http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 72, Number 1, Pages 105–117 (2023)
DOI:10.31801/cfsuasmas.1061950
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received: January 23, 2022; Accepted: May 31, 2022

HADAMARD PRODUCT OF HOLOMORPHIC MAPPINGS ASSOCIATED WITH THE CONIC SHAPED DOMAIN

Syed Zakar Hussain BUKHARI[1] and Aamar SHAHZAD[2]

[1,2]Department of Mathematics, Mirpur University of Science and Technology (MUST)
Mirpur-10250(AJK), PAKISTAN

Abstract. We define certain subclasses δ–$\mathcal{UM}(\ell, \eta_1, \eta_2)$ and δ–$\mathcal{UM}_{\Im}(\ell, \eta_1, \eta_2)$ of holomorphic mappings involving some differential inequalities. These functions are actually generalizations of some basic families of starlike and convex mappings. We study sufficient conditions for $f \in \delta - \mathcal{UM}(\ell, \eta_1, \eta_2)$. We also discuss the characterization for $f \in \delta - \mathcal{UM}_{\Im}(\ell, \eta_1, \eta_2)$ along with the coefficient bounds and other problems. Using certain conditions for functions in the class $\delta - \mathcal{UM}(\ell, \eta_1, \eta_2)$, we also define another class $\delta - \mathcal{UM}^*(\ell, \eta_1, \eta_2)$ and study some subordination related result.

1. Introductory Concept

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ denote the family of mappings f holomorphic in the open unit disc $\mathbb{U} := \{z \in \mathbb{C} \text{ and } |z| < 1\}$. For $m \in \mathbb{N}$ and $\alpha \in \mathbb{C}$, let $f \in \mathcal{H}[\alpha, m] \subset \mathcal{H} : f(z) = \alpha + \sum_{m=1}^{\infty} \alpha_m z^m$ and $f \in \mathcal{A} \subset \mathcal{H}[\alpha, m]$:

$$f(z) = z + \sum_{m=2}^{\infty} \alpha_m z^m, z \in \mathbb{U}. \tag{1}$$

Let $\mathcal{P}$ denote the family of Carathéodory mappings q with $\Re(q(z)) > 0$ and

$$q(z) = 1 + \sum_{m=1}^{\infty} q_m z^m, z \in \mathbb{U}.$$

The Möbius transformation $l_0(z) = \frac{1+z}{1-z}, z \in \mathbb{U}$ is an extremal mapping for the family $\mathcal{P}$. For $f, \ell \in \mathcal{H}$, we say the mapping f *is subordinate to* ℓ and mathematically write $f(z) \prec \ell(z)$, if for $w \in \mathcal{H}(\mathbb{U}) : w(0) = 0$ and $|w(z)| < 1$, we have

2020 *Mathematics Subject Classification.* 30A10, 30C45, 30C80.

Keywords. Carathéodory functions, convolution, subordination.

[1] fatmi@must.edu.pk-Corresponding author; 0000-0002-5243-4252

[2] aamarshahzad17@gmail.com; 0000-0001-5255-0176.

$f(z) = \ell\big(w(z)\big)$. For reference, see [10]. Applying subordination, Janowski [8] introduced the family $\mathcal{P}[\eta_1, \eta_2]$ for $-1 \leq \eta_2 < \eta_1 \leq 1$. A mapping $q \in \mathcal{P}[\eta_1, \eta_2]$, if

$$q(z) \prec \frac{1+\eta_1 z}{1+\eta_2 z} \text{ or } q(z) = \frac{1+\eta_1 w(z)}{1+\eta_2 w(z)}, z \in \mathbb{U},$$

where w is a *Schwarz mapping*. For detail of some work related to subordination, we refer, [2–6, 8, 10]. Clearly, $\mathcal{P}[\eta_1, \eta_2]$ is contained in $\mathcal{P}\left(\frac{1-\eta_1}{1-\eta_2}\right)$. This family is related with the class $\mathcal{P}$. A mappings $q \in \mathcal{P}$ iff

$$\frac{(\eta_1+1)\, q(z) - (\eta_1 - 1)}{(\eta_2+1)\, q(z) - (\eta_2 - 1)} \in \mathcal{P}[\eta_1, \eta_2].$$

The simplest representation of a conic domain Δ_δ, $\delta \geq 0$ is given in the following:

$$\Delta_\delta = \left\{ w = u + iv : u > \delta \sqrt{(u-1)^2 + v^2} \right\}.$$

A mapping $f \in \delta - \mathcal{US}(\beta)$ if the following inequality holds:

$$\Re\left\{ \frac{zf'(z)}{f(z)} - \beta \right\} > \delta \left| \frac{zf'(z)}{f(z)} - 1 \right| \qquad (z \in \mathbb{U}), \tag{2}$$

where $-1 \leq \beta < 1$ and $\delta \geq 0$.

A mapping $f \in \delta - \mathcal{UC}(\beta)$ iff $zf' \in \delta - \mathcal{US}(\beta)$.

