}(h^3). \tag{2.28}$$

Using (2.27) and (2.13) we get

$$s_k = t_k^{-1} = I + 2aQ_k + 3a^2\bar{d}_k + 4a^2Q_k^2 + \mathcal{O}(h^3). \tag{2.29}$$

From (2.27) and (2.28) we find that

$$Q_k = \frac{q_k - t_k}{4a} + \mathcal{O}(h^3), \tag{2.30}$$

and

$$\bar{d}_k = \frac{2I - q_k - t_k}{6a^2} + \mathcal{O}(h^3). \tag{2.31}$$

Substituting (2.30) and (2.31) into (2.9) and (2.15), we obtain

$$\begin{aligned} \bar{\tau}_k = I &+ (1+a)\frac{q_k - t_k}{4a} + (a^2 + 2a + 2)\left(\frac{q_k - t_k}{4a}\right)^2 \\ &+ (1 + a + a^2)\frac{2I - q_k - t_k}{6a^2} + \mathcal{O}(h^3), \end{aligned} \tag{2.32}$$

and

$$\alpha_k = I + \frac{q_k - t_k}{2a} + \frac{2I - q_k - t_k}{2a^2} + 6\left(\frac{q_k - t_k}{4a}\right)^2 + \mathcal{O}(h^3). \tag{2.33}$$

When $a = 1$ the formulas (2.14) and (2.15) are expressed only through t_k. Indeed, according to (2.27) we have

$$2Q_k + 3\bar{d}_k = I - t_k + \mathcal{O}(h^3) \quad \Longrightarrow \quad Q_k = \frac{1}{2}(I - t_k) - \frac{3}{2}\bar{d}_k + \mathcal{O}(h^3), \tag{2.34}$$

and

$$Q_k^2 = \frac{1}{4}(I - t_k)^2 + \mathcal{O}(h^3), \tag{2.35}$$

because of $I - t_k = \mathcal{O}(h)$ and $\bar{d}_k = \mathcal{O}(h^2)$. Substituting (2.34), (2.35) into (2.14) and (2.15) we obtain

$$\bar{\tau}_k = 2I - t_k + \mathcal{O}(h^2),$$
$$\alpha_k = \frac{7}{2}I - 4t_k + \frac{3}{2}t_k^2 + \mathcal{O}(h^3).$$

Thus, we arrive at the result of [4]. We denote it by BA7.

Remark 2.6. One can replace Q_k^2 in (2.9) and (2.15) by $\left(\dfrac{I - t_k}{2a}\right)^2$ without loss of accuracy.

Theorem 2.5 remains true for $\bar{\tau}_k$ and α_k given by (2.32) and (2.33) respectively, which is the first main finding of this section. The seventh-order methods proposed in [1, 15, 19, 20, 24] are different from our method (2.3) with (2.9), (2.14), and (2.15). The use of sufficient convergence condition allows us not only to prove convergence of methods but also to drive new iterations. As an example, we consider the following iterations [1]

$$\begin{aligned} y_k &= x_k - [\mathcal{F}'(x_k)]^{-1}\mathcal{F}(x_k),\\ z_k &= y_k - G(t_k)\,[x_k, y_k; \mathcal{F}]^{-1}\mathcal{F}(y_k),\\ x_{k+1} &= z_k - H(t_k)\,[y_k, z_k; \mathcal{F}]^{-1}\mathcal{F}(z_k), \end{aligned} \tag{2.36}$$

where $G(t_k)$ and $H(t_k)$ are some weight functions to be determined properly and the symmetric divided difference $[x, y; \mathcal{F}]$ of $\mathcal{F}$. The divided difference $[x, y; \mathcal{F}]$ of $\mathcal{F}$ is an $n \times n$ matrix with elements (see [4, 15, 19])

$$[x, y; \mathcal{F}]_{i,j} = \frac{\mathcal{F}_i(x_1, \dots, x_j, y_{j+1}, \dots, y_n) - \mathcal{F}_i(x_1, \dots, x_{j-1}, y_j, \dots, y_n)}{x_j - y_j},$$

where $1 \le i, j \le n$. It is easy to show that

$$[x_k, y_k; \mathcal{F}] = \mathcal{F}'(x_k)(I - Q_k - \bar{d}_k) + \mathcal{O}(h^3),$$

and

$$[x_k, y_k; \mathcal{F}]^{-1} = (I + Q_k)[\mathcal{F}'(x_k)]^{-1} + \mathcal{O}(h^2). \tag{2.37}$$

Hence,

$$\gamma_k := I - [\mathcal{F}'(x_k)]^{-1}[x_k, y_k; \mathcal{F}(x_k)] = Q_k + \mathcal{O}(h^2). \tag{2.38}$$

If we take (2.37) into account, then, using condition (2.14), we have

$$\bar{\tau}_k = G(t_k)(I + Q_k) = I + 2Q_k + \mathcal{O}(h^2).$$

From this and from (2.38), we can obtain

$$G(t_k) = (I + 2Q_k)(I + Q_k)^{-1} + \mathcal{O}(h^2) = I + Q_k + \mathcal{O}(h^2) = I + \gamma_k. \tag{2.39}$$

Analogously, using (2.36), (2.39) we obtain

$$[y_k, z_k; \mathcal{F}] = \mathcal{F}'(y_k)(I - Q_k^2) + \mathcal{O}(h^4). \tag{2.40}$$

