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A New Iterative Algorithm for the Time-Fractional Fisher Equation Including Small Delay

Ali Demir[1*] and Mine Aylin Bayrak[1]

[1]*Department of Mathematics, Faculty of Science and Arts, Kocaeli University, Kocaeli, Turkey*
[*]*Corresponding author*

Abstract

This study aims to establish a numerical solution of time fractional Fisher equation with small delay by utilizing residual power series method (RPSM). First of all, replacing the term including small delay by in Taylor series expansion of it, we reduce the problem into a fractional Fisher equation without delay. Secondly, applying RPSM, the coefficients of the series are determined which converges to the solution of the equation rapidly. Effectiveness and accuracy of this algorithm are illustrated by presented examples.

Keywords: *Caputo derivative; Fisher equation; Fractional delay differential equation; Residual power series method*
2010 Mathematics Subject Classification: *26A33;34K10;40G10*

1. Introduction

Last couple of decades fractional differential equations draw immense interest of researchers from various branches of science. Therefore, it is used to model many problems in different branches such as bioengineering, thermodynamics, visco-elasticity, control theory, aerodynamics, electro-magnetics, signal processing, chemistry, finance [1, 2, 3, 4, 5, 6, 7]. Consequently, different methods have been developed to establish the solution of fractional differential equations in various senses such as Riemann-Liouville , Caputo-Fabrizio, Caputo sense etc. [6, 7, 8, 9, 10, 11, 12].
Since delay is a natural part of almost all processes, in the modelling delay differential equations (DDE's) play a significant role especially in the modelling of physical systems with memory such as kinetics, controllers, signal processing and damping behaviour of viscoelastic materials[13, 14, 15, 16, 17].
RPSM is a common method to solve various kinds of differential equations such as [18, 19, 20, 21, 22, 23, 34, 25]. By this method, the approximate solutions of the mathematical models including differential equations are constructed in the form of Maclaurin series.
In this study, we propose a new algorithm including RPSM to construct an approximate solution of time-fractional Fisher equation with small delay in Caputo sense. By this algorithm, the approximate solutions is established for time fractional Fisher equation with small delay ε which is in the neighbourhood of zero

$$D_t^{\alpha}u(x,t) = D_{xx}u(x,t) + 6u(x,t-\varepsilon)(1-u(x,t)), \quad x \in R, \quad t > 0, \quad 0 < \alpha \leqslant 1 \tag{1.1}$$

subject to initial condition

$$u(x,0) = A_1(x) \tag{1.2}$$

which was originally proposed by Fisher [26]. The time-fractional Fisher equation have been solved analytically and numerically by various methods [27, 28, 29, 30, 31].
The mathematical problem including Fisher equation have been used in the modelling of diverse processes such as chemical and biological processes. In this study, by applying RPSM, series solution of the time-fractional Fisher equation with small delay is constructed. Since delay ε is small, expanding the term including small delay in Taylor series, the problem is reduced into a perturbation problem. Making use of RPSM, the solutions of the equations which is the obtained by means of the coefficients of ε are determined. A approximate solution of the problem are established in terms of these solutions in the form of Taylor series. The rest of the paper is organized as follows: In Section 2, fundamental notions are given. The implementation of RPSM for time fractional Fisher equation with small delay is presented in Section 3. Numerical results are illustrated in Section 4. In Section 5, concluded results are explained.

Email addresses: **ademir@kocaeli.edu.tr (Ali Demir),aylin@kocaeli.edu.tr (Mine Aylin Bayrak)**

2. Preliminaries

In this section, the basic definitions and various features of fractional calculus theory are given [6, 32, 33, 34].

Definition 2.1. *The Riemann-Liouville fractional integral of order α $(\alpha \geq 0)$ is given as [18, 21]*

$$J^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)dt,\ \alpha > 0,\ x > 0$$

$$J^0 f(x) = f(x)$$

Definition 2.2. *The Caputo fractional derivative with order α is given as [18, 21]*

$$D^{\alpha}f(x) = J^{m-\alpha}D^m f(x) = \int_0^x (x-t)^{m-\alpha-1}\frac{d^m}{dt^m}f(t)dt,\ m- < \alpha < m,\ x > 0$$

where D^m is the classic differential operator with order m.

By the Caputo derivative in terms of Gamma function [6], we have

$$D^{\alpha}x^{\beta} = 0, \quad \beta < \alpha$$

$$D^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}, \quad \beta \geq \alpha$$

Definition 2.3. *The Caputo's time fractional derivative of order α for $u(x,t)$ is defined as [18, 21]*

$$D_t^{\alpha}u(x,t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\varsigma)^{m-\alpha-1}\frac{\partial^m u(x,\varsigma)}{\partial t^m}d\varsigma, & m-1 < \alpha < m \\ \frac{\partial^m u(x,t)}{\partial t^m} & ,\alpha = m \in N \end{cases}$$

Definition 2.4. *A power series expansion of the form*

$$\sum_{m=0}^{\infty} c_m(t-t_0)^{m\alpha} = c_0 + c_1(t-t_0)^{\alpha} + c_2(t-t_0)^{2\alpha} + ..., \quad 0 \leq m-1 < \alpha \leq m, \quad t \geq t_0$$

is called fractional power series about $t = t_0$ [35].

Definition 2.5. *A power series expansion of the form*

$$\sum_{m=0}^{\infty} f_m(x)(t-t_0)^{m\alpha} = f_0(x) + f_1(x)(t-t_0)^{\alpha} + f_2(x)(t-t_0)^{2\alpha} + ..., \quad 0 \leq m-1 < \alpha \leq m, \quad t \geq t_0$$

is called multiple fractional power series about $t = t_0$ [35].