The above families are studied by Goodman [7] and Rönning [13]. For mappings $f, \ell \in \mathcal{A}$, the convolution $f * \ell$ is defined by

$$f(z) * \ell(z) = z + \sum_{m=2}^{\infty} \alpha_m \gamma_m z^m = \ell(z) * f(z) \qquad (z \in \mathbb{U}),$$

where the mapping f is given by (1) and

$$\ell(z) = z + \sum_{m=2}^{\infty} \gamma_m z^m \qquad (z \in \mathbb{U}). \tag{3}$$

In 2008, Raina [12] introduced the family $\delta - \mathcal{US}(\ell, \beta)$ which may be defined as follows:

Definition 1. *Let ℓ be given by (3) with $\gamma_m \geq 0$, we say that $f \in \delta - \mathcal{US}(\ell, \beta)$ if $f(z) * \ell(z) \neq 0$ and*

$$\Re\left\{ \frac{z\,(f*\ell)'(z)}{f(z) * \ell(z)} - \beta \right\} > \delta \left| \frac{z\,(f*\ell)'(z)}{f(z) * \ell(z)} - 1 \right| \qquad (z \in \mathbb{U}),$$

where

$$(f * \ell)(z) = z + \sum_{m=2}^{\infty} \alpha_m \gamma_m z^m \qquad (\gamma_m \geq 0, z \in \mathbb{U}), \tag{4}$$

$-1 \leq \beta < 1$ *and* $\delta \geq 0$.

Generally this family consists of uniformly δ-starlike mappings $f * \ell$ of order β in $\mathbb{U}$.

In 2011, Noor and Malik [11] introduced the family $\delta - \mathcal{UM}(\eta_1, \eta_2)$ which is defined as:

Definition 2. *A mapping $f \in \mathcal{A}$ given by (1), is in the family $\delta - \mathcal{UM}(\eta_1, \eta_2)$ provided that $f(z) \neq 0$ and*

$$\Re\left\{\frac{(\eta_2 - 1)\frac{zf'(z)}{f(z)} - (\eta_1 - 1)}{(\eta_2 + 1)\frac{zf'(z)}{f(z)} + (\eta_1 - 1)}\right\} > \delta\left|\frac{(\eta_2 - 1)\frac{zf'(z)}{f(z)} - (\eta_1 - 1)}{(\eta_2 + 1)\frac{zf'(z)}{f(z)} + (\eta_1 - 1)} - 1\right| \qquad (z \in \mathbb{U}),$$

where $-1 \leq \eta_2 < \eta_1 < 1$ and $\delta \geq 0$.

This family consists of mappings f which are associated with uniformly δ-starlike mappings in $\mathbb{U}$. Extending the idea of Noor and Malik [11], we define a new family $\delta - \mathcal{UM}(\ell, \eta_1, \eta_2)$ of holomorphic mappings.

Definition 3. *Let $f \in \mathcal{A}$. Then $f \in \delta - \mathcal{UM}(\ell, \eta_1, \eta_2)$, if it satisfies the condition:*

$$\Re\left\{\frac{(\eta_2 - 1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)} - (\eta_1 - 1)}{(\eta_2 + 1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)} + (\eta_1 - 1)}\right\} > \delta\left|\frac{(\eta_2 - 1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)} - (\eta_1 - 1)}{(\eta_2 + 1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)} + (\eta_1 - 1)}\right|, (z \in \mathbb{U}), \tag{5}$$

where $f * \ell$ is given by (4), $-1 \leq \eta_2 < \eta_1 < 1$ and $\delta \geq 0$.

The mapping $f * \ell$ converges as a convolution of holomorphic mappings defined in $\mathbb{U}$. Clearly $f * \ell$ is associated with uniformly δ-starlike mappings in $\mathbb{U}$.

Let $\Im$ be the family of holomorphic mappings f of positive coefficients and having the series representation of the form:

$$f(z) = z - \sum_{m=2}^{\infty} \alpha_m z^m, \; \alpha_m \geq 0, z \in \mathbb{U}. \tag{6}$$

For details of this family, we refer [14].

Let f be given by (1). Then $f \in \delta - \mathcal{UM}_{\Im}(\ell, \eta_1, \eta_2)$, if and only if

$$f \in \delta - \mathcal{UM}(\ell, \eta_1, \eta_2) \cap \Im,$$

where $-1 \leq \eta_2 < \eta_1 < 1, \delta \geq 0$ and $\Im$ is given by (6).

For some special choices, we obtain the following known classes:

i. $\delta - \mathcal{UM}\left(\frac{z}{1-z}, \eta_1, \eta_2\right) = \delta - \mathcal{US}(\eta_1, \eta_2)$ and $\delta - \mathcal{UM}\left(\frac{z}{(1-z)^2}, 1, \eta_1, \eta_2\right) = \delta - \mathcal{UC}(\eta_1, \eta_2)$.

ii. $\delta - \mathcal{UM}\left(\frac{z}{1-z}, 1, -1\right) = \delta - \mathcal{US}$ and $\delta - \mathcal{UM}\left(\frac{z}{(1-z)^2}, 1, -1\right) = \delta - \mathcal{UC}$..

iii. $\delta - \mathcal{UM}\left(\frac{z}{1-z}, 1 - 2\beta, -1\right) = \delta - \mathcal{US}(\beta)$ and $\delta - \mathcal{UM}\left(\frac{z}{(1-z)^2}, 1 - 2\beta, -1\right) = \delta - \mathcal{UC}(\beta)$.

iv. $0 - \mathcal{UM}\left(\frac{z}{1-z}, \eta_1, \eta_2\right) = \mathcal{S}^*(\eta_1, \eta_2)$ and $0 - \mathcal{UM}\left(\frac{z}{(1-z)^2}, \eta_1, \eta_2\right) = \mathcal{C}(\eta_1, \eta_2)$.