Using (2.25) and $\mathcal{F}(y_k) = \frac{1}{2}\mathcal{F}''(x_k)\xi_k^2 + \mathcal{O}(h^3)$ in (2.40) we have

$$[y_k, z_k; \mathcal{F}] = \mathcal{F}'(x_k)(I - 2Q_k - 3\bar{d}_k - Q_k^2) + \mathcal{O}(h^3).$$

Hence, by (2.13), we obtain

$$[y_k, z_k; \mathcal{F}]^{-1} = (I + 2Q_k + 3\bar{d}_k + 5Q_k^2)[\mathcal{F}'(x_k)]^{-1} + \mathcal{O}(h^3).$$

The comparison of (2.3) and (2.36), and the sufficient convergence condition (2.15) give

$$\alpha_k = H(t_k)(I + 2Q_k + 3\bar{d}_k + 5Q_k^2) = I + 2Q_k + 3\bar{d}_k + 6Q_k^2 + \mathcal{O}(h^3).$$

From this and from (2.38), we get

$$H(t_k) = I + Q_k^2 + \mathcal{O}(h^3) = I + \gamma_k^2 + \mathcal{O}(h^3). \tag{2.41}$$

Thus, we arrive at the scheme (M7) proposed in [1], where the weight functions $G(t_k)$ and $H(t_k)$ are given by (2.39) and (2.41) respectively. Let us consider another three-step iteration

$$\begin{aligned} y_k &= x_k - [w_k, s_k; \mathcal{F}]^{-1}\mathcal{F}(x_k), \\ z_k &= y_k - G\,[w_k, s_k; \mathcal{F}]^{-1}\mathcal{F}(y_k), \\ x_{k+1} &= z_k - H\,[w_k, s_k; \mathcal{F}]^{-1}\mathcal{F}(z_k), \end{aligned} \tag{2.42}$$

where $w_k = x_k + \gamma\mathcal{F}(x_k)$, $s_k = x_k - \gamma\mathcal{F}(x_k)$ and G, H, as in (2.36), are weight functions to be determined properly. The comparison of (2.3) and (2.42) yields

$$\bar{\tau}_k = G\,[w_k, s_k; \mathcal{F}]^{-1}\mathcal{F}'(x_k), \tag{2.43}$$

and

$$\alpha_k = H\,[w_k, s_k; \mathcal{F}]^{-1}\mathcal{F}'(x_k). \tag{2.44}$$

We call $[w_k, s_k; \mathcal{F}]^{-1}\mathcal{F}'(x_k)$ transition factor that allows to pass from (3.1) to (2.42) and vice versa. It is easy to show that

$$[w_k, s_k; \mathcal{F}]^{-1}\mathcal{F}'(x_k) = I - B_k + \mathcal{O}(h^4), \tag{2.45}$$

where

$$B_k = \frac{1}{6}[\mathcal{F}'(x_k)]^{-1}\mathcal{F}'''(x_k)\gamma_k^2\mathcal{F}^2(x_k) + \mathcal{O}(h^2). \tag{2.46}$$

It is known from [25] the sufficient seventh-order convergence conditions for (2.3) are (2.14) and (2.15). Using (2.14), (2.15), and (2.45), from (2.43) and (2.44) we obtain

$$G = I + 2Q_k, \quad H = I + 2Q_k + 6Q_k^2 + 3\bar{d}_k + B_k. \tag{2.47}$$

Thus, by virtue of sufficient conditions (2.14) and (2.15) we obtain a new seventh-order iteration (2.42) with weight-functions given by (2.47). Moreover, when $a = 1$, the order of convergence of our scheme increases by one and become equal to eight (see below Theorem 3.1). This is another advantage of our scheme.

3. Eighth-order iterations

Let $a = 1$ in (2.3). Then the iteration (2.3) becomes as:

$$\begin{aligned} y_k &= x_k - \left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(x_k), \\ z_k &= \psi_p(x_k, y_k) = y_k - \bar{\tau}_k\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(y_k), \\ x_{k+1} &= z_k - \alpha_k\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(z_k). \end{aligned} \tag{3.1}$$

When $a = 1$, the condition (2.9) leads to

$$\bar{\tau}_k = I + 2Q_k + 5Q_k^2 + 3\bar{d}_k + \mathcal{O}(h^3). \tag{3.2}$$

In this case, one can expect stronger results than Theorem 2.5.

Theorem 3.1. *Let the assumptions of Theorem 2.3 be satisfied. Then the convergence order of iteration* (3.1) *equal to eight, when $\bar{\tau}_k$ and α_k satisfy* (3.2) *and* (2.15) *respectively.*

Proof. The Taylor expansion of $\mathcal{F}(z_k)$ around the point y_k is given as:

$$\begin{aligned} \mathcal{F}(z_k) &= \mathcal{F}(y_k) - \mathcal{F}'(y_k)\bar{\tau}_k\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(y_k) \\ &+ \frac{\mathcal{F}''(y_k)}{2}(\bar{\tau}_k\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(y_k))^2 + \mathcal{O}((\mathcal{F}(y_k))^3). \end{aligned} \tag{3.3}$$

