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Research Article

## SOLUTION FOR STEKLOV BOUNDARY VALUE PROBLEM INVOLVING THE p(x)-LAPLACIAN OPERATORS

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***Abstract:*** *In this paper, we are concerned with Steklov boundary value problem involving* $p(x)$*-Laplacian operator. By means of the Mountain Pass theorem together with Ambrosetti- Rabinowitz condition, we prove the existence of a nontrivial weak solution in Sobolev spaces with variable exponent under appropriate conditions* $f(x,u)$.

***Keywords****: Variable exponent Lebesgue-Sobolev spaces, variational methods, Ambrosetti- Rabinowitz condition.*

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## 1. Introduction

In this article, we study the following Steklov boundary value problem involving $p(x)$-Laplacian operator

$$\begin{cases} div\left(a\left(x,\nabla u\right)\right)=0, \ x\in\Omega, \\ a\left(x,\nabla u\right)\dfrac{\partial u}{\partial \upsilon}=\lambda\ f\left(x,u\right), \ x\in\partial\Omega, \end{cases} \quad (P)$$

where $\Omega\subset R^N\ (N\geq 2)$ is a bounded with smooth boundary, $p\in C(\overline{\Omega})$, $f:\partial\Omega\times\mathrm{R}\to R$ is a Carathéodory function, $\lambda$ is a positive parameter, $\frac{\partial u}{\partial \upsilon}$ is the outer unit normal derivative on $\partial\Omega$, $a\left(x,\varepsilon\right):\overline{\Omega}\times\mathrm{R}^N\to R^N$ is the continuous derivative with respect to $\varepsilon$ the mapping $A\left(x,\varepsilon\right):\overline{\Omega}\times\mathrm{R}^N\to R^N$, and $div\left(a\left(x,\nabla u\right)\right)$ is $p(x)-$Laplacian type operator.

The operator $div(a(x,\nabla u))$, which appears in (**P**), is a more general operator than the $p(x)$-Laplacian operator

$$\Delta_{p(x)}u:=div\left(\left|\nabla u\right|^{p(x)-2}\nabla u\right).$$

In this study, we assume that $A$, $a$ and $f$ satisfy the following conditions:

**(A1)** The following inequality holds:

$$\left|a\left(x,\varepsilon\right)\right|\leq c\left(1+\left|\varepsilon\right|^{p(x)-1}\right), \text{ for all } x\in\overline{\Omega} \text{ and all } \varepsilon\in R^N.$$

**(A2)** $A\left(x,0\right)=0$, for all $x\in\overline{\Omega}$.

**(A3)** The monotonicity condition holds:
$\left( a\left(x,\varepsilon\right)-a\left(x,\eta\right)\right)\left(\varepsilon-\eta\right)\geq 0,$
for all $x\in\overline{\Omega}$ and all $\varepsilon,\eta\in R^N$ with equality if and only if $\varepsilon=\eta$.
**(A4)** The following inequality holds:

$$|\varepsilon|^{p(x)}\leq a\left(x,\varepsilon\right)\varepsilon\leq p\left(x\right)A\left(x,\varepsilon\right),$$

for all $x\in\overline{\Omega}$ and all $\varepsilon\in R^N$.
**(A5)** $A$ is $p(x)-$uniformly convex: There exists a constant $m>0$ such that

$$A\left(x,\frac{\xi+\zeta}{2}\right)\leq\frac{1}{2}A\left(x,\xi\right)+\frac{1}{2}A\left(x,\zeta\right)-m\left\|\xi-\zeta\right\|^{p(x)},$$

for all $x\in\overline{\Omega}$ and all $\xi,\zeta\in R^N$.
**(f1)** $f:\partial\Omega\times R\rightarrow R$ is a Carathéodory condition such that

$$f\left(x,t\right)\leq c+c_1|t|^{q(x)-1}$$

where $c$ and $c_1$ are positive constants and $q(x)\in C(\partial\Omega)$ such that $p^+<q^-\leq q(x)<p^{\partial}(x)$.
**(f2)** $f\left(x,t\right)=o\left(|t|^{p^+-1}\right)$ as $t\rightarrow 0$, for all $x\in\partial\Omega$ and $p^+<q^-$.
**(AR)** Ambrosetti-Rabinowitz's condition: there exist $t^*>0$ and $p^+<\theta$ such that

$$0<\theta F\left(x,t\right)\leq f\left(x,t\right)t,\ \ |t|\geq|t|^*,\ \text{for all } x\in\partial\Omega.$$

