Digital Uniform Spaces

Ozgur Ege[1*], Ismet Karaca[2]

[1]Celal Bayar University, Faculty of Science and Letters, Department of Mathematics, Muradiye Campus, 45140, Manisa, Turkey, +90 236 2013228,
ozgur.ege@cbu.edu.tr
[2]Ege University, Faculty of Science, Department of Mathematics, Bornova, 35100, Izmir, Turkey, +90 232 3112335,
ismet.karaca@ege.edu.tr
*Corresponding author / İletişimden sorumlu yazar

Recieved: 18th May (Mayıs) 2016
Accepted: 27th August (Ağustos) 2016
DOI: http://dx.doi.org/10.18466/cbujos.31942

Abstract

In this paper, we introduce the concept of digital uniform space. We give a digital uniform structure for the set of integers $\mathbb{Z}$. We also prove a theorem and give some examples on the digital uniform spaces. Finally, we define the notion of digital uniform base and present two theorems related to new concept.

Keywords — Digital image, digital continuity, digital uniform base, uniform space, uniform continuity.

Dijital Düzgün Uzaylar

Özet

Bu çalışmada, dijital düzgün uzay kavramı tanıtılmıştır. Tamsayılar kümesi $\mathbb{Z}$ için bir dijital düzgün yapı verilmiş, ayrıca dijital düzgün uzaylar üzerine bir teorem ispatlanmış ve birkaç örnek verilmiştir. Son olarak dijital düzgün baz kavramı tanımlanarak bu yeni kavram ile ilgili iki teoreme açıklık getirilmiştir.

Anahtar Kelimeler — Dijital görüntü, dijital süreklilik, dijital düzgün baz, düzgün uzay, düzgün süreklilik.

1 Introduction

Digital topology is an active area of great theoretical interest having the additional bonus of significant applications in imaging science and related areas. The main goal of this area is to determine topological properties of discrete objects. Many researchers such as Rosenfeld [14], Kong [13], Boxer [2-7], Han [9,10], Karaca [12] and others have contributed to this field.

Up to now, many developments have occurred in digital topology. The notions of digital image and digital continuous map have been studied in [2-4]. Some results and characteristic properties on the digital homology groups of 2D digital images are given in [8] and [12].

On the other hand, the first definition of a uniform structure was given by Andre Weil [16] in 1937. Tukey [15] defined uniform structures using coverings in 1940. Bourbaki [1] developed Weil's approach. He used a system of neighbourhoods of the diagonal of $X \times X$. The uniform spaces generalize metric spaces and topological groups. We would like to carry the notion of a digital image on to the uniform spaces and construct a structure in digital images.

This paper is organized as follows. In the first part, we give necessary background on digital topology. In the next section, we define a digital uniform space and give some examples. Finally, we state and prove some results on digital uniform spaces.

2 Preliminaries

A *digital image* is a pair (X,κ), where $X \subseteq \mathbb{Z}^n$ for some positive integer n and κ is an adjacency relation for the members of X. There are various adjacency relations [10,11].

Definition 2.1. [5]. For a positive integer l with $1 \le l \le n$ and two distinct points $p=(p_1,p_2,...,p_n), q=(q_1,q_2,...,q_n) \in \mathbb{Z}^n$, p and q are c_l*-adjacent*, if

(1) there are at most l indices i such that $|p_i - q_i| = 1$, and

(2) for all other indices j such that $|p_j - q_j| \neq 1$, $p_j - q_j$.

The notation c_l represents the number of points $q \in \mathbb{Z}^n$ that are adjacent to a given point $p \in \mathbb{Z}^n$. Thus, in $\mathbb{Z}$, we have $c_1 = 2$-adjacency; in $\mathbb{Z}^2$, we have $c_1 = 4$-adjacency and $c_2 = 8$-adjacency; in $\mathbb{Z}^3$, we have $c_1 = 6$-adjacency, $c_2 = 18$-adjacency, and $c_3 = 26$-adjacency [5]. Given a natural number l in conditions (1) and (2) with $1 \le l \le n$, l determines each of the κ-adjacency relations of $\mathbb{Z}^n$ in terms of (1) and (2) as follows [9]:

$$\kappa \in \left\{ 2n (n \ge 1), 3^n - 1 (n \ge 2), 3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1 \right.$$
$$\left. (2 \le r \le n-1, n \ge 3) \right\} \text{ where } C_t^n = \frac{n!}{(n-t)!.t!}.$$

Definition 2.2. [10]. Given two points $x_i, y_i \in (X_i, \kappa_i)$, $i \in \{0,1\}$, (x_0, x_1) and (y_0, y_1) are adjacent in $X_0 \times X_1$ if and only if one of the following is satisfied:

(i) $x_0 = y_0$ and x_1 and y_1 are κ_1-adjacent; or

(ii) x_0 and y_0 are κ_0-adjacent and $x_1 = y_1$; or

(iii) x_0 and y_0 are κ_0-adjacent and x_1 and y_1 are κ_1-adjacent.