Keeping in mind (2.6), (2.8), (3.2), and $\mathcal{F}''(y_k) = \mathcal{F}''(x_k) + \mathcal{O}(h)$, $\mathcal{F}(y_k) = \mathcal{O}(h^2)$ and neglecting small term $O(h^5)$ in (3.3) we can reach

$$\mathcal{F}(z_k) = \mathcal{F}'(x_k)(I - t_k\bar{\tau}_k + Q_k^2)\left[\mathcal{F}'(x_k)\right]^{-1}\mathcal{F}(y_k) + \mathcal{O}(h^5). \tag{3.4}$$

Since $\mathcal{F}(y_k) = \mathcal{O}(h^2)$, then from expression (3.4) it is obvious that

$$\mathcal{F}(z_k) = \mathcal{O}(h^5), \tag{3.5}$$

provided

$$I - t_k\bar{\tau}_k + Q_k^2 = \mathcal{O}(h^3)$$

i.e.,

$$\bar{\tau}_k = t_k^{-1}(I + Q_k^2) + \mathcal{O}(h^3). \tag{3.6}$$

Using (2.29) with $a = 1$ in (3.6) we get (3.2). Thus, (3.5) holds under (3.2). Due to Theorem 2.3, the order of iteration (3.1) equals $p + 3 = 8$. $\square$

Remark 3.2. Repeating application of Theorem 2.3 and Theorem 3.1 gives iterative methods of arbitrary order of convergence as in [11, 12].

The result of Theorem 3.1 is a new contribution for constructing eighth-order iterations for solving nonlinear systems of equations. When $a = 1$, the formulas (2.27), (2.28), and (2.29) are simplified as

$$t_k = I - 2Q_k - 3\bar{d}_k + \mathcal{O}(h^3), \;\; s_k = I + 2Q_k + 3\bar{d}_k + 4Q_k^2 + \mathcal{O}(h^3), \tag{3.7}$$

and

$$q_k = I + 2Q_k - 3\bar{d}_k + \mathcal{O}(h^3).$$

From (3.7), we have

$$\begin{aligned} t_k^2 &= I - 4Q_k - 6\bar{d}_k + 4Q_k^2 + \mathcal{O}(h^3), \\ t_k^3 &= I - 6Q_k - 9\bar{d}_k + 12Q_k^2 + \mathcal{O}(h^3). \end{aligned} \tag{3.8}$$

We search for $\bar{\tau}_k$ as linear combination of different degree of t_k as:

$$I + 2Q_k + 5Q_k^2 + 3\bar{d}_k = \beta_1 I + \beta_2 t_k + \beta_3 t_k^2 + \beta_4 t_k^3. \tag{3.9}$$

Equating the coefficients of similar terms in both sides,

$$\begin{aligned} &\beta_1 + \beta_2 + \beta_3 + \beta_4 = 1, \qquad \beta_2 + 2\beta_3 + 3\beta_4 = -1, \\ &4\beta_3 + 12\beta_4 = 5, \end{aligned}$$

and solving the last system, we obtain

$$\beta_2 = \frac{25}{4} - 3\beta_1, \quad \beta_3 = 3\beta_1 - \frac{17}{2}, \quad \beta_4 = \frac{13}{4} - \beta_1. \tag{3.10}$$

Then, we have

$$\begin{aligned} \bar{\tau}_k =& \beta_1 I + (\frac{25}{4} - 3\beta_1)t_k + (3\beta_1 - \frac{17}{2})t_k^2 + (\frac{13}{4} - \beta_1)t_k^3 \\ =& \frac{25}{4}t_k - \frac{17}{2}t_k^2 + \frac{13}{4}t_k^3 + \beta_1(I - t_k)^3, \;\; \beta_1 \in R. \end{aligned} \tag{3.11}$$

Similarly,

$$\begin{aligned} \alpha_k =& \bar{\tau}_k + Q_k^2 = \lambda_1 I + (\frac{13}{2} - 3\lambda_1)t_k + (3\lambda_1 - 9)t_k^2 + (\frac{7}{2} - \lambda_1)t_k^3 \\ =& \frac{13}{2}t_k - 9t_k^2 + \frac{7}{2}t_k^3 + \lambda_1(I - t_k)^3, \;\; \lambda_1 \in R. \end{aligned} \tag{3.12}$$

When $\beta_1 = \frac{13}{4}$ and $\lambda_1 = \frac{7}{2}$ the formulas (3.11) and (3.12) lead to

$$\bar{\tau}_k = \frac{13}{4}I - \frac{7}{2}t_k + \frac{5}{4}t_k^2, \tag{3.13}$$

and

$$\alpha_k = \frac{7}{2}I - 4t_k + \frac{3}{2}t_k^2, \tag{3.14}$$

respectively. For the choices of $\bar{\tau}_k$, α_k in (3.13)-(3.14), the iteration in (3.1) becomes NLM8 given by Sharma et al. in [21] or 8MBJ ($r = 1$) in [12]. When $\beta_1 = \lambda_1 = 0$ then the formulas in (3.11) and (3.12) lead to

$$\bar{\tau}_k = \frac{25}{4}t_k - \frac{17}{2}t_k^2 + \frac{13}{4}t_k^3,$$

and

$$\alpha_k = \frac{13}{2}t_k - 9t_k^2 + \frac{7}{2}t_k^3,$$

respectively.