Moreover, throughout this paper, we define

$$p^*(x)=\begin{cases}\dfrac{Np(x)}{N-p(x)}, & if\ \ N>p(x)\\ \infty, & if\ \ N\leq p(x)\end{cases}\quad\text{and}\quad p^{\partial}(x)=\begin{cases}\dfrac{(N-1)p(x)}{N-p(x)}, & if\ \ N>p(x)\\ \infty, & if\ \ N\leq p(x).\end{cases}$$

In recent years, the study of variational problems in the variable exponent Lebesgue-Sobolev spaces is an interesting topic of research due to its significant role in many fields of mathematics. These types of problems have been interesting topics like electrorheological fluids, elastic mechanics, stationary thermo-rheological viscous flows of non-Newtonian fluids, and image processing [2,5,12,13,18].

Recently, many authors have intensively studied the nonlinear boundary value problems involving $p(x)$-Laplacian operator [1,4,7,8,14,15,17]. For example, in [3], the author studied the existence and multiplicity of solutions by using a variation of the Mountain Pass for the following Steklov problem standard growth condition,

$$\begin{cases}\Delta_{p(x)}u=|u|^{p(x)-2}u,\ \ x\in\Omega,\\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial\upsilon}=f\left(x,u\right),\ \ x\in\partial\Omega\end{cases}$$

where $f$ is not satisfy the Ambrosetti-Rabinowitz type condition.

In [9], the authors constrained the existence and multiplicity of solutions by using the Mountain Pass theorem and Ricceri's three critical points theorem under the appropriate conditions for the following Steklov problem standard growth condition,

$$\begin{cases}div(a(x,\nabla u))=0,\ \ x\in\Omega,\\ a(x,\nabla u)v=f\left(x,u\right),\ \ x\in\partial\Omega,\end{cases}$$

Motivated by the above paper, we get some existing results of weak solutions to the problem (**P**). This paper is organized as follows. In Section 2, we recall the definition of the variable exponent Lebesgue - Sobolev spaces. In Section 3, we give the main results.

## 2. Preliminaries

We state some definitions and basic properties of variable exponent Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ [8,10,13,16].

Set

$$C_+(\overline{\Omega}) = \{p: p(x) \in C(\overline{\Omega}), inf p(x) > 1, for\ all\ x \in \overline{\Omega}\}$$

For any $p(x) \in C_+(\overline{\Omega})$, we write

$$1 < p^- := \inf_{x \in \overline{\Omega}} p(x) \text{ and } p^+ := \sup_{x \in \overline{\Omega}} p(x) < \infty$$

Define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \{u \mid u : \Omega \to R \text{ is measurable such that } \int_\Omega |u(\mathrm{x})|^{p(x)} dx < \infty \},$$

with the norm

$$|u|_{p(x)} := \inf\left\{\beta > 0 : \int_\Omega \left|\frac{u(\mathrm{x})}{\beta}\right|^{p(x)} dx \le 1 \right\},$$

and $\left(L^{p(x)}(\partial\Omega), |u|_{p(x)}\right)$ becomes a Banach space.