The class $\delta-\mathcal{UM}(\ell,\eta_1,\eta_2)$ also reduces to the families mentioned in (2), see [13]. For detail of the above classes and various other cases related to the earlier contributions, see [1,3,8,9,11,15] with references therein.

2. Preliminaries

Subsequently, we define the subordinating factor sequence.

Definition 4. *A sequence $\langle c_m : m=1,2,3,...\rangle$ is termed as a subordinating factor sequence for some mappings in $\mathcal{C}$, if for each $f\in\mathcal{C}$, we have*

$$\sum_{m=1}^{\infty}\alpha_m c_m z^m \prec f(z) \qquad (\alpha_1=1, z\in\mathbb{U}). \tag{7}$$

Lemma 1. *The sequence $\langle c_m : m=1,2,3,...\rangle$ is a subordinating factor sequence, iff*

$$\Re\left\{1-2\sum_{m=2}^{\infty}c_m z^m\right\}>0.$$

For detail, see [9,16]. Throughout, we assume $\delta\geq 0$ and $-1\leq\eta_2<\eta_1\leq 1$.

3. Main Discussion

Theorem 1. *For a given mapping ℓ defined by (3) with $\gamma_m\geq 0$, if a mapping $f\in\mathcal{A}$ satisfies the inequality*

$$\sum_{m=2}^{\infty}\left[\{3+2\delta+\eta_2\}(m-1)+\eta_2-\eta_1\right]|\alpha_m|\gamma_m\leq\eta_1-\eta_2, \tag{8}$$

then $f\in\delta-\mathcal{UM}(\ell,\eta_1,\eta_2)$, where $m\geq\frac{1+\eta_1}{1+\eta_2}$ for $-1\leq\eta_2<\eta_1\leq 1$ and $\delta\geq 0$.

Proof. To have the desired proof, we only show that

$$\delta\left|\frac{(\eta_2-1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-(\eta_1-1)}{(\eta_2+1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}+(\eta_1-1)}-1\right|-\Re\left\{\frac{(\eta_2-1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-(\eta_1-1)}{(\eta_2+1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}+(\eta_1-1)}-1\right\}\leq 1$$

where $f*\ell$ is given by (4), $-1\leq\eta_2<\eta_1<1$ and $\delta\geq 0$. For $f*\ell$ given by (4), we see that

$$z\left(f\left(z\right)*\ell\left(z\right)\right)'=z+\sum_{m=2}^{\infty}m\left|\alpha_m\right|\gamma_m z^m.$$

Consider that

$$\delta\left|\frac{(\eta_2-1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-(\eta_1-1)}{(\eta_2+1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-(\eta_1+1)}-1\right|-\Re\left\{\frac{(\eta_2-1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-(\eta_1-1)}{(\eta_2+1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-(\eta_1+1)}-1\right\}$$

$$\leq(1+\delta)\left|\frac{(\eta_2-1)\,z\left(f\left(z\right)*\ell\left(z\right)\right)'-(\eta_1-1)\,f\left(z\right)*\ell\left(z\right)}{(\eta_2+1)\,z\left(f\left(z\right)*\ell\left(z\right)\right)'-(\eta_1+1)\,f\left(z\right)*\ell\left(z\right)}-1\right|$$

$$= 2(1+\delta)\left|\frac{z\left(f\left(z\right)*\ell\left(z\right)\right)'-f\left(z\right)*\ell\left(z\right)}{\left(\eta_2+1\right)z\left(f\left(z\right)*\ell\left(z\right)\right)'-\left(\eta_1+1\right)f\left(z\right)*\ell\left(z\right)}\right|$$

$$\leq \frac{2\sum\limits_{m=2}^{\infty}\left(1+\delta\right)\left(m-1\right)\left|\alpha_m\right|\gamma_m}{\eta_1-\eta_2-\sum\limits_{m=2}^{\infty}\left\{m\eta_2-\eta_1+m-1\right\}\left|\alpha_m\right|\gamma_m} \qquad \left(m\geq\frac{1+\eta_1}{1+\eta_2}\right).$$

The last expression is bounded by 1 if

$$\sum_{m=2}^{\infty}\left[\left(3+2\delta+\eta_2\right)\left(m-1\right)+\eta_2-\eta_1\right]\left|\alpha_m\right|\gamma_m\leq\eta_1-\eta_2.$$

□

We next prove the characterization of the mapping f as below.

Theorem 2. *A mapping f given by* (6) *belongs to the family $\delta-\mathcal{UM}_{\Im}(\ell,\eta_1,\eta_2)$ if and only if*

$$\sum_{m=2}^{\infty}\left\{\left(m-1\right)\left(1+2\delta-\eta_2\right)+\eta_1-\eta_2\right\}\alpha_m\gamma_m\leq\eta_1-\eta_2, \tag{9}$$

where $-1\leq\eta_2<\eta_1\leq1,\gamma_m>0$ and $\delta\geq0$.