Analogously, the conditions (2.15) and (3.2) can be rewritten in terms of s_k as:

$$\begin{aligned}\bar{\tau}_k &= \sigma I + \left(\frac{5}{4} - 3\sigma\right)s_k + \left(3\sigma - \frac{1}{2}\right)s_k^2 + \left(\frac{1}{4} - \sigma\right)s_k^3 \\ &= \frac{5}{4}s_k - \frac{1}{2}s_k^2 + \frac{1}{4}s_k^3 + \sigma(I - s_k)^3, \qquad \sigma \in R,\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\alpha_k &= \varsigma I + \left(\frac{3}{2} - 3\varsigma\right)s_k + (3\varsigma - 1)s_k^2 + \left(\frac{1}{2} - \varsigma\right)s_k^3 \\ &= \frac{3}{2}s_k - s_k^2 + \frac{1}{2}s_k^3 + \varsigma(I - s_k)^3, \qquad \varsigma \in R,\end{aligned} \tag{3.16}$$

respectively. When $\sigma = \varsigma = 0$ in (3.15) and (3.16) then we get

$$\begin{aligned}\bar{\tau}_k &= \frac{5}{4}s_k - \frac{1}{2}s_k^2 + \frac{1}{4}s_k^3, \\ \alpha_k &= \frac{3}{2}s_k - s_k^2 + \frac{1}{2}s_k^3.\end{aligned}$$

In this case, the iteration in (3.1) becomes CCGT1 given by Cordero et al. in [5].

Remark 3.3. From (2.15) and (3.2) it is clear that it suffices to determine $\bar{\tau}_k$ and α_k with accuracy $\mathcal{O}(h^3)$. In some cases, it may be useful to include terms like $\beta_1(I - t_k)^3$ and $\sigma(I - s_k)^3$ in (3.11), (3.12), and (3.15), (3.16) since $I - t_k = \mathcal{O}(h)$, $I - s_k = \mathcal{O}(h)$.

When $\sigma = \frac{1}{4}$, $\varsigma = \frac{1}{2}$ in (3.15) and (3.16), we get

$$\begin{aligned}\bar{\tau}_k &= \frac{1}{4}I + \frac{1}{2}s_k + \frac{1}{4}s_k^2, \\ \alpha_k &= \frac{1}{2}I + \frac{1}{2}s_k^2.\end{aligned}$$

In this case, the iteration in (3.1) becomes CCGT2 given by Cordero et al. in [5].
Using, (2.30), (2.31), the condition in (3.2) can be rewritten in terms of q_k and t_k as follows:

$$\bar{\tau}_k = \eta I + \left(\frac{9}{4} - 3\eta\right)q_k + \left(3\eta - \frac{5}{2}\right)q_k^2 + \left(\frac{5}{4} - \eta\right)q_k^3 + 6\bar{d}_k + \mathcal{O}(h^3), \quad \eta \in R. \tag{3.17}$$

When $\eta = \frac{5}{4}$, we have

$$\bar{\tau}_k = \frac{13}{4}I - \frac{7}{2}q_k + \frac{5}{4}q_k^2 + (q_k - t_k) + \mathcal{O}(h^3). \tag{3.18}$$

Analogously, we have

$$\alpha_k = \xi I + \left(\frac{5}{2} - 3\xi\right)q_k + (3\xi - 3)q_k^2 + \left(\frac{3}{2} - \xi\right)q_k^3 + 6\bar{d}_k + \mathcal{O}(h^3), \quad \xi \in R. \tag{3.19}$$

When $\xi = \frac{3}{2}$, we obtain

$$\alpha_k = \frac{7}{2}I - 4q_k + \frac{3}{2}q_k^2 + (q_k - t_k) + \mathcal{O}(h^3). \tag{3.20}$$

Thus, we also propose eighth-order iteration (3.1) with parameter matrices given by (3.18) and (3.20) respectively. We denote it by ZMO1.
Analogously, it is straightforward to verify that

$$\bar{\tau}_k = -\frac{1}{2}I + \frac{5}{4}s_k + \frac{1}{4}t_k, \tag{3.21}$$

$$\alpha_k = -I + \frac{3}{2}s_k + \frac{1}{2}t_k, \tag{3.22}$$

for the satisfaction of conditions (2.15) and (3.2) respectively. The iteration (3.1) with $\bar{\tau}_k$ and α_k given by (3.21) and (3.22) respectively has also an eighth-order of convergence, which we denote by ZMO2.

Another way to find $\bar{\tau}_k$ and α_k is to use generating function method [26]. Thus, we can easily show that $\bar{\tau}_k$ and α_k given by the following formulas:

$$\bar{\tau}_k = (I + \delta Q_k)^{-1}\left(I + (\delta+2)Q_k + 3\bar{d}_k + (5+2\delta)Q_k^2\right), \tag{3.23}$$

and

$$\alpha_k = (I + \tilde{\delta} Q_k)^{-1}\left(I + (\tilde{\delta}+2)Q_k + 3\bar{d}_k + (6+2\tilde{\delta})Q_k^2\right) \tag{3.24}$$

satisfies the condition (2.15) and (3.2) respectively. The iteration (3.1) with $\bar{\tau}_k$ and α_k as in (3.23) and (3.24) respectively has eighth-order of convergence, which we denote by ZMO3.

