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# TIMELIKE ROTATIONAL HYPERSURFACES WITH TIMELIKE AXIS IN MINKOWSKI FOUR-SPACE

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Abstract. We introduce the timelike rotational hypersurfaces $\mathbf{x}$ with timelike axis in Minkowski 4-space $\mathbb{E}_1^4$. We obtain the equations for the curvatures of the hypersurface. Moreover, we present a theorem for the rotational hypersurfaces with timelike axis supplying $\Delta \mathbf{x} = \mathcal{T}\mathbf{x}$, where $\mathcal{T}$ is a $4 \times 4$ real matrix.

## 1. Introduction

Geometers have been focused on the geometry of the surfaces and hypersurfaces in space forms for years. Some of the works on the topic are indicated in alphabetical order:

Arslan et al. [1] studied the generalized rotation surfaces in Euclidean four space $\mathbb{E}^4$; Arslan, et al. [2] worked the Weyl pseudosymmetric hypersurfaces; Arslan and Milousheva [3] focused the meridian surfaces of elliptic or hyperbolic type in Minkowski 4-space $\mathbb{E}_1^4$; Arvanitoyeorgos et al. [4] introduced the Lorentz hypersurfaces satisfying $\Delta H = \alpha H$ in $\mathbb{E}_1^4$; Beneki et al. [5] served the helicoidal surfaces in Minkowski 3-space; Chen [6] worked the total mean curvature and the finite type submanifolds; Cheng and Wan [7] presented the complete hypersurfaces in $\mathbb{R}^4$ with CMC; Cheng and Yau [8] studied the hypersurfaces with constant scalar curvature; Dillen et al. [9] stated the rotation hypersurfaces in $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$; Do Carmo and Dajczer [10] derived the rotation hypersurfaces in spaces of constant curvature; Dursun [11] considered the hypersurfaces with pointwise 1-type Gauss map.

On the other hand, Dursun and Turgay [12] worked the space-like surfaces in $\mathbb{E}_1^4$ with pointwise 1-type Gauss map; Ferrandez et al. [13] focused some class of conformally at Euclidean hypersurfaces; Ganchev and Milousheva [14] considered the general rotational surfaces in $\mathbb{E}_1^4$; Güler [15] introduced the helical hypersurfaces in

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$\mathbb{E}_1^4$; Güler [16] obtained the fundamental form $IV$ with the curvatures of the hypersphere; Güler [17] worked the rotational hypersurfaces satisfying $\Delta^I R = AR$ in $\mathbb{E}^4$; Güler et al. [18] examined the Gauss map and the third Laplace−Beltrami operator of the rotational hypersurface in 4-space; Güler et al. [19] studied Laplace−Beltrami operator of a helicoidal hypersurface in four-space; Güler and Turgay [20] focused the Cheng−Yau operator and Gauss map of the rotational hypersurfaces in 4-space.

Hasanis and Vlachos [21] worked hypersurfaces in $\mathbb{E}^4$ with harmonic mean curvature vector field; Kim and Turgay [22] served the surfaces with $L_1$-pointwise 1-type Gauss map in $\mathbb{E}^4$; Lawson [23] introduced the minimal submanifolds; Magid et al. [24] focused the affine umbilical surfaces in $\mathbb{R}^4$; Moore [25] revealed the surfaces of rotation in $\mathbb{E}^4$; Moore [26] also indicated the rotation surfaces of constant curvature in $\mathbb{E}^4$; O'Neill [27] presented the semi-Riemannian geometry.

Takahashi [28] served that minimal surfaces and spheres are the only surfaces in $\mathbb{E}^3$ satisfying the condition $\Delta r = \lambda r$, $\lambda \in \mathbb{R}$; Turgay and Upadhyay [29] worked the biconservative hypersurfaces in 4-dimensional Riemannian space forms.

In this work, we consider the timelike rotational hypersurfaces $\mathbf{x}$ with timelike axis in Minkowski 4-space $\mathbb{E}_1^4$. We give the notions of $\mathbb{E}_1^4$ in Section 2. In Section 3, we present the definition of the timelike rotational hypersurfaces with timelike axis, and compute its curvatures. In addition, we give the timelike rotational hypersurfaces with timelike axis supplying $\Delta\mathbf{x} = \mathcal{T}\mathbf{x}$, where $\mathcal{T}$ is a $4 \times 4$ real matrix in Section 4. Finally, we serve the results and conclusion in the last section.