Proof. Suppose that $f\in\delta-\mathcal{UM}_{\Im}(\ell,\eta_1,\eta_2)$. Then, making use of the fact that

$$\Re w>\delta\left|w-1\right|\Leftrightarrow\Re\left\{w(1+\delta e^{i\theta})-\delta e^{i\theta}\right\}>0$$

and taking

$$w\left(z\right)=\frac{\left(\eta_2-1\right)\frac{z\left(f\left(z\right)*\ell\left(z\right)\right)'}{f\left(z\right)*\ell\left(z\right)}-\left(\eta_1-1\right)}{\left(\eta_2+1\right)\frac{z\left(f\left(z\right)*\ell\left(z\right)\right)'}{f\left(z\right)*\ell\left(z\right)}-\left(\eta_1+1\right)},$$

where $f*\ell$ is given by (4) with $\alpha_m\geq0,-1\leq\eta_2<\eta_1<1$, and $\delta\geq0$ in (5), we obtain

$$\Re\left\{(1+\delta e^{i\theta})\frac{\left(\eta_2-1\right)\frac{z\left(f\left(z\right)*\ell\left(z\right)\right)'}{f\left(z\right)*\ell\left(z\right)}-\left(\eta_1-1\right)}{\left(\eta_2+1\right)\frac{z\left(f\left(z\right)*\ell\left(z\right)\right)'}{f\left(z\right)*\ell\left(z\right)}-\left(\eta_1+1\right)}-\delta e^{i\theta}\right\}>0,$$

or equivalently

$$\Re\left\{(1+\delta e^{i\theta})\frac{\left(\eta_2-1\right)z\left(f\left(z\right)*\ell\left(z\right)\right)'-\left(\eta_1-1\right)f\left(z\right)*\ell\left(z\right)}{\left(\eta_2+1\right)z\left(f\left(z\right)*\ell\left(z\right)\right)'-\left(\eta_1+1\right)f\left(z\right)*\ell\left(z\right)}-\delta e^{i\theta}\right\}>0,$$

which on simple manipulation yields

$$\Re\left\{\frac{\left(\eta_1-\eta_2\right)+\sum\limits_{m=2}^{\infty}\left\{m\eta_2-m-2\delta me^{i\theta}+1-\eta_1+2\delta e^{i\theta}\right\}\alpha_m\gamma_mz^{m-1}}{\left(\eta_1-\eta_2\right)+\sum\limits_{m=2}^{\infty}\left\{m\left(\eta_2+1\right)-1-\eta_1\right\}\alpha_m\gamma_mz^{m-1}}\right\}>0.$$

This result holds true for all $z \in \mathbb{U}$. Taking the limit $z \to 1^-$ through real values, we thus obtain that

$$\Re\left\{\frac{(\eta_1-\eta_2)+\sum\limits_{m=2}^{\infty}\left\{m\eta_2-m-2\delta m e^{i\theta}+1-\eta_1+2\delta e^{i\theta}\right\}\alpha_m\gamma_m}{(\eta_1-\eta_2)+\sum\limits_{m=2}^{\infty}\left\{m\left(\eta_2+1\right)-1-\eta_1\right\}\alpha_m\gamma_m}\right\}>0,$$

which further implies that

$$\left\{\eta_1-\eta_2-\sum_{m=2}^{\infty}\left\{\left(1+2\delta-\eta_2\right)(m-1)+\eta_1-\eta_2\right\}\alpha_m\gamma_m\right\}>0,$$

so we have

$$\sum_{m=2}^{\infty}\left\{\left(1+2\delta-\eta_2\right)(m-1)+\eta_1-\eta_2\right\}\alpha_m\gamma_m<\eta_1-\eta_2.$$

Conversely, we let the inequality (9) hold true. Then, in view of the fact that $\Re\left(w\left(z\right)\right)>0$ if and only if $|w(z)-1|<|w(z)+1|$, where

$$w\left(z\right)=\frac{\left(\eta_2-1\right)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-\left(\eta_1-1\right)}{\left(\eta_2+1\right)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-\left(\eta_1+1\right)}-\delta\left|\frac{\left(\eta_2-1\right)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-\left(\eta_1-1\right)}{\left(\eta_2+1\right)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-\left(\eta_1+1\right)}-1\right|. \tag{10}$$

we consider

$$\begin{aligned}
&|w(z)+1| \\
&=\left|\frac{\left(\eta_2-1\right)\frac{z(f*\ell)'(z)}{(f*\ell)(z)}-\left(\eta_1-1\right)}{\left(\eta_2+1\right)\frac{z(f*\ell)'(z)}{(f*\ell)(z)}-\left(\eta_1+1\right)}-\delta\left|\frac{\left(\eta_2-1\right)\frac{z(f*\ell)'(z)}{(f*\ell)(z)}-\left(\eta_1-1\right)}{\left(\eta_2+1\right)\frac{z(f*\ell)'(z)}{(f*\ell)(z)}-\left(\eta_1+1\right)}-1\right|+1\right| \\
&=\frac{2\left|z\right|}{|G|}\left|\eta_1-\eta_2+\sum_{m=2}^{\infty}\left\{m\eta_2-\eta_1+\delta m-\delta\right\}\alpha_m\gamma_m z^{m-1}\right| \\
&>\frac{2}{|G|}\left[\eta_1-\eta_2-\sum_{m=2}^{\infty}\left\{m\eta_2-\eta_1+\delta m-\delta\right\}\alpha_m\gamma_m\right],
\end{aligned} \tag{11}$$

