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Quadrilaterals as Geometric Loci

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ABSTRACT

We give necessary and sufficient conditions, both algebraic and geometric, for a quadrilateral to be the level set of the sum of the distances to $m \geq 2$ different lines.

Keywords: *Quadrilaterals, geometric loci, harmonic point.*

AMS Subject Classification (2020): Primary: 51M04; Secondary: 51N20 53A04.

1. Introduction

In [6] the authors set out to design a single Cartesian equation in variables (x, y) whose set of solutions is a quadrilateral in the Euclidean plane $\mathbb{R}^2$ whose vertices are given by their coordinates. Apart from the four basic arithmetic operations, the equation contains only the absolute value as a further operation. The method presented in the said article works well for most convex quadrilaterals (though not all) but is cumbersome for non-convex or crossed quadrilaterals. We briefly describe the approach in [6]: Let (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3) be the Cartesian coordinates of the vertices of a quadrilateral where its perimeter is traversed in the corresponding order of the vertices. Solve the linear system

$$\underbrace{\begin{pmatrix} x_0 & y_0 & 1 & 0 & 0 & 0 & 0 & 0 \\ x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1 & -y_1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ x_3 & y_3 & 1 & 0 & 0 & 0 & x_3 & y_3 \\ 0 & 0 & 0 & x_0 & y_0 & 1 & -x_0 & -y_0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & x_2 & y_2 & 1 & x_2 & y_2 \\ 0 & 0 & 0 & x_3 & y_3 & 1 & 0 & 0 \end{pmatrix}}_{=:M} \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \tag{1}$$

Then the equation which describes the boundary of the quadrilateral is given by

$$\left|\frac{Ax + By + C}{Gx + Hy + I}\right| + \left|\frac{Dx + Ey + F}{Gx + Hy + I}\right| = 1. \tag{2}$$

Observe, however, that for given $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ the equation $\det M = 0$ is quadratic in the variables (x_3, y_3) and describes a conic through the points $(x_0, y_0), (x_1, y_1)$, (x_2, y_2). For example, for $(x_0, y_0) = (1, 1)$, $(x_1, y_1) = (-1, 2)$, $(x_2, y_2) = (-1, 1)$, we obtain the conic $3 + x^2 - 6y + 2y^2 = 0$ (see Figure 1). For all points (x_3, y_3) on this conic (different from the three given points), the equation (1) has no solution.

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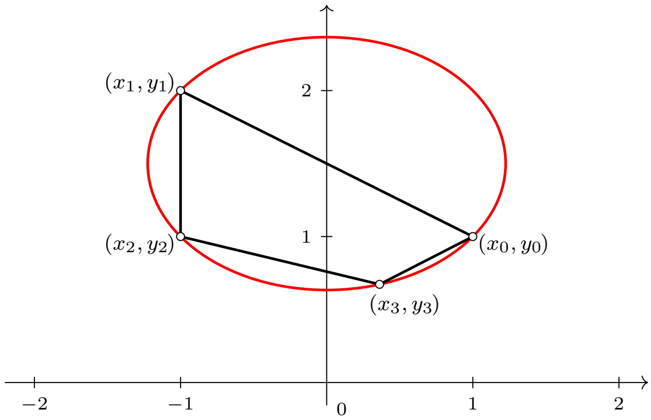

Figure 1. For all convex quadrilaterals with fixed vertices (x_0, y_0), (x_1, y_1), (x_2, y_2) and fourth point (x_3, y_3) on the red ellipse the equation (1) has no solution.

The problem with a non-convex or a crossed quadrilateral is, that equation (2) draws a convex solution in the projective plane that passes over the ideal line (see Figure 2).

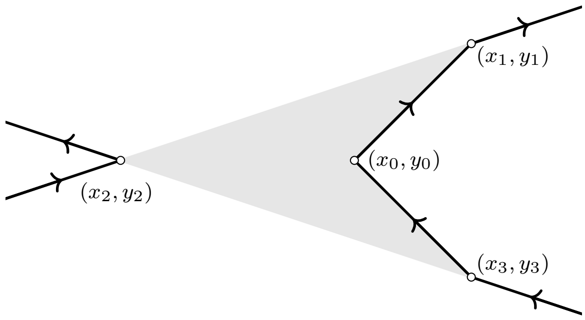

Figure 2. A non-convex quadrilateral.