## 2. Preliminaries

In this section, we indicate the first and second fundamental forms, matrix of the shape operator $\mathbf{S}$, the curvatures $\mathfrak{C}_i$ of the hypersurface $\mathbf{x} = \mathbf{x}(u, v, w)$ in Minkowski 4-space $\mathbb{E}_1^4$. We identify a vector (a,b,c,d) with its transpose $(\text{a,b,c,d})^t$ in the rest of this paper.

Let $\mathbf{x}$ be an isometric immersion of a hypersurface from $M_1^3$ to $\mathbb{E}_1^4 = (\mathbb{R}^4, \cdot)$, where $\overrightarrow{x} \cdot \overrightarrow{y} = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$ is a Lorentzian inner product of vectors $\overrightarrow{x} = (x_1, x_2, x_3, x_4)$, $\overrightarrow{y} = (y_1, y_2, y_3, y_4)$, and $x_i$ are the pseudo-Euclidean coordinates of type $(3, 1)$. The vector product of $\overrightarrow{x} = (x_1, x_2, x_3, x_4)$, $\overrightarrow{y} = (y_1, y_2, y_3, y_4)$, $\overrightarrow{z} = (z_1, z_2, z_3, z_4)$ of $\mathbb{E}_1^4$ is defined by

$$\overrightarrow{x} \times \overrightarrow{y} \times \overrightarrow{z} = \det \begin{pmatrix} e_1 & e_2 & e_3 & -e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}.$$

A vector $\overrightarrow{x}$ is called timelike if $\overrightarrow{x} \cdot \overrightarrow{x} < 0$, and a hypersurface $\mathbf{x}$ is timelike if $\mathbf{x} \cdot \mathbf{x} < 0$. In $\mathbb{E}_1^4$, the hypersurface $\mathbf{x}$ has the following first and second fundamental form matrices, resp.,

$$I=\begin{pmatrix} E & F & A \\ F & G & B \\ A & B & C \end{pmatrix}, \quad II=\begin{pmatrix} L & M & P \\ M & N & T \\ P & T & V \end{pmatrix},$$

and

$$\begin{aligned} \det I &= (EG-F^2)C - EB^2 + 2FAB - GA^2, \\ \det II &= \left(LN-M^2\right)V - LT^2 + 2MPT - NP^2, \end{aligned}$$

where

$$\begin{aligned} E &= \mathbf{x}_u\cdot\mathbf{x}_u, F=\mathbf{x}_u\cdot\mathbf{x}_v, G=\mathbf{x}_v\cdot\mathbf{x}_v, A=\mathbf{x}_u\cdot\mathbf{x}_w, B=\mathbf{x}_v\cdot\mathbf{x}_w, C=\mathbf{x}_w\cdot\mathbf{x}_w, \\ L &= \mathbf{x}_{uu}\cdot e, M=\mathbf{x}_{uv}\cdot e, N=\mathbf{x}_{vv}\cdot e, P=\mathbf{x}_{uw}\cdot e, T=\mathbf{x}_{vw}\cdot e, V=\mathbf{x}_{ww}\cdot e, \end{aligned}$$

and also $e$ is the Gauss map of the hypersurface $\mathbf{x}$:

$$e=\frac{\mathbf{x}_u\times\mathbf{x}_v\times\mathbf{x}_w}{\|\mathbf{x}_u\times\mathbf{x}_v\times\mathbf{x}_w\|}. \tag{1}$$

The shape operator matrix $\mathbf{S}=I^{-1}\cdot II$ is defined by

$$\mathbf{S}=\frac{1}{\det I}\begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix},$$

where

$$\begin{aligned} s_{11} &= ABM - CFM - AGP + BFP + CGL - B^2L, \\ s_{12} &= ABN - CFN - AGT + BFT + CGM - B^2M, \\ s_{13} &= ABT - CFT - AGV + BFV + CGP - B^2P, \\ s_{21} &= ABL - CFL + AFP - BPE + CME - A^2M, \\ s_{22} &= ABM - CFM + AFT - BTE + CNE - A^2N, \\ s_{23} &= ABP - CFP + AFV - BVE + CTE - A^2T, \\ s_{31} &= -AGL + BFL + AFM - BME + GPE - F^2P, \\ s_{32} &= -AGM + BFM + AFN - BNE + GTE - F^2T, \\ s_{33} &= -AGP + BFP + AFT - BTE + GVE - F^2V. \end{aligned}$$