where $G=\left(\eta_2+1\right)z\left(f*\ell\right)'(z)-\left(\eta_1+1\right)f\left(z\right)*\ell\left(z\right)$. Also for $|w(z)-1|=W$

$$\begin{aligned}
W&=\left|\frac{\left(\eta_2-1\right)\frac{z(f*\ell)'(z)}{(f*\ell)(z)}-\eta_1\text{-}1}{\left(\eta_2+1\right)\frac{z(f*\ell)'(z)}{\frac{z(f*\ell)'(z)}{(f*\ell)(z)}}-\eta_1-1}-1-\delta\left|\frac{\left(\eta_2-1\right)\frac{z(f*\ell)'(z)}{(f*\ell)(z)}-\left(\eta_1-1\right)}{\left(\eta_2+1\right)\frac{z(f*\ell)'(z)}{(f*\ell)(z)}-\eta_1-1}-1\right|\right| \\
&=2\left|\frac{-\frac{z(f*\ell)'(z)}{(f*\ell)(z)}+1}{\left(\eta_2+1\right)\frac{z(f*\ell)'(z)}{(f*\ell)(z)}-\eta_1-1}-\delta\left|\frac{-\frac{z(f*\ell)'(z)}{(f*\ell)(z)}+1}{\left(\eta_2+1\right)\frac{z(f*\ell)'(z)}{(f*\ell)(z)}-\eta_1-1}\right|\right|
\end{aligned}$$

$$< \frac{2|z|}{|G|} \sum_{m=2}^{\infty} (m + m\delta - 1 - \delta)\, \alpha_m \gamma_m. \tag{12}$$

where $G = (\eta_2 + 1)\, z\, (f(z) * \ell(z))' - (\eta_1 + 1)\, f(z) * \ell(z)$. From the condition (9) and the inequalities (11) and (12), we deduce that

$$|w(z) + 1| - |w(z) - 1| > 0,$$

where w is defined by (10). This completes the proof of Theorem 2. □

We next provide coefficient bound for a given mapping f to belong to the family $\delta - \mathcal{UM}_{\Im}(\ell, \eta_1, \eta_2)$.

Corollary 1. *A mapping f belongs to the family $\delta - \mathcal{UM}_{\Im}(\ell, \eta_1, \eta_2)$ if*

$$\sum_{m=2}^{\infty} \alpha_m < \frac{\eta_1 - \eta_2}{\{1 + 2\delta - 2\eta_2 + \eta_1\}\, \gamma_2}, \gamma_2 > 0.$$

where $-1 \leq \eta_2 < \eta_1 < 1$, and $\delta \geq 0$.

Corollary 2. *For a mapping f belonging to the family $\delta - \mathcal{UM}_{\Im}(\ell, \eta_1, \eta_2)$, we have*

$$\alpha_m < \frac{\eta_1 - \eta_2}{\{1 + 2\delta - 2\eta_2 + \eta_1\}\, \gamma_2}, \gamma_2 > 0.$$

where $-1 \leq \eta_2 < \eta_1 < 1$ and $\delta \geq 0$.

The subsequent theorem deals with the integral representation for a given mapping $f \in \delta - \mathcal{UM}_{\Im}(\ell, \eta_1, \eta_2)$.

Theorem 3. *If a mapping f given by (6) belongs to the family $\delta - \mathcal{UM}_{\Im}(\ell, \eta_1, \eta_2)$, then f has the following representation:*

$$f(z) = \ell^{(-1)}(z) * \exp\left(\int_0^z \frac{2\delta\eta_1 - Q(t)(\eta_1 - 1)}{t\, \{2\delta + Q(t)(\eta_2 - 1)\}} \mathrm{d}t \right),$$

where $-1 \leq \eta_2 < \eta_1 < 1$ and $\delta \geq 0$.

Proof. For $\delta = 0$, the assertion of the Theorem 3 is obvious. Let $\delta > 0$. Then, for $f \in \delta - \mathcal{UM}_{\Im}(\ell, \eta_1, \eta_2)$ and

$$w(z) = \frac{(\eta_2 - 1)\, \frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)} - (\eta_1 - 1)}{(\eta_2 + 1)\, \frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)} + (\eta_1 - 1)}$$

we have

$$Re(w) > \delta |w - 1|,$$

which implies that

$$|\frac{w - 1}{w}| < \frac{1}{\delta}.$$

We suppose that

$$\frac{w - 1}{w} = \frac{Q(z)}{\delta}$$

and

$$w(z)=\frac{\delta}{\delta-Q(z)},$$

which yields

$$\frac{(\eta_2-1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-(\eta_1-1)}{(\eta_2-1)\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}-(\eta_1+1)}=\frac{\delta}{\delta-Q(z)}.$$