Theorem 2.6. *Suppose that $u(x,t)$ has a multiple fractional power series representation at $t = t_0$ of the following form*

$$u(x,t) = \sum_{m=0}^{\infty} f_m(x)(t-t_0)^{m\alpha}, \quad x \in I, \quad t_0 \leq t \leq t_0 + R$$

If $D_t^{m\alpha}u(x,t),\ m = 0,1,2,...$ are continuous on $I \times (t_0, t_0+R)$, then $f_m(x) = \frac{D_t^{m\alpha}u(x,t_0)}{\Gamma(m\alpha+1)}$.

3. RPSM of the time-fractional Fisher equation with small delay

Let us consider the following time-fractional Fisher equation with small delay

$$D_t^{\alpha}u(x,t) = D_{xx}u(x,t) + 6u(x,t-\varepsilon)(1-u(x,t)), \quad x \in R, \quad t > 0, \quad 0 < \alpha \leqslant 1 \tag{3.1}$$

subject to initial condition

$$u(x,0) = A_1(x) \tag{3.2}$$

Since the delay term is very small, we replace $u(x,t-\varepsilon)$ Eq.(1.1) by the following Taylor series expansion

$$u(x,t-\varepsilon) = u(x,t) - \varepsilon D_t u(x,t) + O(\varepsilon^2) \tag{3.3}$$

where we ignore the higher order terms, which leads to

$$D_t^{\alpha}u(x,t) = D_{xx}u(x,t) + 6u(x,t)(1-u(x,t)) - 6\varepsilon(1-u(x,t))D_t u(x,t) \tag{3.4}$$

and assuming that $n\alpha = 1$, we obtain

$$D_t^{\alpha}u(x,t) = D_{xx}u(x,t) + 6u(x,t)(1-u(x,t)) - 6\varepsilon(1-u(x,t))D_t^{n\alpha}u(x,t) \tag{3.5}$$

In order to construct the solution of time-fractional Fisher equation with small delay for $n=2$, we use the following equation

$$D_t^{\alpha}u(x,t)-D_{xx}u(x,t)-6u(x,t)(1-u(x,t))+6\varepsilon(1-u(x,t))D_t^{2\alpha}u(x,t)=0, \tag{3.6}$$

Using (3.3) in (3.1) leads the problem to a perturbation problem. Using Poincare's expansion [36, 37, 38, 39, 40, 41] which is the asymptotic expansion of functions $h(x)$ to $h(x,\varepsilon)$ leads to the solution in the following series form

$$u=\sum_{k=0}^{\infty}\varepsilon^k u_k \tag{3.7}$$

for the solution of Eq.(3.5). Substituting (3.7) into (3.6) leads to the following equation:

$$D_t^{\alpha}\Big(\sum_{k=0}^{\infty}\varepsilon^k u_k\Big)-D_{xx}\Big(\sum_{k=0}^{\infty}\varepsilon^k u_k\Big)-6\Big(\sum_{k=0}^{\infty}\varepsilon^k u_k\Big)\Big(1-\Big(\sum_{k=0}^{\infty}\varepsilon^k u_k\Big)\Big)+6\varepsilon(1-\Big(\sum_{k=0}^{\infty}\varepsilon^k u_k\Big))D_t^{2\alpha}\Big(\sum_{k=0}^{\infty}\varepsilon^k u_k\Big)=0, \tag{3.8}$$

Hence, we obtain the following equations by making the coefficients of ε terms equal to zero

$$D_t^{\alpha}u_0-D_{xx}u_0-6u_0+6u_0^2=0, \tag{3.9}$$

$$D_t^{\alpha}u_1-D_{xx}u_1-6u_1+12u_0u_1+6D_t^{2\alpha}u_0-6u_0D_t^{2\alpha}u_0=0, \tag{3.10}$$

$$D_t^{\alpha}u_2-D_{xx}u_2-6u_2+6u_1^2+12u_0u_2+6D_t^{2\alpha}u_1-6u_0D_t^{2\alpha}u_1-6u_1D_t^{2\alpha}u_0=0, \tag{3.11}$$

$$D_t^{\alpha}u_3-D_{xx}u_3-6u_3+12u_0u_3+12u_1u_2+6D_t^{2\alpha}u_2-6u_0D_t^{2\alpha}u_2-6u_1D_t^{2\alpha}u_1-6u_2D_t^{2\alpha}u_0=0, \tag{3.12}$$

and so on. We apply the RPSM to find out series solution for these equations subjected to given initial conditions by replacing its fractional power series expansion with its residual function. From each equation, a recurrence relation for the calculation of coefficients is supplied, while coefficients in fractional power series expansion can be calculated by repeatedly fractional differentiation of the residual function [18, 19, 20, 21, 22, 23, 24, 25]. The RPSM is proposed for the solutions for Eq. (3.9)-(3.12) in the form of a fractional power series about the initial point $t=0$ [18]

$$u_i(x,t)=\sum_{k=0}^{\infty}f_{i,k}(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, i=0,1,2,3,...x\in I, 0\leq t<R \tag{3.13}$$

To obtain the numerical values from this series, let $u_m(x,t)$ denotes the m^{th} truncated series of $u(x,t)$. That is

$$u_{i,m}(x,t)=\sum_{k=0}^{m}f_{i,k}(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}, i=0,1,2,3,...x\in I, 0\leq t<R \tag{3.14}$$

By the initial condition, the 0^{th} residual power series approximate solution of $u(x,t)$ can be written as follows:

$$u_0(x,t)=f_0(x)=u(x,0)=A_1(x) \tag{3.15}$$

Eq.(3.13) can be written as

$$u_m(x,t)=A_1(x)+\sum_{k=2}^{m}A_k(x)\frac{t^{k\alpha}}{\Gamma(k\alpha+1)},\ 0<\alpha\leq 1,\quad x\in I,\quad 0\leq t,\quad k=2,3,... \tag{3.16}$$