**Theorem 1.** *The hypersurface* $\mathbf{x}$ *in* $\mathbb{E}_1^4$ *has the following curvature formulas,* $\mathfrak{C}_0=1$ *(by definition),*

$$\mathfrak{C}_1 = \frac{\left\{\begin{array}{c} (EN+GL-2FM)C+(EG-F^2)V-LB^2-NA^2 \\ -2(APG-BPF-ATF+BTE-ABM) \end{array}\right\}}{3\left[(EG-F^2)C-EB^2+2FAB-GA^2\right]}, \tag{2}$$

$$\mathfrak{C}_2 = \frac{\left\{\begin{array}{c} (EN+GL-2FM)V+\left(LN-M^2\right)C-ET^2-GP^2 \\ -2\left(APN-BPM-ATM+BTL-PTF\right) \end{array}\right\}}{3\left[(EG-F^2)C-EB^2+2FAB-GA^2\right]}, \tag{3}$$

$$\mathfrak{C}_3 = \frac{\left(LN - M^2\right)V - LT^2 + 2MPT - NP^2}{(EG - F^2)C - EB^2 + 2FAB - GA^2}. \tag{4}$$

See [16] for Euclidean details. Next, we define the rotational hypersurface in $\mathbb{E}_1^4$.

**Definition 1.** *Let $\gamma : I \subset \mathbb{R} \longrightarrow \Pi$ be a curve in a plane $\Pi$ and $\ell$ be a straight line in $\Pi$ in $\mathbb{E}_1^4$. A rotational hypersurface in $\mathbb{E}_1^4$ is defined as a hypersurface rotating a curve $\gamma$ around a line $\ell$ (called the profile curve and the axis, respectively).*

Therefore, we introduce the rotational hypersurfaces with timelike axis in $\mathbb{E}_1^4$ in the following section.

## 3. Timelike Rotational Surfaces with Timelike Axis

We may suppose $\ell$ spanned by the timelike vector $(1,0,0,0)^t$. The orthogonal matrix is given by

$$\mathbf{A}(v,w) = \begin{pmatrix} \cos v \cos w & -\sin v & -\cos v \sin w & 0 \\ \sin v \cos w & \cos v & -\sin v \sin w & 0 \\ \sin w & 0 & \cos w & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5}$$

where $v, w \in \mathbb{R}$. The following holds: $\det \mathbf{A} = 1$, $\mathbf{A}.\ell = \ell$, $\mathbf{A}^t \varepsilon \mathbf{A} = \varepsilon$, where $\varepsilon = diag(1,1,1,-1)$. Supposing the axis of rotation is $\ell$, there is a Lorentz transformation that the axis is $\ell$ transformed to the $x_4$-axis of $\mathbb{E}_1^4$. The parametrization of the timelike profile curve is given by $\gamma(u) = (u, 0, 0, \varphi(u))$. Here, we assume that the profile curve is timelike, i.e., $\gamma' \cdot \gamma' = 1 - \varphi'^2 < 0$, $\varphi(u) : \mathrm{I} \subset \mathbb{R} \longrightarrow \mathbb{R}$ is a differentiable function for all $u \in$I. So, the rotational hypersurface spanned by the vector $(0,0,0,1)$, is given by $\mathbf{x}(u,v,w) = \mathbf{A}(v,w)\gamma(u)^t$ in $\mathbb{E}_1^4$, where $u \in$I, $v, w \in \mathbb{R}$. If $w = 0$, we get the rotational surface with timelike axis as in three dimensional Minkowski space $\mathbb{E}_1^3$.

Next, we obtain the curvatures of the following rotational hypersurface with timelike axis

$$\mathbf{x}(u,v,w) = \begin{pmatrix} u \cos v \cos w \\ u \sin v \cos w \\ u \sin w \\ \varphi(u) \end{pmatrix}, \tag{6}$$

where $u \in \mathbb{R} - \{0\}$ and $0 \leq v, w \leq 2\pi$. Using the first differentials of $(6)$, we get the first fundamental form matrix as follows

$$I = \begin{pmatrix} 1 - \varphi'^2 & 0 & 0 \\ 0 & u^2 \cos^2 w & 0 \\ 0 & 0 & u^2 \end{pmatrix},$$

where $\varphi = \varphi(u)$, $\varphi' = \frac{d\varphi}{du}$. We obtain $\det I = u^4(1 - \varphi'^2)\cos^2 w < 0$. So, the hypersurface is the timelike rotational hypersurface with timelike axis. Using the second differentials with respect to $u, v, w$, we have the second fundamental form matrix as follows

$$II=\begin{pmatrix} -\frac{\varphi''}{(\varphi'^2-1)^{1/2}} & 0 & 0 \\ 0 & -\frac{u\varphi'\cos^2 w}{(\varphi'^2-1)^{1/2}} & 0 \\ 0 & 0 & -\frac{u\varphi'}{(\varphi'^2-1)^{1/2}} \end{pmatrix},$$