The adjacency of the cartesian product of digital images (X_0, κ_0) and (X_1, κ_1) is denoted by κ_*.

Definition 2.3. [2]. The set $[a,b]_{\mathbb{Z}} = \{ z \in \mathbb{Z} : a \le z \le b \}$ is called a *digital interval* where $a, b \in \mathbb{Z}$ and $a \le b$. Also if $a = b$, then $[a,a]_{\mathbb{Z}} = \{a\}$, $a \in \mathbb{Z}$.

A κ*-neighbor* of $p \in \mathbb{Z}^n$ [3] is a point $\mathbb{Z}^n$ that is κ-adjacent to p. A digital image $X \subset \mathbb{Z}^n$ is κ-*connected* [11] if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, ..., x_r\}$ of points of a digital image X such that $x = x_0$, $y = x_r$ and x_i and x_{i+1} are κ-neighbors where $i = 0,1,...,r-1$.

Definition 2.4. [3]. Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0-adjacency and κ_1-adjacency, respectively. A function $f : X \to Y$ is said to be (κ_0, κ_1)*-continuous* if for every κ_0-connected subset U of X, $f(U)$ is a κ_1-connected subset of Y. We say that such a function is digitally continuous.

Proposition 2.5. [3]. Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0-adjacency and κ_1-adjacency, respectively. The function $f : X \to Y$ is (κ_0, κ_1)-*continuous* if and only if for every κ_0-adjacent points $\{x_0, x_1\}$ of X, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are κ_1-adjacent in Y.

Definition 2.6. [6]. Let $X \subset \mathbb{Z}^{n_0}$ and $Y \subset \mathbb{Z}^{n_1}$ be digital images with κ_0-adjacency and κ_1-adjacency, respectively. A function $f : X \to Y$ is a (κ_0, κ_1)-*isomorphism*, if f is (κ_0, κ_1)-continuous and bijective and further f^{-1} is (κ_1, κ_0)-continuous. It is denoted by $X \cong_{(\kappa_0, \kappa_1)} Y$.

3 Digital Uniform Spaces

In this section, we define the concept of digital uniform space and give its some important properties. Let X be a nonempty digital image with κ-adjacency and A be a subset of $(\mathrm{X}\times\mathrm{X},\kappa_*)$. If $A = A^{-1}$ where $A^{-1} = \{(\mathrm{x},\mathrm{y}) : (\mathrm{y},\mathrm{x}) \in \mathrm{A}\}$, then it is called a *symmetric image*. If $A, B \subset X \times X$, then the composition is defined as follows:

$$A \circ B = \{(\mathrm{x},\mathrm{y}) : \text{there exists } z \in X \text{ such that } (\mathrm{x},\mathrm{z}) \in \mathrm{B}, (\mathrm{z},\mathrm{y}) \in \mathrm{A}\}.$$

Definition 3.1. Let Γ be a nonempty collection of subsets $U \subseteq (\mathrm{X}\times\mathrm{X},\kappa_*)$ and let the following hold:

a) If $U \in \Gamma$, then $\Delta \subset U$ where $\Delta = \{(\mathrm{x},\mathrm{x}) : \mathrm{x} \in \mathrm{X}\}$ is the diagonal in $(\mathrm{X}\times\mathrm{X},\kappa_*)$.

b) If $U_1, U_2 \in \Gamma$, then $U_1 \cap U_2 \in \Gamma$.

c) If $U \in \Gamma$, then there exists a digital image $V \in \Gamma$ such that $V \circ V \subset U$.

d) If $U \in \Gamma$, then there exists a digital image $V \in \Gamma$ such that $V^{-1} \in U$.

e) If $U \in \Gamma$ and $U \subset V$, then $V \in \Gamma$.

Then Γ is called a digital uniform structure and the pair $(\mathrm{X},\Gamma,\kappa)$ is called a digital uniform space.