Define the residual function for Eq.(3.8) as

$$Res_0(x,t)=D_t^{\alpha}u_0-D_{xx}u_0-6u_0+6u_0^2 \tag{3.17}$$

and the m^{th} residual function can be expressed as

$$Res_m(x,t)=D_t^{\alpha}u_m-D_{xx}u_m-6u_m+6u_m^2 \tag{3.18}$$

By making use of some results such as $Res(x,t)=0,\ \lim_{m\to\infty}Res_m(x,t)$ for each $x\in I$ and $t\geq 0$ and $D_t^{r\alpha}Res(x,0)=D_t^{r\alpha}Res_m(x,0)=0,\ r=0,1,2,...,m$ are used to obtain the solution [18, 19, 20, 21, 22, 23, 24, 25],.
Substituting the m^{th} truncated series of $u(x,t)$ into Eq. (3.8), calculating the fractional derivative $D_t^{(m-1)\alpha}$ of $Res(x,t), m=1,2,3,...$ at $t=0$ and solving the following obtained algebraic system

$$D_t^{(m-1)\alpha}Res_m(x,0)=0, 0<\alpha\leq 1, m=1,2,3,... \tag{3.19}$$

the required coefficients $A_k(x), k=2,3,...,m$ in Eq.(3.15) are determined.
In order to determine $A_2(x)$, the 1^{st} residual function in Eq. (3.17) can be written as follows:

$$Res_{0,1}(x,t)=D_t^{\alpha}u_{0,1}-D_{xx}u_{0,1}-6u_{0,1}+6u_{0,1}^2 \tag{3.20}$$

where $u_{0,1}(x,t)=A_1(x)+A_2(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}$. Therefore, we have

$$Res_{0,1}(x,t)=A_2-\left(A_1''+A_2''\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)-6\left(A_1+A_2\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)+6\left(A_1+A_2\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^2. \quad (3.21)$$

From Eq.(3.18), we deduce that $Res_1(x,0)=0$, which leads to

$$A_2(x)=A_1''+6A_1-6A_1^2. \quad (3.22)$$

Similarly, to obtain $A_3(x)$, the 2^{nd} residual function in Eq. (3.17) can be written in the following form

$$Res_{0,2}(x,t)=D_t^{\alpha}u_{0,2}-D_{xx}u_{0,2}-6u_{0,2}+6u_{0,2}^2 \quad (3.23)$$

where $u_{0,2}(x,t)=A_1(x)+A_2(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}+A_3(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$. Therefore,

$$Res_{0,2}(x,t)=\left(A_2(x)+A_3(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)-\left(A_1''+A_2''\frac{t^{\alpha}}{\Gamma(1+\alpha)}+A_3''\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)-6\left(A_1+A_2\frac{t^{\alpha}}{\Gamma(1+\alpha)}+A_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)$$
$$+6\left(A_1+A_2\frac{t^{\alpha}}{\Gamma(1+\alpha)}+A_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)^2 \quad (3.24)$$

The operator D_t^{α} is applied to both sides of Eq.(3.23) as follows:

$$D_t^{\alpha}Res_{0,2}(x,t)=A_3-\left(A_2''+A_3''\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)-6\left(A_2+A_3\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)+6\left(A_2+A_3\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)\left(A_1+A_2\frac{t^{\alpha}}{\Gamma(1+\alpha)}+A_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)$$
$$+6\left(A_1+A_2\frac{t^{\alpha}}{\Gamma(1+\alpha)}+A_3\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)\left(A_2+A_3\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) \quad (3.25)$$

From Eqs. (3.18) and (3.24),

$$A_3(x)=A_2''+6A_2-12A_1A_2. \quad (3.26)$$

The same manner is repeated as above, the following recurrence results are obtained

$$A_4(x)=A_3''+6A_3-12A_1A_3-12A_2^2 \quad (3.27)$$

and so on.
Thus, we have

$$u_0(x,t)=A_1+\left(A_1''+6A_1-6A_1^2\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)}+\left(A_2''+6A_2-12A_1A_2\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+\left(A_3''+6A_3-12A_1A_3-12A_2^2\right)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\ldots \quad (3.28)$$

Define the residual function for Eq.(3.9) as

$$Res_1(x,t)=D_t^{\alpha}u_1-D_{xx}u_1-6u_1+12u_0u_1+6D_t^{2\alpha}u_0-6u_0D_t^{2\alpha}u_0. \quad (3.29)$$

Suppose that $u_1(x,t)$ has the following form

$$u_1(x,t)=B_1(x)+B_2(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}+B_3(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+B_4(x)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\ldots. \quad (3.30)$$

By the initial condition, the 0^{th} truncated solution of $u(x,t)$ can be written as follows:

$$u_1(x,0)=0=B_1(x) \quad (3.31)$$

We apply the RPSM to find out $B_k(x), k=1,2,3,\ldots,m$ in Eq. (9). Thus, we have

$$u_1(x,t)=B_1+\left(B_1''+6B_1-12A_1B_1-6A_3+6A_1A_3\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)}+\left(B_2''+6B_2-12A_2B_1-12A_1B_2-6A_4+6A_2A_3+6A_1A_4\right)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$
$$+\left(B_3''+6B_3-12A_3B_1-12A_1B_3+6A_3^2-24A_2B_2+12A_2A_4\right)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\ldots \quad (3.32)$$

Define the residual function for Eq.(10)

$$Res_2(x,t)=D_t^{\alpha}u_2-D_{xx}u_2-6u_2+6u_1^2+12u_0u_2+6D_t^{2\alpha}u_1-6u_0D_t^{2\alpha}u_1-6u_1D_t^{2\alpha}u_0. \quad (3.33)$$