Nevertheless, it is also possible to write the non-convex boundary of the quadrilateral in Figure 2 as the level set of a single Cartesian equation: The vertices are $(x_0, y_0) = (0, 0)$, $(x_1, y_1) = (1, 1)$, $(x_2, y_2) = (-2, 0)$, $(x_3, y_3) = (1, -1)$. Then the quadrilateral is the level set

$$\left\{x \in \mathbb{R}^2 : \max\big(\max(\langle n_1, x\rangle - d_1, \langle n_2, x\rangle - d_2), \min(\langle n_3, x\rangle - d_3, \langle n_4, x\rangle - d_4)\big) = 1\right\} \tag{3}$$

Here

$$n_1 = \sqrt{\tfrac{1}{10}}(-1, -3)^t,\ n_2 = \sqrt{\tfrac{1}{10}}(-1, 3)^t,\ n_3 = \sqrt{\tfrac{1}{2}}(1, -1)^t,\ n_4 = \sqrt{\tfrac{1}{2}}(1, 1)^t$$

are the outer unit normal vectors of the sides of the quadrilateral, $\langle\cdot,\cdot\rangle$ is the Euclidean inner product, and $d_1 = d_2 = \sqrt{\frac{2}{5}} - 1$, $d_3 = d_4 = -1$. Notice also that the minimum and the maximum function in (3) can be expressed with the absolute value:

$$\min(a, b) = \tfrac{1}{2}(a + b - |a - b|), \quad \max(a, b) = \tfrac{1}{2}(a + b + |a - b|).$$

We refrain from giving a general formula for this problem here, but focus now on the actual goal of this article: Let $f : \mathbb{R}^2 \to \mathbb{R}, x \mapsto f(x)$, denote the (weighted) sum of the distances of a point x to a set of given straight lines $\ell_1, \dots, \ell_m$. We then ask, which quadrilaterals can be written as the level set of such a function f.

This question is also motivated by Descartes' solution of Pappus' problem as described in Chapter 23 of [2]: Given m straight lines ℓ_i in the plane, n angles θ_i, and a line segment a. For any point x in the plane, the oblique distances δ_i to the lines ℓ_i are defined as the (positive) lengths of segments that are drawn from x toward ℓ_i

making angle θ_i with ℓ_i. Find the locus of points x for which the following ratios are constant:

for $m = 3$ lines	$\delta_1^2 : \delta_2\delta_3$
for $m = 2k \geq 4$ lines	$\delta_1 \ldots \delta_k : \delta_{k+1} \ldots \delta_{2k}$
for $m = 2k+1 \geq 5$ lines	$\delta_1 \ldots \delta_{k+1} : a\delta_{k+2} \ldots \delta_{2k+1}$

Instead of oblique distances, we can equivalently work with weighted normal distances (see Figure 3).

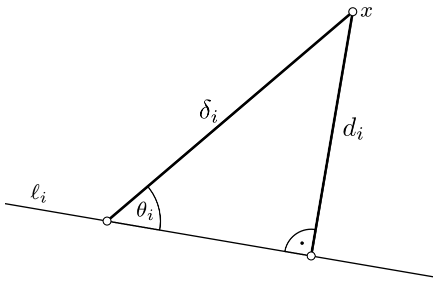

Figure 3. Oblique distances interpreted as weighted normal distances: $\delta_i = d_i \csc(\theta_i)$

The classical Greek geometry has considered the following loci:

- the sum of the distances to two given points is a constant (this gives an ellipse),
- the ratio of the distances to two points is a constant (this gives a circle of Apollonius),
- the ratio of (products of) distances to straight lines is a constant (this is Pappus' problem).

But *the sum of the distances to straight lines* appears then only in Viviani's theorem from 1649, and in its generalizations (see, e.g., [1]). However, there the question is not about the locus. In this sense we close a gap here by considering the locus of the set of points for each of which the sum of the distances to given straight lines is a constant.

2. Weighted distances to three lines

Before we start we fix some notation which we will use throughout this text. The vertices of the quadrilateral will be denoted by A, B, C, D. We will consider the corresponding complete quadrangle and denote by E the intersection of AB and CD, and by F the intersection of AD and BC (see Figure (5). ℓ_1 is the diagonal AC, ℓ_2 the diagonal BD, and ℓ_3' the diagonal EF. The intersections of the diagonals are $O = \ell_1 \cap \ell_2$, $P = \ell_1 \cap \ell_3'$, and $Q = \ell_2 \cap \ell_3'$. When we work with vectors, O will be the origin. Moreover, we use the notation $a = |OA|$, $b = |OB|$, $c = |OC|$, $d = |OD|$, $p = |OP|$, $q = |OQ|$ for the lengths of the respective segments.

In this section we treat the question which quadrilaterals can be described as level sets of the *weighted* sum of the distances to three lines. Suppose we are given a convex quadrilateral $ABCD$ in the Euclidean plane. By choosing unit normal vectors n_1, n_2 to ℓ_1, ℓ_2, and $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ we may write $A = -aMn_1$, $B = bMn_2$, $C = cMn_1$, $D = -dMn_2$.