Similarly, we can define for $p(x) \in C_+(\partial\Omega)$,

$$L^{p(x)}(\partial\Omega) = \{u \mid u : \partial\Omega \to R \text{ is measurable such that } \int_{\partial\Omega} |u(\mathrm{x})|^{p(x)} d\sigma < \infty \}$$

with the norm

$$|u|_{L^{p(x)}(\partial\Omega)} := \inf\left\{\varsigma > 0 : \int_\Omega \left|\frac{u(\mathrm{x})}{\varsigma}\right|^{p(x)} d\sigma \le 1 \right\},$$

where $d\sigma$ is the measure on the boundary. $\left(L^{p(x)}(\partial\Omega), |u|_{p(x)}\right)$ becomes a Banach space.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is denied by

$$W^{1,p(x)}(\Omega) = \{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

and equipped with the norm,

$$|u|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \ \forall u \in W^{1,p(x)}(\Omega)$$

The space $W_0^{1,p(x)}(\Omega)$ is denoted as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $|u|_{1,p(x)}$. For $u \in W_0^{1,p(x)}(\Omega)$, we can define an equivalent norm $\|u\| = |\nabla u|$.

**Proposition 2.1**.[ 3,6,12 ] $L^{p'(x)}(\Omega)$ is the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Then, we write Hölder-Type inequality

$$\left| \int_\Omega uv dx \right| \le \left(\frac{1}{p^-} + \frac{1}{p^+}\right) |u|_{p(x)} |v|_{p'(x)} \quad ,$$

for any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$.

The modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)}(u): L^{p(x)}(\Omega) \to R$ defined by

$$\rho_{p(x)}(u) = \int_\Omega u(x)\,dx,\ \forall\, u \in L^{p(x)}(\Omega).$$

**Proposition 2.2.** [6,10,16] If $u, u_n \in L^{p(x)}(\Omega)$ $(n = 1,2,\dots)$ and $p^+ < \infty$, we denote

(i) $|u|_{p(x)} = 1\ (<1, >1) \Leftrightarrow \rho_{p(x)}(u) = 1\ (<1,\ >1)$,

(ii) $\min\left(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}\right) \le \rho_{p(x)}(u) \le \max\left(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}\right)$,

(iii) $|u_n|_{p(x)} \to 0\ (\to \infty) \Leftrightarrow\ \rho_{p(x)}(u_n) \to 0\ (\to \infty)$,

(iv) $|u_n - u|_{p(x)} \to 0\ (\to \infty) \Leftrightarrow\ \rho_{p(x)}(u_n - u) \to 0\ (\to \infty)$.

**Proposition 2.3.** [14,15,17] We define $\varphi_{p(x)}(u) = \int_{\partial\Omega} |u|^{p(x)} d\sigma,\ \forall\, u \in L^{p(x)}(\partial\Omega)$. Then

*(i)* $|u|_{L^{p(x)}(\partial\Omega)} \ge 1 \Rightarrow |u|^{p^-}{}_{L^{p(x)}(\partial\Omega)} \le \varphi_{p(x)}(u) \le |u|^{p^+}{}_{L^{p(x)}(\partial\Omega)}$

*(ii)* $|u|_{L^{p(x)}(\partial\Omega)} < 1 \Rightarrow\ |u|^{p^+}{}_{L^{p(x)}(\partial\Omega)} \le \varphi_{p(x)}(u) \le |u|^{p^-}{}_{L^{p(x)}(\partial\Omega)}$.

**Proposition 2.4.** [10,15]

(i) If $1 < p^- \le p^+ < \infty$ then the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces,

(ii) If $q(x) \in C_+(\overline{\Omega})$ and $1 \le q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding $W^{1,p(x)}(\Omega) \to L^{q(x)}(\Omega)$ is compact and continuous,

(iii) If $q(x) \in C_+(\partial\Omega)$ and $1 \le q(x) < p^\partial(x)$ for all $x \in \partial\Omega$, then the trace embedding $W^{1,p(x)}(\Omega) \to L^{q(x)}(\partial\Omega)$ is compact and continuous,