Definition 3.2. Let $(\mathrm{X},\Gamma,\kappa)$ be a digital uniform space. If $\cap\{U : U \in \Gamma\} = \Delta$, then $(\mathrm{X},\Gamma,\kappa)$ is called a Hausdorff digital uniform space.

Let us give some examples related to digital uniform spaces.

Example 3.3. Let $X = [\mathrm{a}]$ be a single-point. Since $X \times X = \{(\mathrm{a},\mathrm{a})\}$, the set $\varphi = X \times X$ has a digital uniform structure and (X,φ) is a digital uniform space. It is called a trivial digital uniform space.

Example 3.4. Let (X,κ) be any digital image and P denote the power set which is the set of all subsets of (X,κ), including the empty set and X itself.

(1) A digital image $(\mathrm{X},\varphi_1,\kappa)$ where $\varphi_1 = \{V \subset X^2 : \Delta \subset V\} \subset \mathrm{P}(\mathrm{X}^2)$ is called discrete digital uniform space.

(2) A digital image $(\mathrm{X},\varphi_2,\kappa)$ where $\varphi_2 = \{X^2\}$ is called indiscrete digital uniform space.

Example 3.5. Let $X = [0,1]$ be the digital unit interval. Then we have $\Delta = \{(0,0),(1,1)\}$ and $X \times X = \{(0,0),(0,1),(1,0),(1,1)\}$ (see Figure 1).

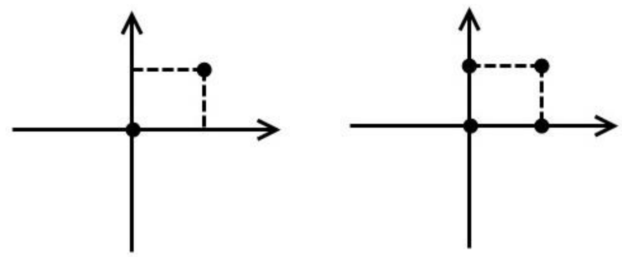

Figure 1. Δ and $X \times X$

Let $A = \{(0,0),(0,1),(1,0)\}$ and $B = \{(0,1),(1,0)\}$ be digital images as in the Figure 2. So we get the following digital image:

$$A \circ B = \{(0,0),(1,0),(1,1)\}.$$

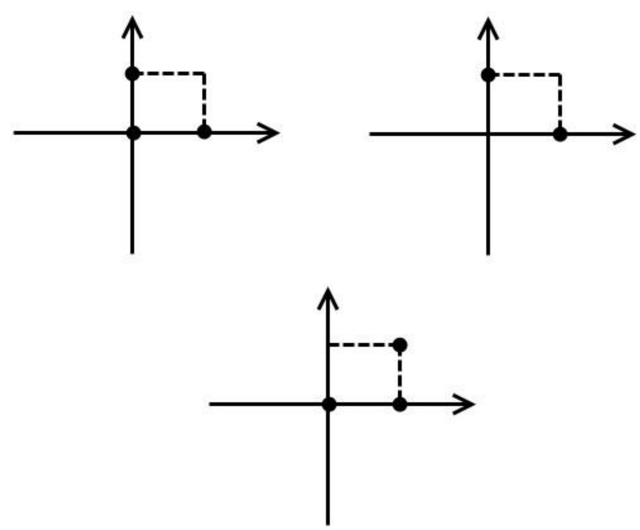

Figure 2. A, B and $A \circ B$

Since $A = A^{-1}$ and $B = B^{-1}$, we can say that A and

B are symmetric digital images. The set

$$\Gamma=\{\{(0,0),(1,1),(1,0)\},\{(0,0),(1,1),(0,1)\},$$
$$\{(0,0),(1,1)\},\{(0,0),(0,1),(1,0),(1,1)\}\}\subset X\times X$$

has a digital uniform structure. As a result, (X,Γ) is a digital uniform space. Since

$$\bigcap\{U:U\in\Gamma\}=\Delta$$

for all $U\in\Gamma$, it is also Hausdorff digital uniform space.

Theorem 3.6. The collection $\varphi=\{B\subset\mathbb{Z}^2 : D_\varepsilon\subset B$ for $\exists\varepsilon>0\}$ is a digital uniform structure on $\mathbb{Z}$ where $D_\varepsilon=\{(x,y):d(x,y)=|x-y|<\varepsilon\}\subset\mathbb{Z}^2$ for every $\varepsilon>0$.