Suppose that $u_2(x,t)$ has the following form

$$u_2(x,t)=C_1(x)+C_2(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}+C_3(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+C_4(x)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\ldots. \quad (3.34)$$

By the initial condition, the 0^{th} truncated solution of $u_2(x,t)$ can be written as follows:

$$u_2(x,0)=0=C_1(x). \quad (3.35)$$

We apply the RPSM to find out $C_k(x), k = 1,2,3,...,m$ in Eq. (10).
Thus, we have

$$u_2(x,t) = C_1 + \Big(C_1'' + 6C_1 - 6B_1^2 - 12A_1C_1 - 6B_3 + 6A_1B_3 + 6A_3B_1\Big)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \Big(C_2'' + 6C_2 - 12B_1B_2 - 12A_2C_1 - 12A_1C_2 - 6B_4 + 6A_2B_3 + 6B_2A_3 + 6A_4B_1 + 6A_1B_4\Big)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \Big(C_3'' + 6C_3 - 12B_3B_1 - 12B_2^2 - 12A_3C_1 - 12A_1C_3 + 12A_3B_3 + 12A_2B_4 + 12A_4B_2 - 24A_2C_2\Big)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + ... \quad (3.36)$$

Define the residual function for Eq.(11) as

$$Res_3(x,t) = D_t^{\alpha}u_3 - D_{xx}u_3 - 6u_3 + 12u_0u_3 + 12u_1u_2 + 6D_t^{2\alpha}u_2 - 6u_0D_t^{2\alpha}u_2 - 6u_1D_t^{2\alpha}u_1 - 6u_2D_t^{2\alpha}u_0 \quad (3.37)$$

Suppose that $u_3(x,t)$ has the following form

$$u_3(x,t) = D_1(x) + D_2(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + D_3(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + D_4(x)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + ... \quad (3.38)$$

By the initial condition, the 0^{th} truncated solution of $u_3(x,t)$ can be written as follows:

$$u_3(x,0) = 0 = D_1(x) \quad (3.39)$$

We apply the RPSM to find out $D_k(x), k = 1,2,3,...,m$ in Eq. (11). Thus, we have

$$u_3(x,t) = D_1 + \Big(D_1'' + 6D_1 - 12A_1D_1 - 12B_1C_1 - 6C_3 + 6A_1C_3 + 6B_1B_3 + 6C_1A_3\Big)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \Big(D_2'' + 6D_2 - 12A_2D_1 - 12A_1D_2 - 12B_2C_1 - 12B_1C_2 - 6C_4 + 6A_2C_3 + 6A_3C_2 + 6B_2B_3 + 6C_1A_4 + 6A_1C_4 + 6B_1B_4\Big)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \Big(D_3'' + 6D_3 - 12A_3D_1 - 12A_1D_3 - 24A_2D_2 - 12B_3C_1 - 12B_1C_3 - 24B_2C_2 + 12A_3C_3 + 12A_2C_4 + 12A_4C_2 + 6B_3^2 + 12B_2B_4\Big)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + ... \quad (3.40)$$

4. Numerical Results

Consider the following time fractional Fisher equation with small delay

$$D_t^{\alpha}u(x,t) = D_{xx}u(x,t) + 6u(x,t)(1-u(x,t)) + 6\varepsilon(1-u(x,t))D_t^{2\alpha}u(x,t)$$

subject to initial condition

$$u(x,0) = \frac{1}{(1+e^x)^2}$$

Then, the exact solution of Eq.(11) for $\alpha = 1$ is given by

$$u(x,t) = \frac{1}{(1+e^{x-5t})^2}$$

Based on the obtained results, we construct the 4^{th} RPSM approximate solution. In Fig 4.1, we give the RPSM approximate solution $u_0(x,t)$ which is the exact solution of Fisher equation without delay as the fractional order α increases to one. As it can be seen from Figs. 4.2-4.5 that convergence of the approximate solution depends on the order of the fractional derivative and the delay.
In Tables 1-4, we constitute values of numerical solution $u_k(x,t)$ for $k = 0,1,2,3$ and $\alpha = 0.5, 0.7, 0.9, 1$. The values of numerical solutions $u_k(x,t), k = 0$ in Table 1 and 4 are the same as the values of the u_{RPSM} solutions for $\alpha = 1$ and $\alpha = 0.5$ in Table 2 and 3 in [42]. We constitute values of numerical solutions $u(x,t)$, obtained by using RPSM, in Table 5. This table presents that as the delay term ε approaches to zero and fractional derivative α approaches to one, approximate solution approaches to the exact solution of Fisher equation without delay for α equal to one. Therefore, it is concluded that the approximate solution converges to the exact solution.
Figs. 4.6-4.9, we plot the RPSM solution $u(x,t)$ for $\alpha = 0.7, 1$ and $\varepsilon = 0.001$ as it can be seen from this figures that as the amount of α enlarges to one, the approximate solution converges to exact solution. It can also be seen in Figs.4.6-4.9, no matter how small delay we have the effect of delay term becomes more clear as time t enlarges.

5. Conclusion

In this research, the combination of Taylor series expansion and RPSM is utilized to establish the solution of the time-fractional Fisher equation with small delay. By expanding the time fractional Fisher equation in powers of ε, the solution is constructed in the rapidly convergent series form . Illustrative examples verify that the implemented algorithm is very effective and accurate. In the future works, the term including small delay will be added to the other mathematical models of scientific processes to analyze their behaviour by establishing approximate solutions by RPSM or other methods.