A further line ℓ_3 which does not meet the quadrilateral and with unit normal vector n_3 will be determined later. We assume that the orientation of n_3 is such that the quadrilateral lies in the half plane with boundary ℓ_3 in which n_3 points. The line ℓ_i is given by the equation

$$\langle n_i, x\rangle - d_i = 0, \quad \|n_i\| = 1,$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ runs along ℓ_i. The distance of a point $x \in \mathbb{R}^2$ from ℓ_i is given by the function

$$f_i(x) = |\langle n_i, x\rangle - d_i|.$$

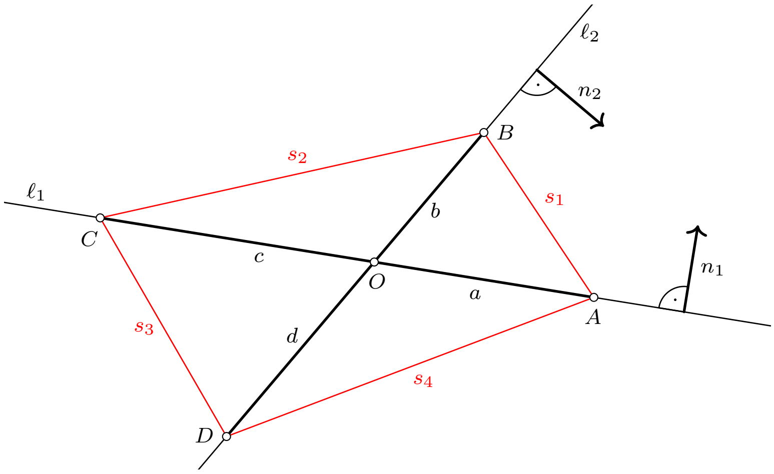

Figure 4. The quadrilateral $ABCD$.

The weighted sum of the distances of x to ℓ_1, ℓ_2 and ℓ_3 is

$$f(x) = \sum_{i=1}^{3} k_i f_i(x)$$

for weights $k_i \geq 0$. Then the gradient of f along the boundary of the quadrilateral is given as follows:

$$\begin{array}{ll} \text{along } s_1 & \nabla f = k_1 n_1 + k_2 n_2 + k_3 n_3 \\ \text{along } s_2 & \nabla f = k_1 n_1 - k_2 n_2 + k_3 n_3 \\ \text{along } s_3 & \nabla f = -k_1 n_1 - k_2 n_2 + k_3 n_3 \\ \text{along } s_4 & \nabla f = -k_1 n_1 + k_2 n_2 + k_3 n_3 \end{array}$$

The gradient of f along s_1 is perpendicular to s_1, and hence there exists $\alpha \in \mathbb{R} \setminus \{0\}$ such that

$$-\alpha M(bMn_2 + aMn_1) = k_1 n_1 + k_2 n_2 + k_3 n_3$$

or equivalently

$$k_3 n_3 = n_1(\alpha a - k_1) + n_2(\alpha b - k_2). \tag{4}$$

Similarly, with s_2, s_3, s_4 in place of s_1, we obtain

$$\begin{aligned} k_3 n_3 &= n_1(\beta c - k_1) + n_2(-\beta b + k_2) & (5)\\ k_3 n_3 &= n_1(-\gamma c + k_1) + n_2(-\gamma d + k_2) & (6)\\ k_3 n_3 &= n_1(-\delta a + k_1) + n_2(\delta d - k_2) & (7) \end{aligned}$$

Since n_1 and n_2 are linearly independent, we infer from (4)–(7)

$$\begin{aligned} \alpha a - k_1 = \beta c - k_1 &= -\gamma c + k_1 = -\delta a + k_1 \\ \alpha b - k_2 = -\beta b + k_2 &= -\gamma d + k_2 = \delta d - k_2. \end{aligned}$$

It follows that

$$\alpha = 2\epsilon cd,\ \beta = 2\epsilon ad,\ \gamma = 2\epsilon ab,\ \delta = 2\epsilon bc$$

for arbitrary $\epsilon > 0$, and

$$k_1 = \epsilon ac(b+d),\ k_2 = \epsilon bd(a+c),\ k_3 n_3 = n_1 \epsilon ac(d-b) + n_2 \epsilon bd(c-a).$$

It turns out that this result has a nice geometric interpretation which can be seen by choosing $\epsilon = \frac{2}{(a+c)(b+d)}$. Then the wights

$$k_1 = \frac{2ac}{a+c}, \quad k_2 = \frac{2bd}{b+d} \tag{8}$$

are the harmonic means of the segments of the diagonals, and

$$k_3 n_3 = n_1 \frac{2ac(d-b)}{(a+c)(b+d)} + n_2 \frac{2bd(c-a)}{(b+d)(c+a)}.$$

We consider the following three cases:

1. Suppose $a = c$ and $b = d$. In this case the quadrilateral is a parallelogram, and we have $k_3 = 0$. Hence, in this case, the third line ℓ_3 is not necessary.