Proof. We shall show that φ holds all conditions of digital uniform structure.

(a) If $U\in\varphi$, then we have $D_\varepsilon\subset U$ for at least $\varepsilon>0$. Since $|x-x|=0<\varepsilon$ where $(x,x)\in\Delta$, we conclude that

$$(x,x)\in D_\varepsilon\subset U \Rightarrow (x,x)\in U$$
$$\Rightarrow \Delta\subset U.$$

(b) If $U_1,U_2\in\varphi$, then $D_{\varepsilon_1}\subset U_1$ and $D_{\varepsilon_2}\subset U_2$ for at least $\varepsilon_1,\varepsilon_2>0$. Taking $\varepsilon=\min\{\varepsilon_1,\varepsilon_2\}$, we obtain $D_\varepsilon\subset D_{\varepsilon_1}\subset U_1$ and $D_\varepsilon\subset D_{\varepsilon_2}\subset U_2$. We take the intersection of the last two statements and the required result is obtained as follows:

$$D_\varepsilon\subset D_{\varepsilon_1}\cap D_{\varepsilon_2}\subset U_1\cap U_2 \Rightarrow D_\varepsilon\subset U_1\cap U_2$$
$$\Rightarrow U_1\cap U_2\in\varphi.$$

(c) If $U\in\varphi$, we know that there exists at least $\varepsilon>0$ such that $D_\varepsilon\subset U$. Let us take $E=D_{\frac{\varepsilon}{\alpha}}$ where $\alpha>2$ and define

$$E\circ E=\{(x,y):(x,z)\in E \text{ and } (z,y)\in E$$
$$\text{for a point } z \text{ of } E\}.$$

We shall show that $E\circ E\subset U$. If $(x,y)\in E\circ E$, then we have

$$|x-z|<\frac{\varepsilon}{\alpha} \text{ and } |z-y|<\frac{\varepsilon}{\alpha}.$$

Since

$$|x-y|<|x-z|+|z-y|<\frac{\varepsilon}{\alpha}+\frac{\varepsilon}{\alpha}=\frac{2\varepsilon}{\alpha}<\varepsilon,$$

we obtain $(x,y)\in D_\varepsilon\subset U$. So we get $E\circ E\subset U$.

(d) If $U\in\varphi$, then $D_\varepsilon\subset U$ for at least $\varepsilon>0$. Let $E=D_\varepsilon$. Since

$$E=D_\varepsilon=\{(x,y):|x-y|<\varepsilon\}$$
$$=\{(y,x):|y-x|=|x-y|<\varepsilon\}$$
$$=D_\varepsilon^{-1}$$
$$=E^{-1},$$

we conclude that there exists $E\in\varphi$ such that

$$E=D_\in=D_\varepsilon^{-1}=E^{-1}\subset U.$$

(e) Let $U\in\varphi$. By the definition of φ, we have $D_\varepsilon\subset U$ for $\varepsilon>0$. If $U\subset V$, then $D_\varepsilon\subset U\subset V$. Thus we have $V\in\varphi$.

Definition 3.7. Let (X,φ,κ) be a digital uniform space and β be a subset of the power set $P(X^2)$. If

$$\varphi=\{U\subset X^2 : B\subset U \text{ for at least one } B\in\beta\},$$

then the family β is called a digital uniform base for φ.

Theorem 3.8. Let (X,φ,κ) be a digital uniform space and $\beta\subset P(X^2)$ be a nonempty family. β is a digital uniform base for φ if and only if

(i) If $U\in\beta$, then $\Delta\subset U$,

(ii) If $U\in\beta$, then U^{-1} contains a member of β.

(iii) If $U\in\beta$, then $V\circ V\subset U$ for some V in β.

(iv) If $U,V\in\beta$, then there exists at least one member $W\in\beta$ such that $W\in U\cap V$.

Proof. ($\Rightarrow$) Let β be a digital uniform base.

(i) Since $U \in \beta \subset \varphi$ and φ is the digital uniform structure, we have $\Delta \subset U.$

(ii) If $U \in \beta \subset \varphi,$ then we get $U \subset \varphi.$ By the Definition 3.1 and 3.7, we have $U^{-1} \in \varphi$ and there exists a member $V \in \beta$ such that $V \subset U^{-1},$ respectively.