Table 1: The values of RPSM solution with $\alpha = 0.5$ and $x = 0.5$ for several values t.

t	u_0	u_1	u_2	u_3
0.1	0.86888	0.46622	281.71110	-9885.89950
0.2	1.43047	14.20107	891.69290	-29658.84397
0.4	2.51465	60.07964	2764.97866	-83876.24503
0.6	3.59378	123.73322	5303.10485	-152424.01898
0.8	4.68138	201.48099	8381.42026	-232397.44369
1	5.78130	291.28413	11927.08012	-322169.27398

Table 2: The values of RPSM solution with $\alpha = 0.7$ and $x = 0.5$ for several values t.

t	u_0	u_1	u_2	u_3
0.1	0.46448	-3.79142	34.76128	-434.27630
0.2	0.80075	-2.83548	162.29522	-5973.66810
0.4	1.58529	11.51386	837.35198	-29626.07359
0.6	2.50398	43.13349	2145.79613	-69770.83693
0.8	3.54332	92.75059	4136.53246	-126505.71658
1	4.69440	160.93458	6841.51018	-200011.32213

Table 3: The values of RPSM solution with $\alpha = 0.9$ and $x = 0.5$ for several values t.

t	u_0	u_1	u_2	u_3
0.1	0.29555	-2.69972	11.22176	856.90753
0.2	0.48919	-4.54462	27.85582	-1.24970
0.4	1.00924	-3.66668	200.11633	-8110.94966
0.6	1.70357	6.66826	705.87023	-27164.15471
0.8	2.57096	30.04098	1695.38097	-59850.80871
1	3.61195	69.68132	3301.69296	-108655.22740

Table 4: The values of RPSM solution with $\alpha = 1$ and $x = 0.5$ for several values t.

t	u_0	u_1	u_2	u_3
0.1	0.25067	-2.07699	9.85944	840.53652
0.2	0.39384	-4.03984	15.88454	804.56154
0.4	0.81646	-5.76976	90.68394	-3414.86454
0.6	1.40528	-1.48314	372.90270	-15694.16140
0.8	2.17322	12.52664	1011.04533	-39069.21223
1	3.12867	39.96619	2153.61633	-76575.90018

Table 5: The values of RPSM solution with $t = 0.5$ and $x = 0.5$ for several values α.

ε	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1$
0.1	6.15328	-0.00133	0.99471	0.98211
0.01	0.89183	0.42961	0.27053	0.23173
0.001	0.86962	0.46072	0.29286	0.24861

Figure 4.1: The figure of u_0 for $\alpha = 0.5, 0.7, 0.9, 1$, $t = 0.1$ and $-5 \leq x \leq 5$.

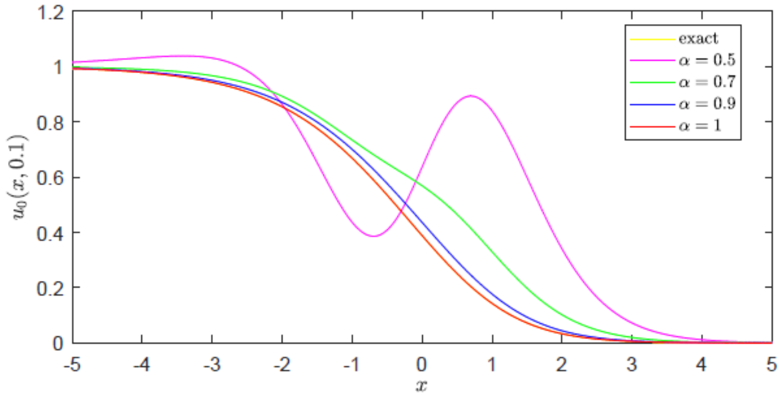


Figure 4.2: The figure of u_1 for $\alpha = 0.5, 0.7, 0.9, 1$, $t = 0.1$ and $-5 \leq x \leq 5$.

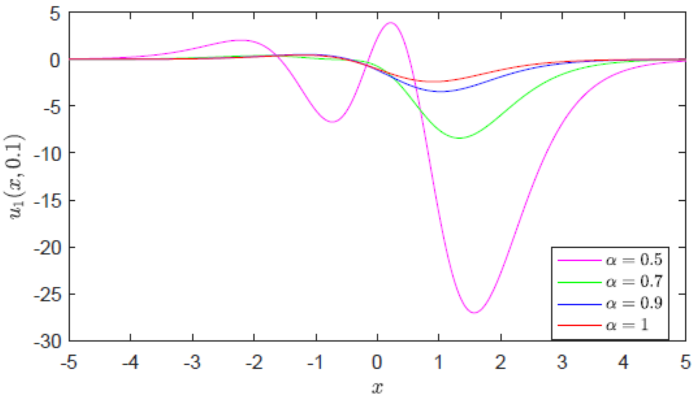


Figure 4.3: The figure of u_2 for $\alpha = 0.5, 0.7, 0.9, 1$, $t = 0.1$ and $-5 \leq x \leq 5$.

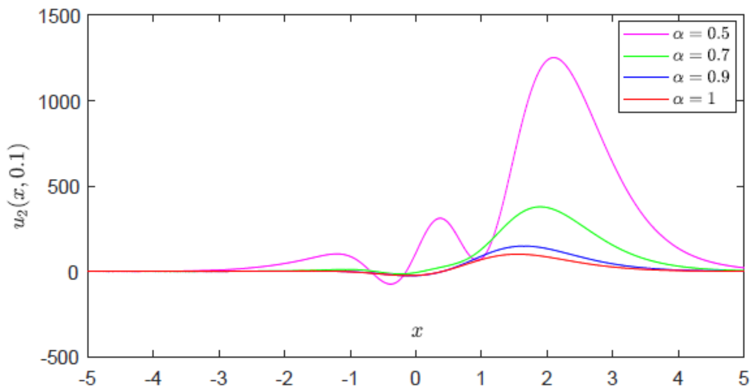


Figure 4.4: The figure of u_3 for $\alpha = 0.5, 0.7, 0.9, 1$, $t = 0.1$ and $-5 \leq x \leq 5$.