and $\det II=-\frac{u^2\varphi'^2\varphi''\cos^2 w}{(\varphi'^2-1)^{3/2}}$. Therefore, the shape operator matrix of the hypersurface is given by

$$\mathbf{S}=\begin{pmatrix} \frac{\varphi''}{(\varphi'^2-1)^{3/2}} & 0 & 0 \\ 0 & -\frac{\varphi'}{u(\varphi'^2-1)^{1/2}} & 0 \\ 0 & 0 & -\frac{\varphi'}{u(\varphi'^2-1)^{1/2}} \end{pmatrix}.$$

Finally, we calculate the curvatures of the timelike rotational hypersurface with timelike axis, and give the results of it in the following.

**Corollary 1.** *The timelike rotational hypersurface* (6) *with timelike axis has the following curvatures*

$$\begin{aligned} \mathfrak{C}_1 &= H=\frac{u\varphi''-2\left(\varphi'^2-1\right)\varphi'}{3u(\varphi'^2-1)^{3/2}}, \\ \mathfrak{C}_2 &= \frac{-2u\varphi'\varphi''+\left(\varphi'^2-1\right)\varphi'^2}{3u^2(\varphi'^2-1)^2}, \\ \mathfrak{C}_3 &= K=\frac{\varphi'^2\varphi''}{u^2(\varphi'^2-1)^{5/2}}. \end{aligned}$$

Proof. By using the eqs. $(2),(3),(4)$ for the timelike rotational hypersurface with timelike axis $(6)$, we get the curvatures.

**Corollary 2.** *When the timelike rotational hypersurface* (6) *with timelike axis has* $\mathfrak{C}_i=0$, $i=1,2,3$, *respectively, then it has the following general* $\varphi$ *solutions, respectively,*

$$\begin{aligned} u\varphi''-2\varphi'^3+2\varphi'=0 \quad &\Leftrightarrow \quad \varphi=\mp i^{1/2}e^{-c_1/2}F\left[i\operatorname{arg\,sinh}\left((ie^{c_1}u)^{1/2}\right),-1\right]+c_2 \\ &\qquad \text{or } \varphi=c_1, \\ 2u\varphi'\varphi''-\varphi'^4+\varphi'^2=0 \quad &\Leftrightarrow \quad \varphi=\mp 2e^{-2c_1}\left(1+e^{2c_1}u\right)^{1/2}+c_2 \\ &\qquad \text{or } \varphi=c_1, \\ \varphi'^2\varphi''=0 \quad &\Leftrightarrow \quad \varphi=c_1u+c_2 \text{ or } \varphi=c_1, \end{aligned}$$

*where* $F[\phi,m]=\int_0^{\phi}\left(1-m\sin^2\theta\right)^{-1/2}d\theta$ *gives the elliptic integral of the first kind for* $-\pi/2<\phi<\pi/2$, $i=(-1)^{1/2}$, $c_1,c_2\in\mathbb{R}$.

## 4. Timelike Rotational Surfaces with Timelike Axis Satisfying $\Delta \mathbf{x} = \mathcal{T}\mathbf{x}$

**Definition 2.** *The Laplace−Beltrami operator of the hypersurface* $\mathbf{x}(u,v,w)\,|_{D\subset\mathbb{R}^4}$ *of class* $C^3$ *is given by*

$$\Delta \mathbf{x} = \frac{1}{(\det I)^{1/2}} \left\{ \begin{array}{c} \frac{\partial}{\partial u}\left( \frac{\left(CG-B^2\right)\mathbf{x}_u + (AB-CF)\mathbf{x}_v + (BF-AG)\mathbf{x}_w}{(\det I)^{1/2}} \right) \\ + \frac{\partial}{\partial v}\left( \frac{(AB-CF)\mathbf{x}_u + \left(CE-A^2\right)\mathbf{x}_v + (AF-BG)\mathbf{x}_w}{(\det I)^{1/2}} \right) \\ + \frac{\partial}{\partial w}\left( \frac{(BF-AG)\mathbf{x}_u + (AF-BG)\mathbf{x}_v + \left(EG-F^2\right)\mathbf{x}_w}{(\det I)^{1/2}} \right) \end{array} \right\}. \tag{7}$$