Thus on simplification, we have

$$\frac{z(f(z)*\ell(z))'}{f(z)*\ell(z)}=\frac{2\delta\eta_1-Q(z)(\eta_1-1)}{2\delta+Q(z)(\eta_2-1)}.$$

which proves that

$$f(z)*\ell(z)=\exp\left(\int_0^z\frac{2\delta\eta_1-Q(t)(\eta_1-1)}{t\left\{2\delta+Q(t)(\eta_2-1)\right\}}\mathrm{d}t\right)$$

or

$$f(z)=\ell^{(-1)}(z)*\exp\left(\int_0^z\frac{2\delta\eta_1-Q(t)(\eta_1-1)}{t\left\{2\delta+Q(t)(\eta_2-1)\right\}}\mathrm{d}t\right).$$

This finishes the proof of Theorem 3. □

Theorem 4. *If f_j is such that*

$$f_j(z)=z-\sum_{m=2}^{\infty}\alpha_{m,j}z^m\in\delta-\mathcal{UM}_{\Im}(\ell,\eta_1,\eta_2),\qquad(j=1,2,z\in\mathbb{U}),$$

then

$$f(z)=(1-\lambda)f_1(z)+\lambda f_2(z)\in\delta-\mathcal{UM}_{\Im}(\ell,\eta_1,\eta_2),\qquad(0\le\lambda\le1,z\in\mathbb{U}).$$

Proof. For the mappings f_j such that $f_j(z)=z-\sum_{m=2}^{\infty}\alpha_{m,j}z^m\in\delta\text{–}\mathcal{UM}_{\Im}(\ell,\eta_1,\eta_2)$, by using Theorem 2, we write

$$\sum_{m=2}^{\infty}\left\{(1+2\delta-\eta_2)(m-1)+\eta_1-\eta_2\right\}\alpha_{m,1}\gamma_m\le\eta_1-\eta_2\tag{13}$$

and

$$\sum_{m=2}^{\infty}\left\{(1+2\delta-\eta_2)(m-1)+\eta_1-\eta_2\right\}\alpha_{m,2}\gamma_m\le\eta_1-\eta_2.\tag{14}$$

In view of (13) and (14), we have

$$\begin{aligned}&(1-\lambda)\sum_{m=2}^{\infty}\left\{(1+2\delta-\eta_2)(m-1)+\eta_1-\eta_2\right\}\alpha_{m,1}\gamma_m\\&+\lambda\sum_{m=2}^{\infty}\left\{(1+2\delta-\eta_2)(m-1)+\eta_1-\eta_2\right\}\alpha_{m,2}\gamma_m\\&\le(1-\lambda)(\eta_1-\eta_2)+\lambda(\eta_1-\eta_2)=\eta_1-\eta_2.\end{aligned}$$

Again by using Theorem 2, we reach the conclusion. □

In the following, we define the family $\delta\text{–}\mathcal{UM}^*(\ell,\eta_1,\eta_2)$ of holomorphic mappings f satisfying the coefficient conditions (8). Assume that

$$f(z) = z + \sum_{m=2}^{\infty} \alpha_m z^m \in \mathcal{A}.$$

Then $f \in \delta - \mathcal{UM}^*(\ell,\eta_1,\eta_2)$, if it satisfies the condition:

$$\sum_{m=2}^{\infty} \left[(3+2\delta+\eta_2)(m-1)+\eta_2-\eta_1\right] |\alpha_m| \gamma_m \le \eta_1 - \eta_2,$$

for some $\gamma_m \ge 0, \delta \ge 0$ and $-1 \le \eta_2 < \eta_1 \le 1$.

For special choices of η_1, η_2, δ and the mapping ℓ, we refer the study of Aouf and Mostafa [2] and others. Clearly

$$\delta - \mathcal{UM}^*(\ell,\eta_1,\eta_2) \subset \delta - \mathcal{UM}(\ell,\eta_1,\eta_2).$$

Adopting the required procedure found in [2, 3, 15], we have:

Theorem 5. *If* $f \in \delta - \mathcal{UM}^*(\ell,\eta_1,\eta_2)$ *and*

$$\Re\left(f(z)\right) > -\frac{\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2}{(3+2\delta+2\eta_2-\eta_1)\gamma_2} \qquad (z \in \mathbb{U}), \tag{15}$$

then

$$\frac{(1+2\delta-2\eta_2+\eta_1)\gamma_2}{2\left[\eta_1-\eta_2+(1+2\delta-2\eta_2+\eta_1)\right]\gamma_2} f(z) * h(z) \prec h(z) \qquad (z \in \mathbb{U}), \tag{16}$$

for all $h \in \mathcal{C}$. *The constant factor* $\frac{(1+2\delta-2\eta_2+\eta_1)\gamma_2}{2[\eta_1-\eta_2+(1+2\delta-2\eta_2+\eta_1)]\gamma_2}$ *in* (16) *cannot be replaced by a larger one.*