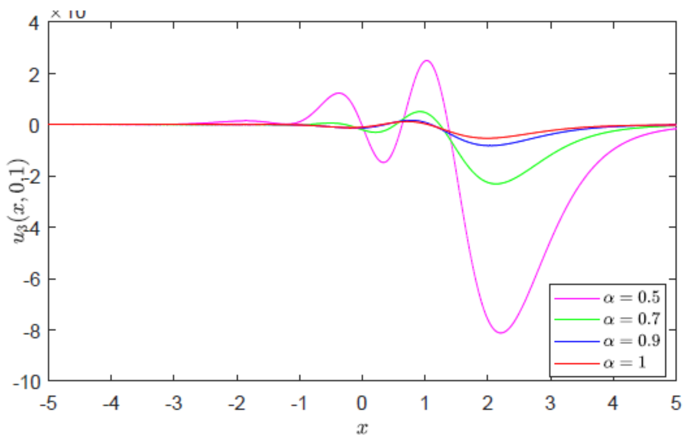


Figure 4.5: An approximate solution for $\alpha = 1$, $t = 0.01$ and for delay $\varepsilon = 0.1, 0.01, 0.001$.

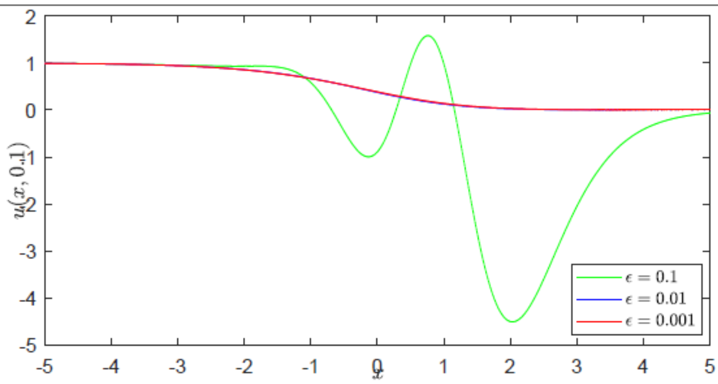


Figure 4.6: An approximate solution for $\alpha = 0.7$,$\varepsilon = 0.001$ and $0 \leq t \leq 0.4$.

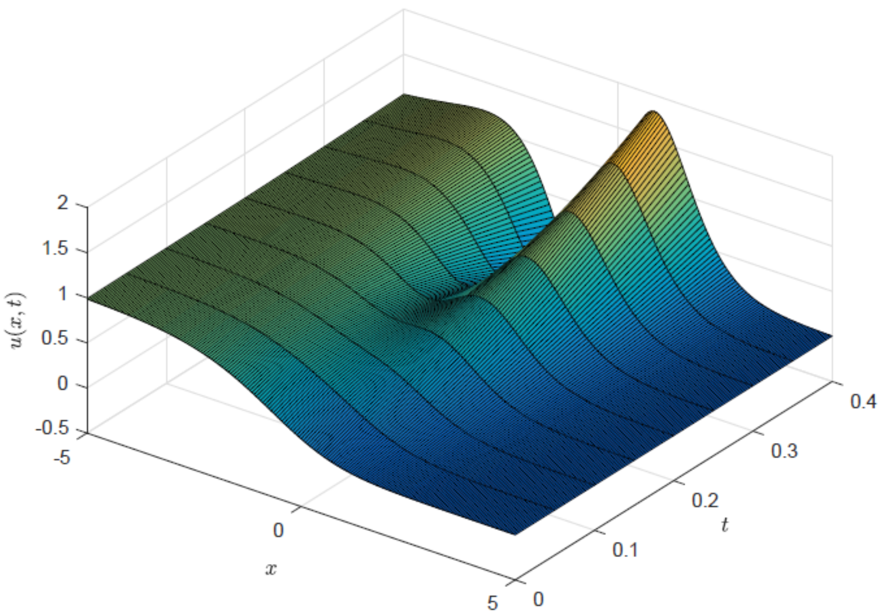


Figure 4.7: An approximate solution for $\alpha = 0.7$,$\varepsilon = 0.001$ and $0 \leq t \leq 1$

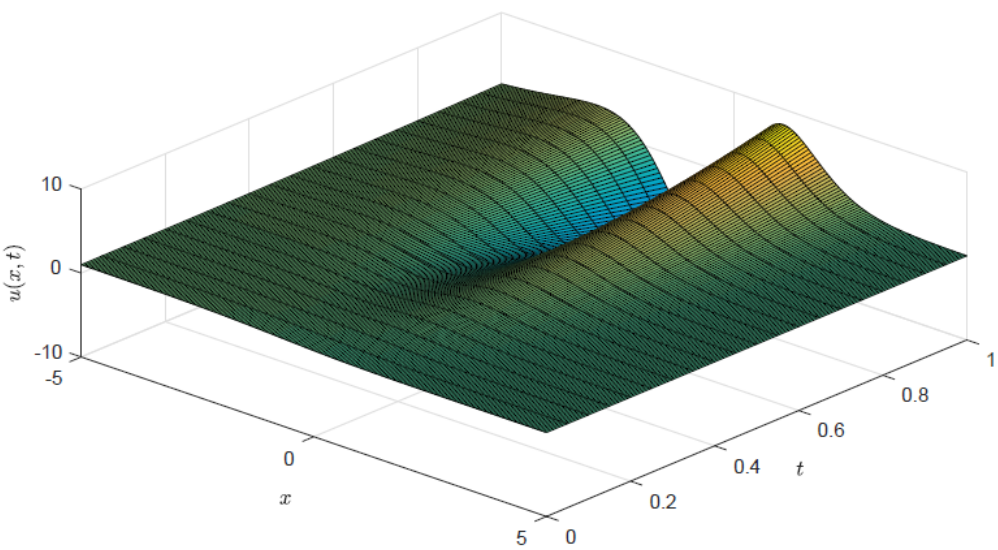


Figure 4.8: An approximate solution for $\alpha = 1, \varepsilon = 0.001$ and $0 \leq t \leq 0.4$.