By using above definition with the hypersurface (6), we have the following Laplace−Beltrami operator

$$\Delta \mathbf{x} = \frac{u\varphi'' - 2\varphi'^3 + 2\varphi'}{uW^2} \begin{pmatrix} \varphi' \cos v \cos w \\ \varphi' \sin v \cos w \\ \varphi' \sin w \\ 1 \end{pmatrix},$$

where $W = \varphi'^2 - 1$. The Gauss map of the rotational hypersurface (6) with timelike axis is given by

$$e = \frac{1}{W^{1/2}} \begin{pmatrix} \varphi' \cos v \cos w \\ \varphi' \sin v \cos w \\ \varphi' \sin w \\ 1 \end{pmatrix}.$$

Considering $3He = \mathcal{T}\mathbf{x}$, we obtain

$$\begin{pmatrix} (\Psi\varphi' - t_{11}u)\cos v \cos w - t_{12}u \cos w \sin v - t_{13}u \sin w \\ t_{21} - u\cos v \cos w + (\Psi\varphi' - t_{22}u)\sin v \cos w - t_{23}u \sin w \\ -t_{31}u \cos v \cos w - t_{32}u \sin v \cos w + (\Psi\varphi' - t_{33}u)\sin w \\ \Psi \end{pmatrix}$$
$$= \begin{pmatrix} t_{14}\varphi(u) \\ t_{24}\varphi(u) \\ t_{34}\varphi(u) \\ t_{41}u\cos v \cos w + t_{42}u \sin v \cos w + t_{43}u \sin w + t_{44}\varphi(u) \end{pmatrix},$$

where $\mathcal{T}$ is a $4\times 4$ real matrix with the components $t_{ij}$, and also $\Psi(u) = 3HW^{-1/2}$. The equation $\Delta \mathbf{x} = \mathcal{T}\mathbf{x}$ with respect to the first quantity $I$, and $\Delta \mathbf{x} = 3He$ give rises to the following system

$$\begin{array}{rcl} (\Psi\varphi' - t_{11}u)\cos v \cos w - t_{12}u \sin v \cos w - t_{13}u \sin w &=& t_{14}\varphi(u), \\ -t_{21}u \cos v \cos w + (\Psi\varphi' - t_{22}u)\sin v \cos w - t_{23}u \sin w &=& t_{24}\varphi(u), \\ -t_{31}u \cos v \cos w - t_{32}u \sin v \cos w + (\Psi\varphi' - t_{33}u)\sin w &=& t_{34}\varphi(u), \\ -t_{41}u \cos v \cos w - t_{42}u \sin v \cos w - t_{43}u \sin w + \Psi &=& t_{44}\varphi(u). \end{array}$$

Differentiating ODE's two times depends on $v$, we have

$$t_{14} = t_{24} = t_{34} = t_{44} = 0, \quad \Psi = 0. \tag{8}$$

From (8) , we see the following

$$\begin{aligned} t_{11}u\cos v + t_{12}u\sin v &= 0, \\ t_{21}u\cos v + t_{22}u\sin v &= 0, \\ t_{31}u\cos v + t_{32}u\sin v &= 0, \\ t_{41}u\cos v + t_{42}u\sin v &= 0. \end{aligned}$$

When we use these equality in the equation system, we get

$$t_{13} = t_{23} = t_{33} = t_{43} = 0. \tag{9}$$

Then, matrix $\mathcal{T}$ becomes zero matrix. So, if $\Delta\mathbf{x} = \mathcal{T}\mathbf{x}$, then $\mathcal{T} = 0$ and the hypersurface is a minimal.

Also, the cos and sin are the linearly independent functions of $v$, then we obtain $t_{ij} = 0$. Since $\Psi = 3HW^{-1/2}$, we find $H = 0$. Therefore, $\mathbf{x}$ is a timelike minimal hypersurface with timelike axis.

Hence, we serve the following theorem:

**Theorem 2.** *Let timelike* $\mathbf{x} : M_1^3 \longrightarrow \mathbb{E}_1^4$ *be an isometric immersion given by* (6). $\Delta\mathbf{x} = \mathcal{T}\mathbf{x}$, *where* $\mathcal{T}$ *is a* $4 \times 4$ *real matrix iff* $\mathbf{x}$ *is a timelike minimal hypersurface with timelike axis, i.e.,* $H = \mathfrak{C}_1 = 0$.

## 5. Results and Conclusion

The concepts of rotational hypersurfaces are studied by many mathematician and geometers. It is shown that the timelike rotational hypersurface has three different curvatures in Minkowski 4-space. Moreover, the minimality condition of it by using the Laplace−Beltrami operator has presented. These concepts propose for the other space forms may be useful in the future.

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