2. Suppose $a = c$ and $b < d$ (the case $b > d$ is symmetric). In this case the quadrilateral is an oblique kite, and we have
$$k_1 = a, \quad k_2 = \frac{2bd}{b+d}, \quad k_3 = a\frac{d-b}{b+d}, \quad n_3 = n_1. \tag{9}$$
This means that the third line ℓ_3 is parallel to n_1.

3. Suppose $a \neq c$ and $b \neq d$. Without loss of generality we assume $a > c, b > d$. $P \in \ell_1$ is the harmonic conjugate of O with respect to A and C, and $Q \in \ell_2$ is the harmonic conjugate of O with respect to B and D (see [4], and Figure 5). Then the distances of P and Q respectively from O are
$$p = \frac{2ac}{a-c}, \quad q = \frac{2bd}{b-d}. \tag{10}$$

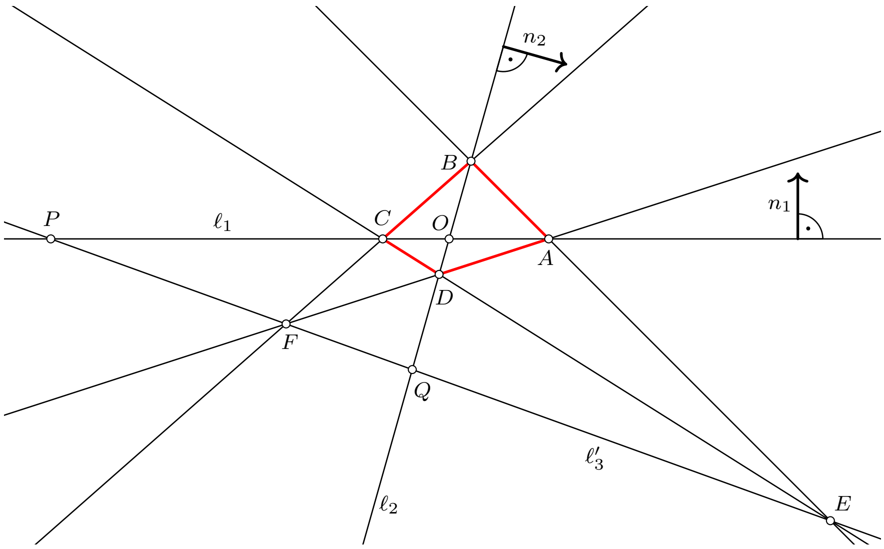

Figure 5. The complete quadrangle $ABCD$.

A simple calculation shows that
$$\frac{(a-c)(b-d)}{(a+c)(b+d)} M\vec{PQ} = -k_3 n_3. \tag{11}$$
Hence, ℓ_3 is parallel to the outer diagonal ℓ_3'. Observe that (11) also shows that $-n_3$ points towards the half plane bounded by ℓ_3' which contains the quadrilateral $ABCD$. Hence we must choose $\ell_3 \parallel \ell_3'$ on the other side of the quadrilateral $ABCD$. What we also learn from (11) is that
$$k_3 = \frac{(a-c)(b-d)}{(a+c)(b+d)} |PQ|. \tag{12}$$

To summarize we have the following result.

Theorem 1. *1. Every parallelogram is the level set of a weighted sum of the distances to its diagonals. The weights are given by (8).*

2. Every convex oblique kite is the level set of a weighted sum of the distances to its diagonals and a third line parallel to the diagonal which is bisected by the other. The weights are given by (9).

3. Every convex quadrilateral which is neither a parallelogram nor an oblique kite is the level set of a weighted sum of the distances to its diagonals and a third line which is parallel to the outer diagonal of the complete quadrangle. The weights are given by (8) and (12).

We remark, that for a parallelogram, the points E, F, P, Q lie on the ideal line (of the projective plane), for an oblique kite, P or Q lies on the ideal line, and for a trapezoid, E or F lies on the ideal line.

3. Distances to an arbitrary number of lines

If we restrict ourselves to the case of an unweighted sum, the question arises which quadrilaterals occur as level sets of the sum of the distances to two or more lines. We start with the general case of $m \geq 2$ lines.