(iii) Poincaré inequality, i.e. there exists a positive constant $C > 0$ such that

$$\|u\| \le C|\nabla u|_{p(x)},\ \text{for all}\ u \in W_0^{1,p(x)}(\Omega).$$

**Definition 2.5.** [16] Let $X$ be a Banach space and the function $I \in C^1(X, R)$. We say that $I$ satisfies the Palais-Smale condition **(PS)** in $X$ if any sequence $\{u_n\}$ in $X$ such that $I(u_n)$ is bounded and $I(u_n) \to 0$ as $n \to \infty$ has a convergent subsequence. $I \in C^1(X, R)$

**Lemma 2.6. (Mountain Pass Theorem)** [16] Let $X$ be a Banach space and the function $I \in C^1(X, R)$ satisfies the Palais-Smale condition. Assume that $I(0) = 0$, and the following conditions hold.

(i) There exists two positive real numbers $\tau$ and $r$ such that $I(u) \geq \tau$ with $\|u\| = r$,

(ii) There exists $u_1 \in X$ such that $\|u_1\| > r$ and $I(u_1) < 0$.

Put $G = \{ g \in C([0,1], X) : g(0) = 0 \text{ and } g(1) = u_1 \}$. Set $\beta = \inf \{ \max I(g([0,1])) : g \in G \}$. Then $\beta \geq \tau$ and $\beta$ is a critical value of $I$.

## 3. Main Results

Let $X$ denote the variable exponent Sobolev space $W_0^{1,p(x)}(\Omega)$. The main results of the present paper is:

**Theorem 3.1.** If **(A1) - (A5)**, **(f1)**, **(f2)**, and **(AR)** hold, then problem **(P)** has a nontrivial weak solution for any $\lambda \in (0, \infty)$.

We say that $u \in X$ is a weak solution of **(P)** if

$$\int_\Omega a(x, \nabla u) \nabla v \, dx - \lambda \int_{\partial\Omega} f(x,u) v \, d\sigma = 0$$

for all $v \in X$.

The energy functional corresponding to the problem **(P)** is defined as $I : X \to R$

$$I(u) = \int_\Omega A(x, \nabla u) \, dx - \lambda \int_{\partial\Omega} F(x,u) \, d\sigma = \Lambda(u) - \lambda J(u),$$

where $\Lambda(u) = \int_\Omega A(x, \nabla u) \, dx$ and $J(u) = \int_{\partial\Omega} F(x,u) \, d\sigma$.

**Proposition 3.2.** [3,9] Let $f : \partial\Omega \times \mathrm{R} \to R$ is a Carathéodory function satisfying **(f1)**. For each $u \in X$ set $J(u) = \int_{\partial\Omega} F(x,u) \, d\sigma$. Then, $J(u) \in C^1(X, R)$) and

$$\langle J'(u), v \rangle = \int_{\partial\Omega} f(x,u) v \, d\sigma,$$

for all $v \in X$. Moreover, the operator $J' : X \to X^*$ is compact.

**Lemma 3.3.** [9, 11]

(i) $A$ verifies the growth condition: for all $x \in \overline{\Omega}$ and all $\varepsilon \in R^N$,
$| A(x, \varepsilon) | \leq c_1 \left( |\varepsilon| + |\varepsilon|^{p(x)} \right),$

(ii) $A$ is $\mathrm{p(x)}$ -homogeneous: for all $z \geq 1$, $x \in \overline{\Omega}$ and $\varepsilon \in R^N$,
$\varepsilon \in R^N$ $A(x, z\varepsilon) \leq A(x, \varepsilon) z^{p(x)}$.

**Lemma 3.4.** [6, 9, 11, 14]

(i) The functional $\Lambda(u)$ is well-defined on $X$.

(ii) The functional $\Lambda(u)$ is of class $C^1(X, R)$ and

$$\langle \Lambda'(u), v \rangle = \int_\Omega a(x, \nabla u) \nabla v \, dx, \text{ for all } u, v \in X.$$

(iii) The functional $\Lambda(u)$ is weakly lower semi-continuous on $X$.