In Table 1 below, we summarize eighth-order iterative methods (3.1) for various choices for parameter matrices $\bar{\tau}_k$ and α_k. The proposed family includes some known methods as special cases. The choices for $\bar{\tau}_k$ and α_k in Table 1 also ensure the eighth-order of convergence of methods (3.1).

Table 1. Choices of parameters

$\bar{\tau}_k$	(3.11)	(3.15)	(3.17)	(3.21)	(3.23)
α_k	(3.12)	(3.16)	(3.19)	(3.22)	(3.24)

As in the preceding section, we can construct a derivative-free variant of (3.1) using sufficient convergence conditions (2.15) and (3.2). We consider the following derivative-free method

$$\begin{aligned} y_k &= x_k - [w_k, s_k; \mathcal{F}]^{-1}\,\mathcal{F}(x_k), \\ z_k &= y_k - T_k\,[w_k, s_k; \mathcal{F}]^{-1}\,\mathcal{F}(y_k), \\ x_{k+1} &= z_k - H_k\,[w_k, s_k; \mathcal{F}]^{-1}\,\mathcal{F}(z_k), \end{aligned} \tag{3.25}$$

where $w_k = x_k + \gamma\mathcal{F}(x_k)$, $s_k = x_k - \gamma\mathcal{F}(x_k)$ and T_k, H_k are parameter matrices to be determined properly. The comparison of (3.1) and (3.25) yields

$$\bar{\tau}_k = T_k\,[w_k, s_k; \mathcal{F}]^{-1}\,\mathcal{F}'(x_k), \qquad \alpha_k = H_k\,[w_k, s_k; \mathcal{F}]^{-1}\,\mathcal{F}'(x_k). \tag{3.26}$$

As before (2.45) and (2.46) hold for transition factor $[w_k, s_k; \mathcal{F}]^{-1}\mathcal{F}'(x_k)$. From (3.26), using (2.15), (3.2), and (2.45) we obtain

$$T_k = I + 2Q_k + 3\bar{d}_k + 5Q_k^2 + B_k + \mathcal{O}(h^3), \tag{3.27}$$

$$H_k = I + 2Q_k + 3\bar{d}_k + 6Q_k^2 + B_k + \mathcal{O}(h^3). \tag{3.28}$$

A direct consequence of Theorem 3.1 is the following result.

Theorem 3.4. *The order of local convergence of iteration* (3.25) *equals eight if* T_k *and* H_k *satisfy the conditions* (3.27) *and* (3.28) *respectively.*

Now we find implementable formulas for T_k and H_k. To this end, we consider the matrix $D_k = [w_k, s_k; \mathcal{F}]^{-1} [u_k, v_k; \mathcal{F}]$, where $u_k = y_k + b\mathcal{F}(y_k)$, $v_k = y_k - b\mathcal{F}(y_k)$.
Since $[u_k, v_k; \mathcal{F}] = \mathcal{F}'(y_k) + \mathcal{O}(h^4)$, using (2.36) and (2.45), we get

$$D_k = (I - B_k)t_k = I - 2Q_k - 3\bar{d}_k - B_k + \mathcal{O}(h^3), \quad I - D_k = I - t_k + \mathcal{O}(h^2).$$

From this, we find

$$2Q_k + 3\bar{d}_k = I - D_k - B_k + \mathcal{O}(h^3), \quad Q_k^2 = \frac{1}{4}(I - D_k)^2 + \mathcal{O}(h^3). \tag{3.29}$$

Substituting (3.29) into (3.27) and (3.28), we obtain

$$T_k = 2I - D_k + \frac{5}{4}(I - D_k)^2 + \mu(I - D_k)^3, \quad \mu \in R, \tag{3.30}$$

$$H_k = 2I - D_k + \frac{3}{2}(I - D_k)^2 + \nu(I - D_k)^3, \quad \nu \in R. \tag{3.31}$$

As in (3.11), (3.12), we include here the $\mathcal{O}(h^3)$ term with μ and ν. Theorem 3.4 once again indicates the efficiency of utilizing sufficient convergence conditions. As a result, we obtain the new eighth-order derivative-free iterations given by (3.25).

4. Computational efficiency

To compare the different iterative methods we use CEI=$p^{\frac{1}{d+op}}$, where p is the order of convergence, d is the total number of function evaluations and op is the total number of operations (products and quotients) per iteration. The total computational cost $C = d + op$ as follows:

- To evaluate function $\mathcal{F}$ and derivative $\mathcal{F}'$, n and n^2 scalar functions are calculated
- Further to evaluate an inverse matrix, an $n \times n$ linear system must be solved. We have to do an LU decomposition, which requires $\frac{2}{3}n^3 + \mathcal{O}(n^2)$ operations (unless the matrix is symmetric, in which case the cost reduces $\frac{1}{3}n^3 + \mathcal{O}(n^2)$) and to solve the obtained triangular systems. To solve m linear systems with the same coefficient matrix, we need to do $\frac{2}{3}n^3 + 2mn^2 + \mathcal{O}(n^2)$ operations.
- We need n^2 products for each matrix-vector multiplications ($M \times V$) and n products for scalar-vector multiplication.
- We add n^2 quotients from first-order divided differences.