(iii) Since $U \in \beta \subset \varphi$ and φ is a digital uniformity, there exists an element $W \in \varphi$ such that $W \circ W \subset U.$ From the Definition 3.7, we have $V \in \beta$ such that $V \subset W.$ So we get the following:

$$V \circ V \subset W \circ W \subset U \;\Rightarrow\; V \circ V \subset U.$$

(iv) Let $U,V \in \beta \subset \varphi.$ From the digital uniformity of $\varphi,$ we have $U \cap V \in \varphi.$ Definition 3.7 shows that there exists an element $W \in \beta$ such that $W \subset U \cap V.$

$(\Leftarrow)$ Let four conditions be hold for $\beta.$ We shall show that the following structure

$$\varphi = \left\{U \subset X^2 : B \subset U \text{ for at least one } B \in \beta\right\}$$

is a digital uniformity for $(X,\kappa).$

(a) If $U \in \varphi,$ there exists $B \in \beta$ such that $B \subset U.$ Using (i), we have $\Delta \subset B \subset U.$

(b) Let $U \in \varphi.$ Then there exists $B \in \beta$ such that $B \subset U.$ From (ii), we obtain that there is an element $V \in \beta$ such that $V \subset B^{-1} \subset U^{-1}.$ Thus we have $U^{-1} \in \varphi.$

(c) If $U,V \in \varphi,$ then there are two elements $B \subset U$ and $B^* \subset V$ such that $B, B^* \in \beta.$ We can say that there exists at least one element $A \in \beta$ such that $A \subset B \cap B^*.$ Since

$$A \subset B \cap B^* \subset U \cap V,$$

we conclude that $U \cap V \in \varphi.$

(d) Assume that $U \in \varphi.$ This shows that there is an element $B \in \beta$ such that $B \subset U.$ We have an element $W \in \beta$ such that $W \circ W \subset B$ by (iii). So we have $W \circ W \subset B \subset U.$

(e) Let $U \in \varphi$ and $U \subset V \subset X^2.$ From the Definition 3.7, there exists $B \in \beta$ such that $B \subset U.$ As a result, we have the following:

$$B \subset U \subset V \;\Rightarrow\; V \in \varphi.$$

Theorem 3.9. Let $(\mathrm{Y},\phi,\kappa_2)$ be a digital uniform space, (X,κ_1) be a digital image and $f : X \to Y$ be a digital function. The family

$$\beta = \left\{(\mathrm{f}\times\mathrm{f})^{-1}(\mathrm{V}) : \mathrm{V} \in \phi\right\}$$

is a digital uniform base for $X.$

Proof. We shall use the Theorem 3.8.

(i) Let $U \in \beta$ Then there exists an element $V \in \phi$ such that $U = (\mathrm{f}\times\mathrm{f})^{-1}(\mathrm{V}).$ Since ϕ is the digital uniformity, we have $\Delta \subset V$ and

$$(\mathrm{f}\times\mathrm{f})^{-1}(\Delta) = \Delta_X \subset (\mathrm{f}\times\mathrm{f})^{-1}(\mathrm{V}).$$

(ii) Let $U = (\mathrm{f}\times\mathrm{f})^{-1}(\mathrm{V}) \in \beta.$ From the digital uniformity of $\phi,$ we get $V^{-1} \in \phi.$ Take $W = (\mathrm{f}\times\mathrm{f})^{-1}(\mathrm{V}^{-1}) \in \beta.$ As a result, we have

$$\begin{aligned}(\mathrm{x},\mathrm{y}) \in W &\Rightarrow (\mathrm{f(x)},\mathrm{f(y)}) \in V^{-1}\\ &\Rightarrow (\mathrm{f(y)},\mathrm{f(x)}) \in V\\ &\Rightarrow (\mathrm{y},\mathrm{x}) \in (\mathrm{f}\times\mathrm{f})^{-1}(\mathrm{V}) = U\\ &\Rightarrow (\mathrm{x},\mathrm{y}) \in U^{-1}\\ &\Rightarrow W \in U^{-1}.\end{aligned}$$