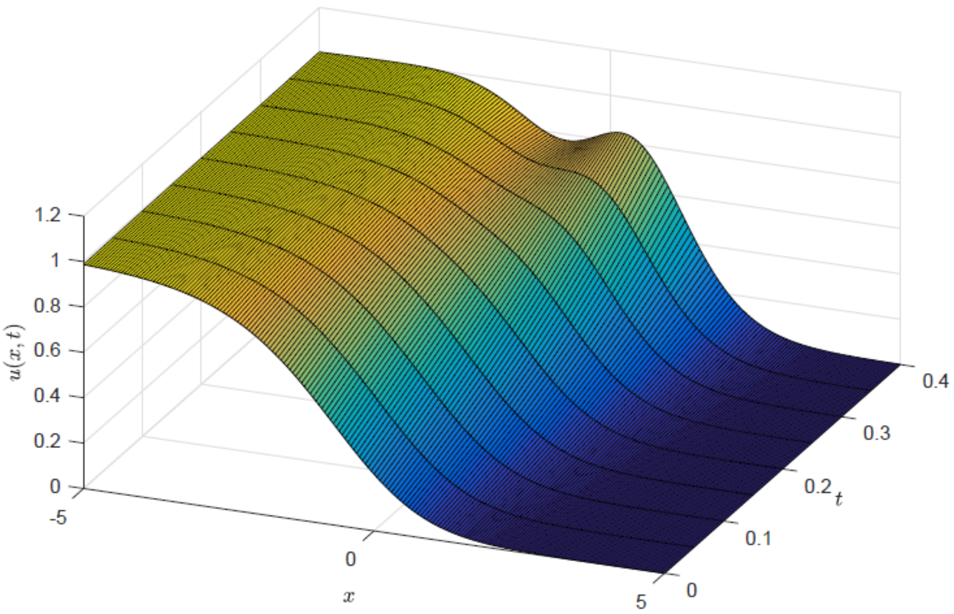
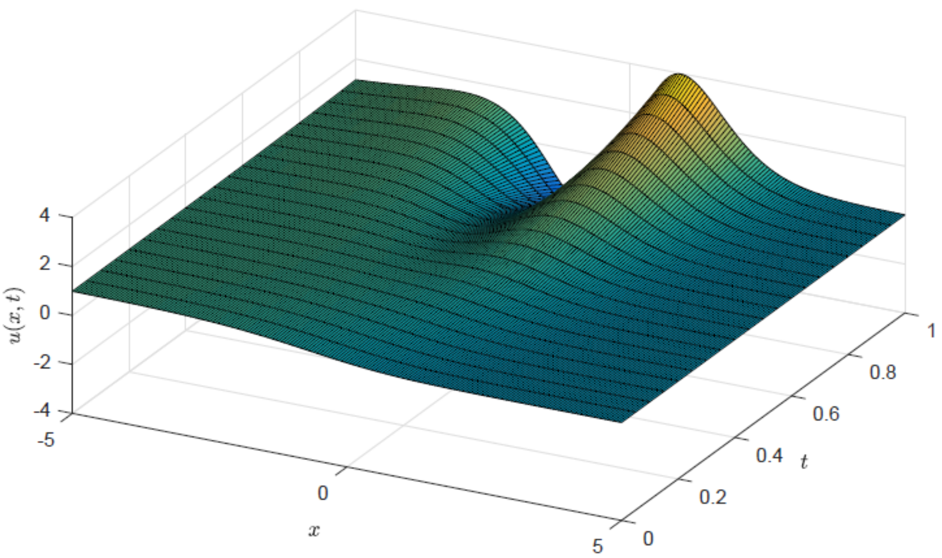


Figure 4.9: An approximate solution for $\alpha = 1, \varepsilon = 0.001$ and $0 \leq t \leq 1$.