In Table 1 we denote by **NLS1**-the number of linear systems with same coefficients matrix $\mathcal{F}'(x_k)$ and by **NLS2**-the number of linear systems with other matrices. The methods presented in Table 1 except ZMO1, ZMO3 require three function evaluations $\mathcal{F}(x_k)$, $\mathcal{F}(y_k)$, $\mathcal{F}(z_k)$ and two derivatives evaluations $\mathcal{F}'(x_k)$, $\mathcal{F}'(y_k)$ for each iteration, so $2n^2 + 3n$ functional evaluations are needed. The methods ZMO1 and ZMO3 require additional derivative evaluations $\mathcal{F}'(w_k)$. Hence, $3n^2 + 3n$ functional evaluations are needed.

From Table 2, we see that the cheapest are NLM8 and ZMO1, the total computational cost is $\frac{n^3}{3}$. All other methods cost $\frac{2n^2}{3}$. So, NLM8 and ZMO1 have a higher CEI than other methods.

5. Numerical examples

In this section, we examine the proposed theoretical outcomes with test examples. The computations are executed in Matlab 2017 with *vpa* precision on the computer Microsoft Windows 10, Intel(R), Dell, Core(TM) i5-8500 CPU, 3.00 GHz (6CPUs) with 8200 MB RAM. We take the following examples.

Table 2. Computational cost

methods	d	**NLS1**	**NLS2**	$M \times V$	C
NLM8	$2n^2+3n$	7	0	4	$\frac{1}{3}n^3+13n^2+\frac{8}{3}n$
CCGT1	$2n^2+3n$	1	6	4	$\frac{2}{3}n^3+13n^2+\frac{7}{3}n$
CCGT2	$2n^2+3n$	3	4	2	$\frac{2}{3}n^3+11n^2+\frac{7}{3}n$
ZMO1	$3n^2+3n$	7	0	6	$\frac{1}{3}n^3+15n^2+\frac{8}{3}n$
ZMO2	$2n^2+3n$	5	2	4	$\frac{2}{3}n^3+13n^2+\frac{7}{3}n$
ZMO3*	$3n^2+3n$	7	2	6	$\frac{2}{3}n^3+18n^2+\frac{7}{3}n$

$*$ *when* $\delta = 0$ *ZMO3 leads to NLM8.*

Table 3. Comparison numerical results of seventh-order methods for Examples 5.1-5.3

Methods	$\Vert x_2-x_1\Vert^*$	$\Vert x_3-x_2\Vert$	$\Vert x_4-x_3\Vert$	$\Vert\mathcal{F}(x_4)\Vert$	ρ
			Example 5.1		
(2.3) with (2.32),(2.33) ($a=0.5$)	3.07e-05	1.07e-35	6.56e-249	5.92e-1741	7.0
(2.3) with (2.32),(2.33) ($a=1$)	5.04e-06	6.32e-47	1.05e-373	2.04e-2991	8.0
(2.3) with (2.32),(2.33) ($a=1.04$)	5.29e-07	8.78e-49	3.04e-341	4.97e-2388	7.0
(2.42) with (2.47) ($\gamma=.01$)	1.31e-06	3.12e-46	1.48e-323	1.32e-2264	7.0
BA7 [4]	1.31e-06	3.16e-46	1.52e-323	2.41e-2264	7.0
M7 [1]	2.11e-06	4.07e-45	4.05e-316	1.05e-2212	7.0
$M_{7,3}(\beta=-.01)$ [19]	1.23e-06	5.74e-47	2.73e-329	4.13e-2305	7.0
$M_{7,5}(a=b=c=-.01)$ [20]	3.29e-07	5.66e-51	2.50e-357	2.23e-2501	7.0
NM7($a=b=.01$) [15]	5.91e-07	5.45e-49	3.06e-343	1.47e-2402	7.0
W71 [24]	6.92e-05	1.78e-33	1.34e-233	4.87e-1634	7.0
			Example 5.2		
(2.3) with (2.32),(2.33) ($a=0.9$)	7.32e-04	1.49e-25	2.16e-177	8.66e-1240	7.0
(2.3) with (2.32),(2.33) ($a=1$)	5.04e-06	6.32e-47	1.48e-231	2.28e-1849	8.0
(2.3) with (2.32),(2.33) ($a=1.2$)	8.42e-05	1.03e-31	4.16e-220	2.20e-1538	7.0
(2.42) with (2.47) ($\gamma=.01$)	2.09e-03	2.09e-21	2.14e-147	7.33e-1029	7.0
BA7 [4]	2.09e-03	2.09e-21	2.11e-147	6.74e-1029	7.0
M7 [1]	4.04e-04	1.95e-27	1.19e-190	1.13e-1332	7.0
$M_{7,3}(\beta=-.01)$ [19]	4.06e-04	1.58e-27	2.12e-191	5.05e-1338	7.0
$M_{7,5}(a=b=c=-.01)$ [20]	1.20e-03	1.59e-23	1.12e-162	3.05e-1136	7.0
NM7($a=b=.01$) [15]	1.50e-03	9.73e-23	4.60e-157	7.34e-1097	7.0
W71 [24]	3.98e-04	3.85e-27	3.06e-188	1.86e-1315	7.0
			Example 5.3		
(2.3) with (2.32),(2.33) ($a=0.8$)	6.22e-03	4.52e-18	4.78e-124	1.39e-847	7.0
(2.3) with (2.32),(2.33) ($a=1$)	5.04e-06	6.32e-47	1.48e-231	4.19e-1156	8.0
(2.3) with (2.32),(2.33) ($a=1.2$)	1.51e-03	3.61e-22	4.90e-151	2.24e-996	6.9
(2.42) with (2.47) ($\gamma=.01$)	8.27e-03	7.17e-17	2.67e-115	3.15e-793	7.0
BA7 [4]	8.27e-03	7.17e-17	2.67e-115	3.15e-793	7.0
M7 [1]	2.63e-03	3.61e-21	2.29e-143	1.08e-947	7.0
$M_{7,3}(\beta=-.01)$ [19]	2.53e-03	1.80e-21	1.65e-148	2.53e-1038	7.0
$M_{7,5}(a=b=c=-.01)$ [20]	4.94e-03	7.69e-19	1.74e-129	1.44e-904	7.0
NM7($a=b=.01$) [15]	6.06e-03	3.74e-18	1.28e-124	8.29e-855	7.0
W71 [24]	6.10e-04	5.46e-30	2.19e-238	4.13e-1906	8.0