3.1. A necessary condition

Let $\ell_1, \dots, \ell_m$ be different straight lines in the Euclidean plane $\mathbb{R}^2$, $m \geq 2$. The line ℓ_i is again given by the equation

$$\langle n_i, x\rangle - d_i = 0, \quad \|n_i\| = 1,$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ runs along ℓ_i, and n is a unit normal vector of ℓ_i. The distance of a point $x \in \mathbb{R}^2$ to ℓ_i is given by the function

$$f_i(x) = |\langle n_i, x\rangle - d_i|.$$

The sum of the distances of x to $\ell_1, \dots, \ell_m$ is

$$f(x) = \sum_{i=1}^{m} f_i(x).$$

As a sum of convex functions, f is also convex. We assume that at least two of the lines ℓ_i are not parallel. Then it follows that f is coercive and hence the level sets of f are bounded. The lines ℓ_i divide the plane into convex poygonal regions. On each such region, f is an affine function. Therefore, f attains its minimum either in a single point (a vertex of one of the mentioned polygons), along a line segment (the side of one of the polygons), or in all points of one of the polygons. Let us assume, that f has a unique minimum in a point $x_0 \in \mathbb{R}^2$. By suitable choice of the coordinate system we can achieve $x_0 = 0$. Let us further assume that only two of the lines, say ℓ_1 and ℓ_2 pass through the origin, which we denote by O. Then the level sets

$$\{x \in \mathbb{R}^2 : f(x) = h\}$$

are quadrilaterals for all $h \in (f(0), f(0) + \varepsilon)$ provided $\varepsilon > 0$ is sufficiently small. Let a, b, c, d continue to denote the positive distances of the vertices of the quadrilateral from O (see Figure 6). Then, with

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

the vertices are given by $-aMn_1$, bMn_2, cMn_1 and $-dMn_2$ if we chose the orientation of the normal vectors n_1, n_2 as indicated in Figure 6.

We choose the orientations of the normal vectors $n_3, \dots, n_m$ such that $d_3, \dots, d_m < 0$. Let $n_0 := \sum_{i=3}^{m} n_i$. Then the gradient of f along the line segments $s_1, \dots, s_4$ is given as follows:

$$\begin{aligned}
&\text{along } s_1: && \nabla f = n_1 + n_2 + n_0 \\
&\text{along } s_2: && \nabla f = n_1 - n_2 + n_0 \\
&\text{along } s_3: && \nabla f = -n_1 - n_2 + n_0 \\
&\text{along } s_4: && \nabla f = -n_1 + n_2 + n_0
\end{aligned}$$

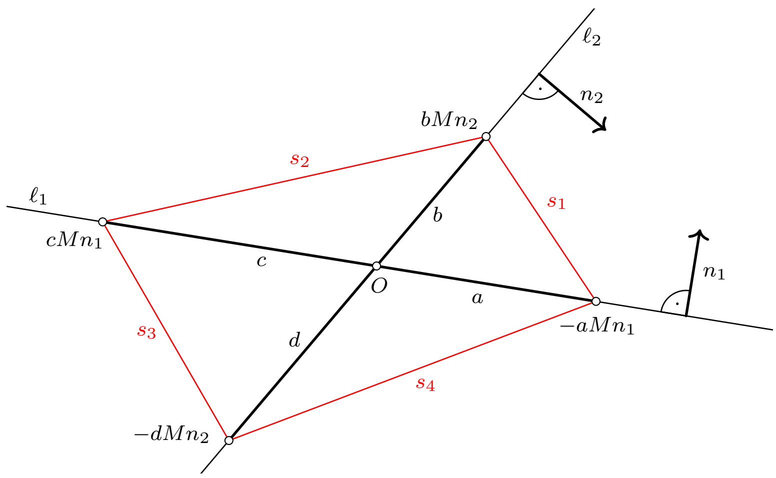

Figure 6. The level set (red) for $h \in (f(0), f(0) + \varepsilon)$, $\varepsilon > 0$ small.

The gradient of f along s_1 is perpendicular to s_1, hence there exists $\alpha \in \mathbb{R} \setminus \{0\}$ such that

$$-M(bMn_2 + aMn_1) = \alpha(n_1 + n_2 + n_0)$$

or equivalently

$$n_1(a - \alpha) + n_2(b - \alpha) = \alpha n_0. \tag{13}$$

In the same way, we have

$$\begin{aligned} n_1(c - \beta) + n_2(-b + \beta) &= \beta n_0 && (14)\\ n_1(-c + \gamma) + n_2(-d + \gamma) &= \gamma n_0 && (15)\\ n_1(-a + \delta) + n_2(d - \delta) &= \delta n_0 && (16) \end{aligned}$$