(iii) If we take $U = (\mathrm{f}\times\mathrm{f})^{-1}(\mathrm{V}) \in \beta,$ we have an element $T \in \phi$ such that $T \circ T \subset V$ by the digital uniformity of $\phi.$

$$(\mathrm{x},\mathrm{y}) \in W \circ W \;\Rightarrow\; (\mathrm{x},\mathrm{z}) \in \mathrm{W} \text{ and } (\mathrm{z},\mathrm{y}) \in \mathrm{W}$$

for at least one element $z \in X.$ Since

$$\begin{aligned}&(\mathrm{f(x)},\mathrm{f(z)}) \in \mathrm{T} \text{ and } (\mathrm{f(z)},\mathrm{f(y)}) \in \mathrm{T}\\ &\Rightarrow (\mathrm{f(x)},\mathrm{f(y)}) \in \mathrm{T}\circ\mathrm{T} \subset \mathrm{V}\\ &\Rightarrow (\mathrm{x},\mathrm{y}) \in (\mathrm{f}\times\mathrm{f})^{-1}(\mathrm{V}),\end{aligned}$$

we have $W \circ W \subset U.$

(iv) Let $U_1, U_2 \in \beta.$ Thus there are $V_1, V_2 \in \phi$ such that

$$U_1 = (\mathrm{f}\times\mathrm{f})^{-1}(\mathrm{V}_1),\; U_2 = (\mathrm{f}\times\mathrm{f})^{-1}(\mathrm{V}_2).$$

By the digital uniformity of ϕ, we find $V_1 \cap V_2 \in \phi$. Taking $U_3 = (\mathrm{f} \times \mathrm{f})^{-1}(\mathrm{V}_1 \cap \mathrm{V}_2)$, it is clear that $U_3 \subset U_1 \cap U_2$. Let $(\mathrm{x}, \mathrm{y}) \in U_3$. Then we have

$$(\mathrm{f}(\mathrm{x}), \mathrm{f}(\mathrm{y})) \in \mathrm{V}_1 \cap \mathrm{V}_2.$$

From the last statement, we obtain the following:

$$(\mathrm{x}, \mathrm{y}) \in (\mathrm{f} \times \mathrm{f})^{-1}(\mathrm{V}_1) = U_1,$$

$$(\mathrm{x}, \mathrm{y}) \in (\mathrm{f} \times \mathrm{f})^{-1}(\mathrm{V}_2) = U_2.$$

As a result, β is the digital uniform base for (X, κ_1).

4 Conclusion

Digital topology has been a major area of mathematics with various applications. Associating with digital topology and the notion of uniform space in general topology, we get interesting results on digital images. We believe that all results in this work will be useful to develop digital images.

5 Acknowledgements

The authors would like to thank the anonymous referees for their helpful suggestions and corrections.

6 References

[1] Bourbaki, N. Elements of Mathematics. General Topology. Springer, 1989.
[2] Boxer, L. Digital Continuous Functions. Pattern Recognit. Lett. 1994; 15, 833-839.
[3] Boxer, L. A Classical Construction for the Digital Fundamental Group. J. Math. Imaging Vision. 1999; 10, 51-62.
[4] Boxer, L. Properties of Digital Homotopy. J. Math. Imaging Vision. 2005; 22, 19-26.
[5] Boxer, L. Homotopy Properties of Sphere-like Digital Images. J. Math. Imaging Vision. 2006; 24, 167-175.
[6] Boxer, L. Digital Products, Wedges and Covering Spaces. J. Math. Imaging Vision. 2006; 25, 159-171.
[7] Boxer, L. Continuous Maps on Digital Simple Closed Curves. Appl. Math. 2010; 1, 377-386.
[8] Ege, O.; Karaca, I. Fundamental Properties of Simplicial Homology Groups for Digital Images. Amer. J. Comput. Technol. Appl. 2013; 1(2), 25-42.
[9] Han, S.E. An Extended Digital $(\mathrm{k}_0, \mathrm{k}_1)$-Continuity. J. Appl. Math. Comput. 2004; 16(1-2), 445-452.
[10] Han, S.E. Non-product Property of the Digital Fundamental Group. Inform. Sci. 2005; 171, 73-91.
[11] Herman, G.T. Oriented Surfaces in Digital Spaces. CVGIP: Graphical Models and Image Processing, 1993; 55, 381-396.
[12] Karaca, I.; Ege, O. Some Results on Simplicial Homology Groups of 2D Digital Images. Int. J. Inf. Comput. Sci. 2012; 1(8), 198-203.
[13] Kong, T.Y. A Digital Fundamental Group. Comput. Graph. 1989; 13, 159-166.
[14] Rosenfeld, A. Continuous Functions on Digital Pictures. Pattern Recognit. Lett. 1986; 4, 177-184.
[15] Tukey, J.W. Convergence and Uniformity in Topology. Princeton University Press, 1940.
[16] Weil, A. Sur les Espaces a Structure Uniforme et sur la Topologie Generale. Hermann, 1938.