* $\Vert\cdot\Vert$ is Euclidean norm of a vector.

Table 4. Numerical results of eighth-order methods for Example 5.1

Methods	Parameter matrices	Scalar parameters		$\|x_2-x_1\|$	$\|x_3-x_2\|$	$\|x_4-x_3\|$	$\|\mathcal{F}(x_4)\|$	ρ
		β_1	λ_1					
(3.1)	(3.11), (3.12)	6.4	8.9	2.42e-11	5.41e-91	3.33e-728	1.87e-5825	8.0
(3.1)		0	0	8.08e-07	1.32e-53	6.85e-428	9.58e-3422	8.0
NLM8 [21]		13/4	3.5	2.42e-07	1.62e-58	6.53e-468	1.24e-3742	8.0
8MBJ [12]		$r=1$						
		σ	ς					
(3.1)	(3.15), (3.16)	-0.9	-1.9	1.12e-12	3.04e-102	9.30e-819	1.92e-6550	8.0
CCGT1 [5]		0	0	4.12e-08	4.45e-66	8.35e-530	3.47e-4239	8.0
CCGT2 [5]		1/4	0.5	6.68e-08	6.06e-64	2.78e-512	1.47e-4098	8.0
		η	ξ					
(3.1)	(3.17), (3.19)	-1.7	-14.9	3.93e-06	5.88e-51	1.54e-409	9.21e-3278	8.0
(3.1)		0	0	3.43e-06	6.86e-48	1.74e-381	8.29e-3050	8.0
ZMO1		5/4	3/2	3.34e-06	6.76e-48	1.89e-381	1.90e-3049	8.0
ZMO2	(3.21), (3.22)			9.55e-08	2.10e-62	1.14e-499	2.28e-3997	8.0
		δ and $\tilde{\delta}$						
ZMO3	(3.23), (3.24)	-4.4		2.80e-06	3.11e-52	7.21e-420	1.62e-3360	8.0
ZMO3		0		5.04e-06	6.32e-47	3.86e-374	2.04e-2991	8.0
ZMO3		1.5		5.90e-06	3.79e-46	1.10e-367	1.56e-2939	8.0
(3.25)	(3.30), (3.31) ($\gamma=.01$, $b=.01$, $\mu=\nu=0$)			2.41e-07	1.59e-58	5.71e-468	4.27e-3743	8.0
ASKS8 [2]		$a_0=3$	$a_5=1$	2.79e-05	1.14e-40	9.88e-324	3.60e-2588	8.0

Table 5. Numerical results of eighth-order methods for Example 5.2

Methods	Parameter matrices	Scalar parameters		$\|x_2-x_1\|$	$\|x_3-x_2\|$	$\|x_4-x_3\|$	$\|\mathcal{F}(x_4)\|$	ρ
		β_1	λ_1					
(3.1)	(3.11), (3.12)	6.3	8.4	1.62e-08	2.78e-65	2.15e-519	8.11e-4152	8.0
(3.1)		0	0	3.25e-03	6.45e-22	1.57e-171	5.81e-1368	8.0
NLM8 [21]		13/4	3.5	9.17e-04	3.68e-27	2.46e-214	2.96e-1711	8.0
8MBJ [12]		$r=1$						
		σ	ς					
(3.1)	(3.15), (3.16)	-0.3	-0.1	9.36e-09	6.58e-69	3.94e-550	1.94e-4399	8.0
CCGT1 [5]		0	0	4.33e-07	7.71e-57	7.78e-455	2.50e-3638	8.0
CCGT2 [5]		1/4	0.5	7.99e-05	2.58e-37	3.09e-297	3.90e-2376	8.0
		η	ξ					
(3.1)	(3.17), (3.19)	0.4	-0.3	4.80e-08	1.73e-62	5.01e-498	7.32e-3982	8.0
(3.1)		0	0	1.95e-05	1.75e-41	7.13e-330	1.66e-2636	8.0
ZMO1		5/4	3/2	2.61e-04	1.61e-32	3.40e-258	4.07e-2063	8.0
ZMO2	(3.21), (3.22)			2.14e-04	3.30e-33	1.05e-263	3.21e-2107	8.0
		δ and $\tilde{\delta}$						
ZMO3	(3.23), (3.24)	-2.2		4.55e-06	1.61e-48	3.91e-388	1.44e-3104	8.0
ZMO3		2.2		1.08e-03	2.13e-26	5.02e-208	1.43e-1660	8.0
(3.25)	(3.30), (3.31) ($\gamma=.01$, $b=.01$, $\mu=\nu=0$)			9.17e-04	3.68e-27	2.50e-214	3.42e-1711	8.0
ASKS8 [2]		$a_0=3$	$a_5=1$	7.54e-02	5.58e-10	4.31e-75	1.64e-595	8.0