Proof. Let $f \in \delta - \mathcal{UM}^*(\ell,\eta_1,\eta_2)$ and let $h(z) = z + \sum\limits_{m=2}^{\infty} c_m z^m$. Then

$$\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2 f(z) * h(z)}{2\left[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\right]\gamma_2} = \frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2 \left(z + \sum\limits_{m=2}^{\infty} \alpha_m c_m z^m\right)}{2\left[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\right]\gamma_2}$$

In view of Definition 4 and Lemma 1, (16) will hold true if

$$\left\langle \frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2\alpha_m}{2\left[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2\right]}, m = 1,2,\ldots \right\rangle, \alpha_1 = 1 \tag{17}$$

is a subordinating factor sequence. Using Lemma 1, we observe that (17) is equivalent to

$$\Re\left\{1 + \sum_{m=1}^{\infty} \frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2\alpha_m z^m}{\left[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2\right]}\right\} > 0. \tag{18}$$

The mapping

$$\varphi(m) = \left\{(3+2\delta+\eta_2)(m-1)+\eta_2-\eta_1\right\}\gamma_m, \gamma_m \ge \gamma_2 > 0.$$

is an increasing mapping for $m \geq 2$. Considering this fact along with (18), we can write

$$\Re\left\{1+\sum_{m=1}^{\infty}\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2\alpha_m z^m}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}\right\}$$

$$=\Re\left\{1+\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2 z}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}+\sum_{m=2}^{\infty}\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2\alpha_m z^m}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}\right\}$$

$$\geq 1-\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2\,|z|}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}-\frac{\sum_{m=2}^{\infty}(3+2\delta+2\eta_2-\eta_1)\gamma_2\,|\alpha_m|\,|z|^m}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}$$

$$\geq 1-\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2 r}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}-\frac{\sum_{m=2}^{\infty}(3+2\delta+2\eta_2-\eta_1)\gamma_2\,|\alpha_m|\,r^m}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}$$

$$\geq 1-\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2 r}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\eta_1)\gamma_2]}-\frac{\sum_{m=2}^{\infty}(3+2\delta+2\eta_2-\eta_1)\gamma_m\,|\alpha_m|\,r}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}$$

On using (8), we see that

$$\Re\left\{1+\sum_{m=1}^{\infty}\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2\,|\alpha_m|\,z^m}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}\right\}$$
$$\geq 1-\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2 r}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}-\frac{(\eta_1-\eta_2)\,r}{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2].}$$
$$=1-r>0, r\to 1.$$

This leads to (18). Thus we have (16). Also (15) is obtained from (16) for the mapping

$$h(z)=\frac{z}{1-z},\qquad (z\in\mathbb{U}).$$

For the sharpness of

$$\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2}{2[(\eta_1-\eta_2)+(3+2\delta+2\eta_2-\eta_1)\gamma_2]},$$

we consider the mapping f_0 such that

$$f_0(z)=z-\frac{(\eta_1-\eta_2)}{(3+2\delta+2\eta_2-\eta_1)}z^2. \tag{19}$$

Combining (16) and (19), we write

$$\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2}{2[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}f_0(z)\prec\frac{z}{1-z},\ z\in\mathbb{U}.$$

Consider

$$\begin{aligned}
&\Re\left\{\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2}{2[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}f_0(z)\right\}\\
&=\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2}{2[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}\Re\left(f_0(z)\right)\\
&\geq-\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2)}{2[(\eta_1-\eta_2)+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}\left(\frac{[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1))\gamma_2]}{(3+2\delta+2\eta_2-\eta_1)\gamma_2}\right).
\end{aligned}$$

Thus, we have

$$\min_{|z|\leq r}\Re\left\{\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2}{2[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}f_0(z)\right\}=-\frac{1}{2}.$$

This proves that the constant $\frac{(3+2\delta+2\eta_2-\eta_1)\gamma_2}{2[\eta_1-\eta_2+(3+2\delta+2\eta_2-\eta_1)\gamma_2]}$ is the best possible. □

4. Concluding Remarks

In this research, we have used convolution between holomorphic mappings in defining some subfamilies $\delta-\mathcal{UM}(\ell,\eta_1,\eta_2)$ and $\delta-\mathcal{UM}_{\Im}(\ell,\eta_1,\eta_2)$ of holomorphic mappings involving starlike and convex mappings and associated with the conic domains. We derived sufficient conditions for the mappings to be in the family $\delta-\mathcal{UM}(\ell,\eta_1,\eta_2)$. We also discussed the characterization of mappings in the family $\delta-\mathcal{UM}_{\Im}(\ell,\eta_1,\eta_2)$ along with the coefficient bounds, integral representation and convex combination. Using the sufficient conditions for mappings belonging to the family $\delta-\mathcal{UM}(\ell,\eta_1,\eta_2)$, we also defined a family $\delta-\mathcal{UM}^*(\ell,\eta_1,\eta_2)$ and then making use of a particular sequence, we discussed some subordination result. Our findings can be related with the existing known results.

5. Research Background and Significance

Goodman studied the uniformly convex and starlike functions, whereas, Kanas and Wisniowska explored k-uniformly convex and k-uniformly starlike functions. While using the convolution technique, Raina introduced the similar family of analytic functions. In view of Janowski functions, Noor and Malik extended their results for the petal like domains. Using Hadamard product used by Raina and in contaxt of Noor and Malik work, we defined new classes of analytic functions and studied them in various aspects.