(iv) $I$ is weakly lower semi-continuous on $X$ .

(v) $I$ is well-defined on $X$ .

(vi) For all $u, v \in X$

$$\Lambda\ (u\ ) - \Lambda\ (v\ ) \geq \left\langle\ \Lambda^{'}(v), u - v\ \right\rangle$$

(vii) For all $u, v \in X$ and $m > 0$ is a constant

$$\Lambda\ \left(\frac{u+v}{2}\right) \leq \frac{1}{2}\Lambda\ (u) + \frac{1}{2}\Lambda\ (v) - m\|u - v\|^{p^-} .$$

Therefore, from Proposition 2.3, Proposition 3.1, and Lemma 3.3, it is easy to see that $I(u) \in C^1(X, R)$ the critical points $I$ are weak solutions of **(P)**. Moreover, the derivate of $I$ is the mapping $I : X \to R$

$$\left\langle I^{'}(u), v\right\rangle = \int_\Omega a\left(x, \nabla u\right)\nabla v\, dx - \lambda\int_{\partial\Omega} f\left(x, u\right)v\, d\sigma,$$

for any $u, v \in X$ **[8,15,17]**.

**Lemma 3.5.** Assume that the conditions **(A1) - (A4)**, **(f1)**, **(f2)**, and **(AR)** hold. Then the following statements hold:

(i) There exist two positive real numbers $\tau$ and $r$ such that $I(u) \geq r > 0$ with $\|u\| = \tau$ ,

(ii) There exists $u_1 \in X$ such that $\|u_1\| > \tau$ and $I(u_1) < 0$.

**Proof (i)**: For $\|u\| < 1$, from **(f1)** and **(f2)**, we have

$$|\ F\left(x, u\right)| \leq \varepsilon\,|u|^{p^+} + c_\varepsilon|u|^{q(x)}, \text{ for all } (x, u) \in \partial\Omega \times R . \tag{1.1}$$

Then, using the above inequality **(1.1)** and **(A4),** we write

$$I(u) = \int_\Omega A(x, \nabla u)\, dx - \lambda\int_{\partial\Omega} F(x, u)\, d\sigma$$

$$\geq \int_\Omega \frac{1}{p(x)}|u|^{p(x)}\, dx - \lambda\int_{\partial\Omega}\left(\varepsilon|t|^{p^+} + c_\varepsilon|t|^{q(x)}\right)d\sigma$$

On the other hand, by Proposition 2.4 (iii)**,** we can write

$$X \to L^{p^+}(\Omega) \text{ and } X \to L^{q^+}(\partial\Omega) \to L^{q^-}(\partial\Omega)$$

Thus, there exist constants $c_2, c_3, c_4 > 0$ for all $u \in X$ ,

$$\int_\Omega |u|^{p^+} dx \leq c_2\,\|u\|^{p^+},\ \int_{\partial\Omega}|u|^{q^+} d\sigma \leq c_3\,\|u\|^{q^+} \text{ and } \int_{\partial\Omega}|u|^{q^-} d\sigma \leq c_4\,\|u\|^{q^-} , \tag{1.2}$$

Moreover, using Proposition 2.1, Proposition 2.2, Proposition 2.3, **(A4)**, the inequalities (1.1) and (1.2), we obtain

$$I(u) \geq \frac{c_5}{p^+}\|u\|^{p^+} - \lambda\left(c_6\ \varepsilon\,\|u\|^{p^+} + c_\varepsilon\ c_7\,\|u\|^{q^-}\right)$$

Choose $\varepsilon > 0$ small enough such that $\lambda\, c_6\ \varepsilon < \dfrac{c_5}{2p^+}$ , we obtain

$$I(u) \geq \|u\|^{p^+}\left(\frac{c_5}{2p^+} - \lambda\, c_7\, c_\varepsilon\, \|u\|^{q^- - p^+}\right).$$

Let us define the function $\eta : [0,1] \to R$ by

$$\eta(t) = \frac{c_5}{2p^+} - \lambda\, c_7\, c_\varepsilon\, t^{q^- - p^+},$$

where $c_5, c_6$ and $c_7$ are positive constants. Since $p^+ < q^-$ the function $\eta$ is strictly positive in a neighborhood of zero.