Example 5.1. Considering the following system of 20 equations (see [3]):

$$x_{(i)} - \cos\left(2x_{(i)} - \sum_{j=1}^{4} x_{(j)}\right) = 0,$$
$$i = 1, 2, \cdots, n.$$

Table 6. Numerical results of eighth-order methods for Example 5.3

Methods	Parameter matrices	Scalar parameters		$\|x_2 - x_1\|$	$\|x_3 - x_2\|$	$\|x_4 - x_3\|$	$\|\mathcal{F}(x_4)\|$	ρ
		β_1	λ_1					
(3.1)	(3.11), (3.12)	7.5	16.7	2.22e-04	6.67e-31	4.66e-243	1.85e-1705	8.0
(3.1)		0	0	1.16e-02	4.02e-17	9.05e-133	5.24e-1045	8.0
NLM8 [21]		13/4	3.5	4.39e-03	2.75e-21	2.58e-166	4.19e-1156	8.0
8MBJ [12]		$r = 1$						
		σ	ς					
(3.1)	(3.15), (3.16)	-1.4	-1.4	6.84e-04	4.87e-28	1.54e-222	5.10e-1494	8.1
CCGT1 [5]		0	0	3.27e-04	2.80e-28	3.76e-189	1.23e-1249	6.7
CCGT2 [5]		1/4	0.5	1.02e-03	1.07e-26	4.23e-184	7.19e-1229	6.8
		η	ξ					
(3.1)	(3.17), (3.19)	-3.1	-11.1	2.72e-04	3.49e-30	1.83e-235	3.48e-1600	7.9
(3.1)		0	0	2.43e-03	3.72e-24	6.79e-177	1.38e-1202	7.3
ZMO1		5/4	3/2	4.39e-03	2.75e-21	2.58e-166	4.19e-1156	8.0
ZMO2	(3.21), (3.22)			1.70e-03	3.22e-25	1.23e-179	2.94e-1211	7.1
		δ and $\tilde{\delta}$						
ZMO3	(3.23), (3.24)	-4.4		1.95e-03	7.18e-25	4.44e-195	5.48e-1327	7.9
ZMO3		-4.5		2.89e-03	1.99e-23	1.01e-184	1.02e-1314	8.0
(3.25)	(3.30), (3.31) ($\gamma = .01$, $b = .01$, $\mu = \nu = 0$)			4.39e-03	2.75e-21	2.58e-166	4.19e-1156	8.0
ASKS8 [2]		$a_0 = 3$	$a_5 = 1$	1.95e-03	2.28e-26	6.40e-210	5.97e-1679	8.0

with initial guess $x_0 = \{0.75, \cdots, 0.75\}^T$ for obtaining the solution $x^* = \{0.514933264, \cdots, 0.514933264\}^T$.

Example 5.2. The second nonlinear system with $n = 20$ is given by [3].

$$\begin{cases} x_{(i+1)}x_{(i)}^2 - 1 = 0, & i = 1, 2, \cdots, n-1, \\ x_{(1)}x_{(n)}^2 - 1 = 0. \end{cases}$$

A solution $x^* = \{1, 1, \cdots, 1\}^T$, initial value $x_0 = \{1.25, 1.25, \cdots, 1.25\}^T$.

Example 5.3. We also consider the boundary value problem (see [21]):

$$u'' + a^2(u')^2 + 1 = 0, \ u(0) = 0, \ u(1) = 0.$$

If we discretize the problem by using the numerical formulas for first and second derivatives

$$u'_j \approx \frac{u_{j-1} - u_{j+1}}{2h}, \ u''_j \approx \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2}, \ (j = 1, 2, 3, \cdots, n-1),$$

we obtain a system of $n - 1$ nonlinear equations in $n - 1$ variables:

$$u_{(j-1)} - 2u_{(j)} + u_{(j+1)} + \frac{a^2}{4}(u_{(j+1)} - u_{(j-1)})^2 + h^2 = 0, \ (j = 1, 2, 3, \cdots, n-1).$$

In particular, we solve this problem for $n = 21$ so that $1 \leq j \leq 20$ by selecting $u_0 = \{-\frac{1}{4}, -\frac{1}{4}, \cdots, -\frac{1}{4}\}^T$ as the initial value and $a = 2$.

All iterations are performed in the first four steps, we can see that proposed theoretical convergence orders are confirmed by numerical outcomes from the columns ρ (computed by the formula given in [8]) of Tables 3-6. Our proposed methods are more effective among the compared existing methods since they give more accurate results in Tables 3-6.

6. Conclusions

We first develop a new family of three-step seventh and eighth-order Newton-type iterative methods. The proposed family includes some existing methods as particular cases. We suggest different choices of parameter matrices of iterations that ensure the expected order of convergence. The presented sufficient convergence condition enables to analytically prove the convergence of the existing methods as well as to construct new iterations. As an example, we derived a new eighth-order derivative-free iterations.

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