Functions with positive real part as well as function with certain assumptions on the arguments are of funcdamental importance in the study of starlike, convex, close-to-convex and Bazilevic functions which are related with the Kufarev differential equation. We study the characterization and bounds on the functions from the differential and integral ineqalities. Same study for the complex valued function is carried out using the idea of differential subordination. The study of the geometric properties of various types of image domains is still a prime focus of the theorists. Techniques of convolutions and other classical methods are still in progress in studying these images of compex analytic univalent and multivalent functions In this research, we have used convolution between holomorphic mappings in defining some subfamilies $\delta - \mathcal{UM}(\ell, \eta_1, \eta_2)$ and $\delta - \mathcal{UM}_{\Im}(\ell, \eta_1, \eta_2)$ of holomorphic mappings involving starlike and convex mappings and associated with the conic domains. We derived sufficient conditions for the mappings to be in the family $\delta - \mathcal{UM}(\ell, \eta_1, \eta_2)$. We also discussed the characterization of mappings in the family $\delta - \mathcal{UM}_{\Im}(\ell, \eta_1, \eta_2)$ along with the coefficient bounds, integral representation and convex combination. Using the sufficient conditions for mappings belonging to the family $\delta - \mathcal{UM}(\ell, \eta_1, \eta_2)$, we also defined a family $\delta - \mathcal{UM}^*(\ell, \eta_1, \eta_2)$ and then making use of a subordinating factor sequence, we discuss some subordination result. Our findings can be related with the existing literature of subject. Various problems like radius of convexity, starlikeness and close-to-convexity are still open.

Author Contribution Statements All authors have equally contributed in this research.

Declaration of Competing Interests The authors declare that the research is original and have not submitted anywhere in any form. Authors have no competing interest to disclose. The authors accept any sort of ethical responsibility regarding the manuscript publication.

Acknowledgements Authors are thankful to the Mirpur University of Science and Technology(MUST), for the provision of the research conducive environment at MUST.

References

[1] Aouf, M. K., Mostafa, A. O., Some properties of a subclass of uniformly convex functions with negative coefficients, *Demonstration Math.*, 2 (2008), 353-370. https://doi.org/10.1515/dema-2008-0212

[2] Aouf, M. K., El-Ashwah, R. M., El-Deeb, S. M., Subordination results for certain subclasses of uniformly starlike and convex functions defined by convolution, *European J. Pure Appl. Math.,* 3(5) (2010), 903-917.

[3] Attiya, A. A., On some application of a subordination theorems, *J. Math. Anal. Appl.,* 311 (2005), 489-494. https://doi.org/10.1016/j.jmaa.2005.02.056

[4] Bukhari, S. Z. H., Bulboacă, T., Shabbir, M. S., Subordination and superordination results for analytic functions with respect to symmetrical points, *Quaest. Math.,* 41(1) (2018), 1-15. https://doi.org/10.2989/16073606.2017.1372528

[5] Bukhari, S. Z. H., Raza, M., Nazir, M., Some generalizations of the class of analytic functions with respect to δ-symmetric points, *Hacet. J. Math. Stat.,* 45(1) (2016), 1-14.

[6] Bukhari, S. Z. H., Sokol, J., Zafar, S., Unified approach to starlike and convex functions involving convolution between analytic functions, *Results in Math.*, 73 (2018), 30. https://doi.org/10.1007/s00025-018-0782-0

[7] Goodman, A. W., On uniformly starlike functions, *J. Math. Anal. Appl.,* 155 (1991), 364-370. https://doi.org/10.1016/0022-247X(91)90006-L

[8] Janowski, W., Some extremal problems for certain classes of analytic functions, *Ann. Polon. Math.,* 28 (1973), 297-326. 10.4064/AP-28-3-297-326

[9] Kanas, S., Wiśniowska, A., Conic domains and starlike functions, *Rev. Roumaine Math. Pures Appl.,* 45 (2000), 647-657.

[10] Miller, S. S., Mocanu, P. T., Differential Subordinations Theory and Applications, Series on Monographs and Textbooks in Pure and Appl Math, No. 255 Marcel Dekker, Inc., New York, 2000. https://doi.org/10.1201/9781482289817

[11] Noor, K. I., Malik, S. N., On coefficient inequalities of functions associated with conic domains, *Comp. Math. Appl.,* 62 (2011), 2209-2217. https://doi.org/10.1016/j.camwa.2011.07.006

[12] Raina, R. K., Deepak, B., Some properties of a new class of analytic functions defined in terms of a Hadamard product, *J. Inequal Pure Appl. Math.,* 9 (2008), 1-9.

[13] Ronning, F., Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Soc.,* 118 (1993), 189-196. https://doi.org/10.2307/2160026

[14] Silverman, H., Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.,* 51 (1975), 109-116.

[15] Srivastava, H. M., Attiya, A. A., Some subordination results associated with certain subclass of analytic functions, *J. Inequal Pure Appl. Math.,* 5(4) (2004), 1-6.

[16] Wilf, H. S., Subordinating factor sequence for convex maps of the unit circle, *Proc. Amer. Math. Soc.,* 129 (1961), 689-693.