**(ii)** Let $\omega \in X/\{0\}$ and $t > 1$. From **(AR)**, we obtain $|F(x,t)| \geq c_8 |t|^\theta$ for all $(x,t) \in \partial\Omega \times R$ and $c_8$ is a positive constant. Then, by Lemma 3.3 (ii), we get

$$I(t\omega) = \int_\Omega A(x, \nabla t\omega)\,dx - \lambda \int_{\partial\Omega} F(x, t\omega)\,d\sigma$$

$$\leq \frac{c_9\, t^{p^-}}{p^-} \int_\Omega A(x, \nabla\omega)\,dx - \frac{\lambda\, c_8\, t^{q^-}}{q^+} \int_{\partial\Omega} |\omega|^{q(x)} d\sigma$$

where $c_8$ is constant. Since $q^- > p^-$ we conclude that $I(t\omega) < 0$ as $t \to \infty$.There exists $u_1 = t\omega \in X$ such that $\| u_1 \| > \tau$ and $I(u_1) < 0$ The proof is completed.

**Lemma 3.6.** Suppose that the conditions **(A1) - (A5)**, **(f1)**, **(f2)** and **(AR)** hold. Then $I$ satisfies the **(PS)** condition.

**Proof;** Suppose that $\{u_n\} \subset X$ is a **(PS)**-sequence that satisfy the properties:

$$I(u_n) \to c_{10} \text{ and } I'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty, \tag{1.3}$$

where $X^*$ is dual space of $X$ and $c_{10}$ is a positive constant.

We prove that $\{u_n\}$ possesses a convergent subsequence. Firstly, we show that $\{u_n\}$ is bounded in $X$. We do the proof by contradiction. That is, we show that $\| u_n \| \to \infty$ as $n \to \infty$.

By using the condition **(A4)**, we can write

$$\int_\Omega |\nabla u_n|^{p(x)} \leq \int_\Omega a(x, \nabla u_n)\nabla u_n\,dx \leq p^+ \int_\Omega A(x, \nabla u_n)\,dx. \tag{1.4}$$

Moreover, using (1.3), (1.4), **(AR)**, and Proposition 2.2 and considering $\| u_n \| > 1$ for $n$ large enough, we have

$$1 + c_{10} \geq I(u_n) - \frac{1}{\theta}\langle I'(u_n), u_n \rangle$$

$$= \int_\Omega A(x, \nabla u_n)\,dx - \lambda \int_{\partial\Omega} F(x, u_n)\,d\sigma - \frac{1}{\theta}\int_\Omega a(x, \nabla u_n)\nabla u_n\,dx + \frac{\lambda}{\theta}\int_{\partial\Omega} f(x, u_n)u_n\,d\sigma$$

$$\geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right)\|u_n\|^{p^-}.$$

Since $\theta > p^+$, we obtain that $\{u_n\}$ is bounded in $X$ and $u \in X$ such that $u_n \to u$ (weak convergent) in $X$. Now, we prove that $\{u_n\}$ strongly convergent to $u$ in $X$.

By relation (1.3), we obtain the following $\langle I'(u_n), u_n - u \rangle \to 0$ as $n \to \infty$.