The equations (13)–(16) can only hold simultaneously if all 2×2 minors of the matrix

$$\begin{pmatrix} a - \alpha & b - \alpha & \alpha \\ c - \beta & -b + \beta & \beta \\ -c + \gamma & -d + \gamma & \gamma \\ -a + \delta & d - \delta & \delta \end{pmatrix} \tag{17}$$

vanish. In particular we have

$$\begin{aligned} \det \begin{pmatrix} b - \alpha & \alpha \\ d - \delta & \delta \end{pmatrix} &= b\delta - d\alpha = 0 \\ \det \begin{pmatrix} -b + \beta & \beta \\ -d + \gamma & \gamma \end{pmatrix} &= -b\gamma + d\beta = 0 \\ \det \begin{pmatrix} a - \alpha & \alpha \\ c - \beta & \beta \end{pmatrix} &= a\beta - c\alpha = 0 \\ \det \begin{pmatrix} c - \beta & \beta \\ -a + \delta & \delta \end{pmatrix} + \det \begin{pmatrix} -b + \beta & \beta \\ d - \delta & \delta \end{pmatrix} &= (a - d)\beta + (c - b)\delta = 0. \end{aligned}$$

This is a linear system for $\alpha, \beta, \gamma, \delta$, and a nontrivial solution exists only if

$$0 = \det \begin{pmatrix} -d & 0 & 0 & b \\ 0 & d & -b & 0 \\ -c & a & 0 & 0 \\ 0 & a - d & 0 & c - b \end{pmatrix} = b(bcd + bda - bca - cda).$$

Recall that $b, c, d, a > 0$ and hence the last condition is equivalent to

$$\frac{1}{a} + \frac{1}{c} = \frac{1}{b} + \frac{1}{d}.$$

In summary, we have obtained the following theorem:

Theorem 2. *Let $\ell_1, \ldots, \ell_m$, $m \geq 2$, be straight lines in the Euclidean plane, not all parallel. Assume that the sum $f(x)$ of the distances of a point x to the lines $\ell_1, \ldots, \ell_m$ attains its minimum in a single point x_0 in which only two of the lines $\ell_1, \ldots, \ell_m$ meet. Then, the level sets $\{x \in \mathbb{R}^2 : f(x) = h\}$ form a family of homothetic convex quadrilaterals for all $h \in (f(0), f(0) + \varepsilon)$ provided $\varepsilon > 0$ is small enough. The intersection of the diagonals divides them into segments of lengths b and d on one diagonal and of lengths c and a on other diagonal. These lengths satisfy the conditon*

$$\frac{1}{a} + \frac{1}{c} = \frac{1}{b} + \frac{1}{d}. \tag{18}$$

Observe that the theorem is trivially valid for $m = 2$ where the level sets are rectangles.

We want to interpret condition (18) geometrically. To this end, we express the points E and F by the vectors n_1 and n_2:

$$E = \frac{1}{ad - bc} M\big(ac(b+d)n_1 + bd(a+c)n_2\big) \tag{19}$$

$$F = \frac{1}{ab - cd} M\big(ac(b+d)n_1 - bd(a+c)n_2\big) \tag{20}$$

Here, we assume for the moment that the quadrilateral is neither a parallelogram nor a trapezoid, i.e., both denominators $ad - bc$ and $ab - cd$ in (19) and (20) are different from zero. Using these expressions, we find for the scalar product

$$\langle E, F\rangle = \frac{(abc - abd + acd - bcd)(abc + abd + acd + bcd)}{(ad - bc)(ab - cd)}.$$

This expression is equal to 0, if and only if the vectors E and F are orthogonal, and if and only if (18) holds. Thus we obtain:

Corollary 3. *Let $ABCD$ be a convex complete quadrilateral which is neither a parallelogram nor a trapezoid. Then, with the notation used before, the condition (18) is equivalent to the fact that O lies on the Thales circle over the segment EF (see Figure 7).*

The condition (18) can be interpreted geometrically in yet another way (see Figure 8): Indeed, it is easy to check that for $a > d$, $b > c$ and $a = |O'A'|$, $b = |O'B'|$, $c = |O'C'|$, $d = |O'D'|$, the line RS passes through O' if and only if the condition (18) holds — apply the intersecting chords theorem for the point O' and for the point C'.

Remark 4. Note that for a rectangle (18) is trivially always satisfied. For a trapezoid we have either $ab = cd$ or $ac = bd$. Thus, (18) is equivalent to $a = d$ or $a = b$, respectively. Hence, a trapezoid satisfies (18) if and only if it is symmetric. Geometrically, the Thales circle over EF degenerates for a trapezoid to the normal to the parallel sides of the trapezoid going through E or F, and hence, under the conditon (18), to the symmetry axis of the trapezoid.