References

[1] R. L. Bagley and P. J. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity, J. Rheol. 27 (1983), 201–210.
[2] R. L. Bagley and P. J. Torvik, Fractional calculus a different approach to the analysis of viscoelastically damped structures, AIAA J. 21(5) (1983), 741–748.
[3] R. L. Bagley and P. J. Torvik, Fractional calculus in the transient analysis of viscoelastically damped structures, AIAA J. 23 (1985), 918–925.
[4] R. L. Magin, Fractional calculus in bioengineering, Crit. Rev. Biomed. Eng. 32 (2004), 1–104.
[5] D. A. Robinson,The use of control systems analysis in neurophysiology of eye movements, Ann. Rev. Neurosci. 4 (1981), 462–503.
[6] I. Podlubny, Fractional Differential equations, Academic Press, San Diego, 1999.
[7] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam, 2006.
[8] V. Daftardar-Gejji and A. Babakhani, Analysis of a system of fractional differential equations, J. Math. Anal. Appl. 293(2)(2004), 511–522.
[9] Z. Odibat, Approximations of fractional integrals and Caputo fractional derivatives, Appl. Math. Comput. 178(2)(2006), 527–533.
[10] J. Sabatier, O. P. Agrawal and JAT Machado (eds), Advances in fractional calculus: theoretical developments and applications in physics and engineering. Dordrecht: Springer, 2007.
[11] S. G. Samko , A. A Kilbas and O. I. Marichev, Fractional integrals and derivatives: theory and applications, Amsterdam: Gordon and Breach, 1993.
[12] D. Baleanu ,K. Diethelm and E. Scalas, et al., Fractional: calculus models and numerical methods (complexity, nonlinearity, and chaos). Boston, MA: World Scientific, 2012.
[13] M. Yavuz, T. A. Sulaiman, F. Usta and H. Bulut, Analysis and Numerical Computations of the Fractional Regularized Long Wave Equation with Damping Term, Math. Meth. Appl. Sci. In Press, https://doi.org/10.1002/mma.6343.
[14] T. A. Sulaiman, M. Yavuz, H. Bulut and H. M. Baskonus, Investigation of the fractional coupled viscous Burgers' equation involving Mittag-Leffler kernel, Physica A. 527 (2019),121-126.
[15] F. Usta, Numerical Analysis of Fractional Volterra Integral Equations via Bernstein Approximation Method, J. Comput.d Appl. Math. 384 (2021),113198, https://doi.org/10.1016/j.cam.2020.113198.
[16] F. Usta and M. Z. Sarıkaya, The Analytical Solution of Van der Pol and Lienard Differential Equations within Conformable Fractional Operator by Retarded Integral Inequalities, Demonstratio Mathematica, 52(1)(2019), 204–212.
[17] F. Usta, Numerical solution of fractional elliptic PDE's by the collocation method, Applications Appl. Math.: An Intern. J., 12(1) (2017), 470-478.
[18] O. Abu Arqub , Series solution of fuzzy differential equations under strongly generalized differentiability, J. Adv. Res. Appl. Math. 5 (2013), 31–52.
[19] O. Abu Arqub , A. El-Ajou and A. Bataineh, et al., A representation of the exact solution of generalized Lane Emden equations using a new analytical method, Abstr. Appl. Anal. 2013(2013), 378593.
[20] A. El-Ajou , O. Abu Arqub and Z. Al Zhour et al., New results on fractional power series: theories and applications, Entropy 16 (2013), 5305–5323.
[21] O. Abu Arqub , A. El-Ajou , Z. Al Zhour , et al., Multiple solutions of nonlinear boundary value problems of fractional order: a new analytic iterative technique, Entropy 16 (2014), 471–493.
[22] O. Abu Arqub , A. El-Ajou and S. Momani , Constructing and predicting solitary pattern solutions for nonlinear time-fractional dispersive partial differential equations, J. Comput. Phys. 293 (2015), 385–399.
[23] A. El-Ajou , O. Abu Arqub and S. Momani , Approximate analytical solution of the nonlinear fractional KdV–Burgers equation: a new iterative algorithm, J. Comput. Phys. 293 (2015), 81–95.
[24] M. Alquran , K. Al-Khaled and J. Chattopadhyay , Analytical solutions of fractional population diffusion model: residual power series, Nonlin Stud 22 (2015), 31–39
[25] A. El-Ajou , O. Abu Arqub , S. Momani , et al., A novel expansion iterative method for solving linear partial differential equations of fractional order, Appl. Math. Comput. 257 (2015), 119–133.
[26] R. A.Fisher, The wave of advance of advantageous genes, Annu. Eugen. 7 (1937), 355-369.
[27] I. C. Sungu and H. Demir , New approach and solution technique to solve time fractional nonlinear reaction-diffusion equations, Math. Probl. Eng. 2015 (2015), 457013.
[28] S. Z. Rida , A.M.A El-Sayed and A.A.M. Arafa, On the solutions of time-fractional reaction–diffusion equations, Commun. Nonlinear Sci. 15 (2010), 3847–3854.
[29] N.A. Khan , F. Ayaz , L. Jin , et al., On approximate solutions for the time-fractional reaction-diffusion equation of Fisher type, Int. J. Phys. Sci. 6 (2011), 2483–2496.
[30] V. K. Baranwal , R. K. Pandey , M. P. Tripathi , et al., An analytic algorithm for time fractional nonlinear reaction–diffusion equation based on a new iterative method, Commun. Nonlinear Sci. 17 (2012), 3906–3921.
[31] M. Merdan , Solutions of time-fractional reaction– diffusion equation with modified Riemann-Liouville derivative, Int. J. Phys. Sci. 7 (2012), 2317–2326.
[32] K. B. Oldham and J. Spanier, The Fractional Calculus: Theory and Application of Differentiation and Integration to Arbitrary order, Academic Press, California, 1974.
[33] K. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.
[34] S. S. Ray, Analytical solution for the space fractional diffusion equation by two-step Adomian decomposition method, Commun. Nonlinear Sci. Numer. Simul. 14 (2009), 1130–1295.
[35] R. Magin , X. Feng and D. Baleanu , Solving fractional order Bloch equation, Concept magnetic Res. 34A (2009), 16-23.
[36] H. Poincaré, Sur les intégrales irrégulières, Acta mathematica 8(1)(1886), 295-344.
[37] M. Abramowitz, I. A. Stegun, Handbook of mathematical functions: with formulas, graphs, and mathematical tables (Vol. 55, p. 319). Washington, DC: National bureau of standards, 1886.
[38] W. Eckhaus, Matched asymptotic expansions and singular perturbations, Elsevier, 2011.
[39] F. Say, Late-order terms of second order ODEs in terms of pre-factors, Hacettepe J. Math. Statistics, 50(2) (2021), 342-350.
[40] M. Van Dyke, Higher approximations in boundary-layer theory part 3. parabola in uniform stream, J. Fluid Mech. 19(1) (1964), 145-159.
[41] E. J. Hinch, Perturbation Methods,Cambridge University Press, 1991.
[42] M. M. Al Qurashi , Z. Korpnar , D. Baleanu and M. Inc, A new iterative algorithm on the time-fractional Fisher equation: Residual power series method, Adv. Mech. Eng. 9(9) (2017), 1-8.