That is,

$$\left\langle I'(u_n), u_n - u \right\rangle$$
$$= \int_\Omega a(x, \nabla u_n)(\nabla u_n - \nabla u)\,dx - \lambda \int_{\partial\Omega} f(x, u_n)(u_n - u)\,d\sigma \to 0.$$

From **(f1)** and Proposition 2.1, we write

$$\left| \int_{\partial\Omega} f(x, u_n)(u_n - u)\,d\sigma \right| \le \left| \int_{\partial\Omega} \left( c + c_1 |u_n|^{q(x)-1} \right)(u_n - u)\,d\sigma \right|$$
$$\le c_{11} \, | u_n - u |_{q(x)} + c_{12} \left| \, | u_n |^{q(x)-1} \right|_{q'(x)} | u_n - u |_{q(x)}.$$

Moreover, thanks to the compact embedding $X \to L^{q(x)}(\partial\Omega)$, we have

$$u_n \to u \text{ (strongly convergent) in } L^{q(x)}(\partial\Omega) \,. \tag{1.5}$$

By Proposition 2.2 and relation (1.5), we get

$$\int_{\partial\Omega} f(x, u_n)(u_n - u)\,d\sigma \to 0 \text{ as } n \to \infty.$$

Taking into account the above inequality, we have

$$\int_\Omega a(x, \nabla u_n)(\nabla u_n - \nabla u)\,dx \to 0 \text{ as } n \to \infty.$$

That is,

$$\lim_{n\to\infty} \left\langle \Lambda'(u_n), u_n - u \right\rangle = 0\,.$$

Then, from Lemma 3.4 (vi), we can write

$$0 = \lim_{n\to\infty} \left\langle \Lambda'(u_n), u_n - u \right\rangle \le \lim_{n\to\infty} \left( \Lambda(u) - \Lambda(u_n) \right) = \Lambda(u) - \lim_{n\to\infty} \Lambda(u_n)$$

or

$$\lim_{n\to\infty} \Lambda(u_n) \le \Lambda(u)$$

and from Lemma 3.4 (iii), we obtain

$$\lim_{n\to\infty} \Lambda(u_n) = \Lambda(u).$$

Now, we assume by contradiction that $\{u_n\}$ does not converge strongly to $u$ in $X$. Then, there exists $\xi > 0$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\| u_{n_k} - u \| \ge \xi$. Moreover, by Lemma 3.4(vii), we can write the following inequality

$$\frac{1}{2}\Lambda(u) + \frac{1}{2}\Lambda(u_{n_k}) - \Lambda\left( \frac{u + u_{n_k}}{2} \right) \ge m \| u - u_{n_k} \|^{p^-} \ge m\xi^{p^-}.$$

Letting $k \to \infty$ in the above inequality, we have

$$\limsup_{n\to\infty} \Lambda \left( \frac{u + u_{n_k}}{2} \right) \le \Lambda(u) - m\xi^{p^-}.$$

We also have $\left\{ \frac{u + u_{n_k}}{2} \right\}$ converges weakly to $u$ in $X$. On the other hand, using Lemma 3.4 (iii), we obtain

$$\Lambda(u) \le \liminf_{n\to\infty} \Lambda \left( \frac{u + u_{n_k}}{2} \right),$$

and this is a contradiction. Hence, it follows that $\{u_n\}$ converges strongly to $u$ in $X$. The proof of Lemma 3.6 is complete.

**Proof of Theorem 3.1**. from Lemma 3.5, Lemma 3.6, and $I(0)=0$ from (A2), $I$ satisfies all statements of Lemma 2.6. Thus, $I$ has a nontrivial critical point, i.e., problem (**P**) has a nontrivial weak solution**.**

## 4. Conclusion

Through this paper, we have studied the existence of a nontrivial weak solution of the nonlinear Steklov boundary value problem in variable exponent Sobolev spaces and using the variational method under appropriate conditions on $f$ and $a$ .

**Conflict of interest**
The authors declare no conflict of interest.
**Authors' Contributions**
Z.Y.: Conceptualization, Methodology, Formal analysis, Writing - Original draft preparation (%50)
V. M.: Conceptualization, Methodology, Resources, Investigation (%50).
All authors read and approved the final manuscript.
**Ethical Statement**
The authors declare that this document does not require ethics committee approval or any special permission. Our study does not cause any harm to the environment.

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