3.2. *The case of two lines*

The case of two lines is simple:

Proposition 5. *Each rectangle is the level set of the sum of the distances to its diagonals. Vice versa, given two intersecting lines ℓ_1, ℓ_2, the level sets of the sum of the distances to ℓ_1 and ℓ_2 are rectangles with ℓ_1, ℓ_2 as diagonals.*

Notice that a rectangle can also be written as the level set of the sum of the distances to $m \geq 4$ lines. Indeed, we can add to the two diagonals

- any number of pairs of parallel lines with the rectangle between them,
- any number of equilateral triangles with the rectangle in its interior,
- the pair ℓ_1, ℓ_2 more than once,

or any combination of these variants.

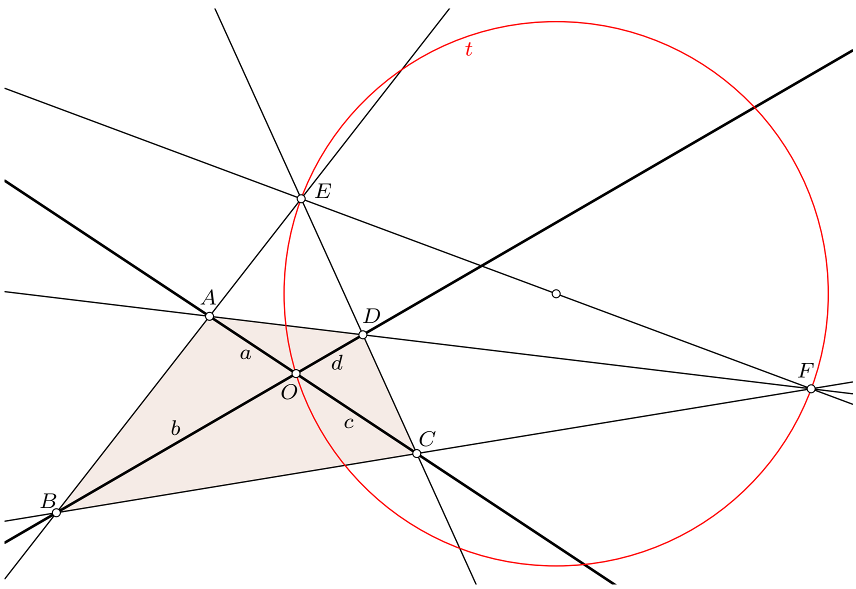

Figure 7. Necessary and sufficient condition for $\frac{1}{a}+\frac{1}{c}=\frac{1}{b}+\frac{1}{d}$: The Thales circle t over the segment EF passes through O.

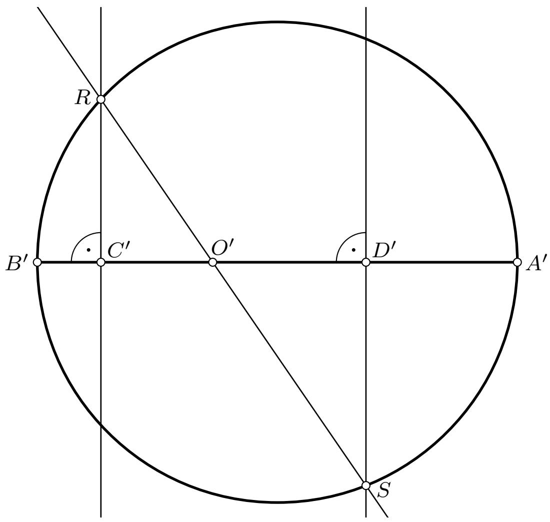

Figure 8. Geometric interpretation of condition (18).

3.3. *The case of three lines*

If we set $k_1 = k_3$ in (9) it follows that $b = 0$. A kite can therefore not be the level set of the sum of the distances to three different lines. If the quadrilateral is not a kite, then the points P and Q exist, and the condition $k_1 = k_3$ in (8) and (12) imply

$$|PQ| = p \cdot \frac{b+d}{b-d} \tag{21}$$

where we assume $b > d$. Similarly, $k_2 = k_3$ yields

$$|PQ| = q \cdot \frac{a+c}{a-c}. \tag{22}$$

From these two equations we deduce

$$\frac{pq}{|PQ|} = |PQ| \cdot \frac{b-d}{b+d} \cdot \frac{a-c}{a+c} = k_3$$

where we have used (12) for the last equality. So, we obtain:

Theorem 6. *A necessary and sufficient condition for a convex quadrilateral $ABCD$ to be the level set of the sum of the distances to three lines is*

$$\frac{|OP||OQ|}{|PQ|} = \frac{2|OA||OC|}{|AC|} = \frac{2|OB||OD|}{|BD|}, \tag{23}$$

where O is the intersection of the diagonals AC and BD, and where P and Q are the intersections of AC and BD with the outer diagonal.

If $ABCD$ is a quadrilateral which satisfies the condition of Theorem 6, then the three triangle inequalities must hold in the triangle OPQ. If we denote $r = |PQ|$, then this means that

$$(p^2 + q^2 + r^2)^2 - 2(p^4 + q^4 + r^4) > 0$$

(see, e.g., [5]). Using (21) and (22) this can be expressed by the inequality

$$(3ab + bc + ad - cd)(ab + 3bc - ad + cd)(ab - bc + 3ad + cd)(-ab + bc + ad + 3cd) < 0. \tag{24}$$

So, we obtain:

Corollary 7. *A quadrilateral which is the level set of the sum of the distances to three lines exists if and only if the diagonal segments a, b, c, d satisfy (18) and (24).*

We give two constructions of quadrilaterals which are the level set of the sum of the distances to three lines.

Construction 1. Start with four segments of lengths a, b, d, c which satisfy (18) such that A', O', C' are collinear with $a = |O'A'| > c = |O'C'|$, and B', O', D' are collinear with $b = |O'B'| > d = |O'D'|$ (see Figure 8). Construct the harmonic conjugate P' of O' with respect to $A'C'$, and the harmonic conjugate Q' of O' with respect to $B'D'$. Construct a segment of length $r = |O'P'|\frac{b+d}{b-d}$. Condition (24) is satisfied if and only if $|O'P'|, |O'Q'|$ and r are the sides of a triangle OPQ. Then the quadrilateral $ABCD$ is easily constructed as can be seen in Figure 5.

Construction 2. We start with three points AOC with $a = |OA| > c = |OC|$ on the diagonal ℓ_1 of the quadrilateral $ABCD$, and its second diagonal ℓ_2 meeting ℓ_1 in O. We need to find the points B and D on ℓ_2. To do so, construct the harmonic conjugate P of AOC. Observe that we have

$$\frac{|OQ|}{|PQ|} \overset{\text{Thm. 6}}{=} \frac{2ac}{(a+c)|OP|} \overset{(10)}{=} \frac{a-c}{a+c} = \frac{c}{\frac{2ac}{a-c} - c} \overset{(10)}{=} \frac{|OC|}{|PC|}.$$

Hence Q is an intersection of ℓ_2 and the Apollonian circle K for this ratio which is the Thales circle over AC as indicated in Figure 9.

It remains to find B, D on ℓ_2 such that B, D, O, Q are harmonic points and such that (18) is satisfied. The construction is given in Figure 10: K_1 is the circle with diameter OQ, and K_2 the circle with diameter OH, where $|OH| = \frac{2ac}{a+c}$, and where O is between H and Q. Then, B is the intersection of the common tangents of K_1 and K_2. If X, Y denote the contact points of these tangents on K_1, then D is the intersection of XY and OQ. Let Z and W denote the contact points of K_2 with the tangents, and D' be the intersection of ZW with OQ. Then B, D, O, Q and B, D', H, O are harmonic points by construction. The harmonic mean of $|OD'|$ and $|OB|$ is $|OH|$. Since the segments on the tangents $|O'O| = |O'X| = |O'Z|$ have equal lengths, it follows that $|OD'| = |OD|$. Hence, the harmonic mean $|OD|$ and $|OB|$ is $|OH|$, and therefore (18) is satisfied, and the construction is completed.

We have learned from Construction 2 that Q on ℓ_2 lies on the Thales circle over AC. By symmetry, the point P on ℓ_1 is a point of the Thales circle over BD. We can therefore reformulate Theorem 6 geometrically as follows:

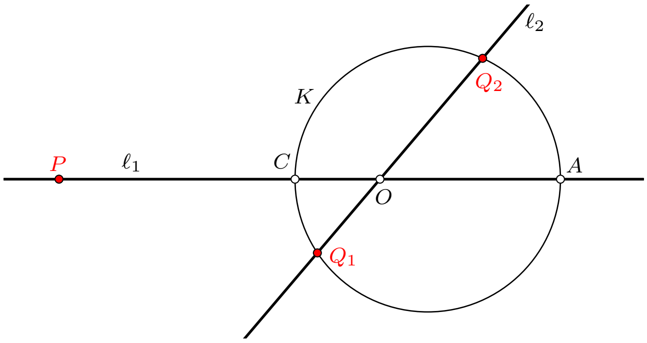

Figure 9. Construction of the point Q (two solutions).

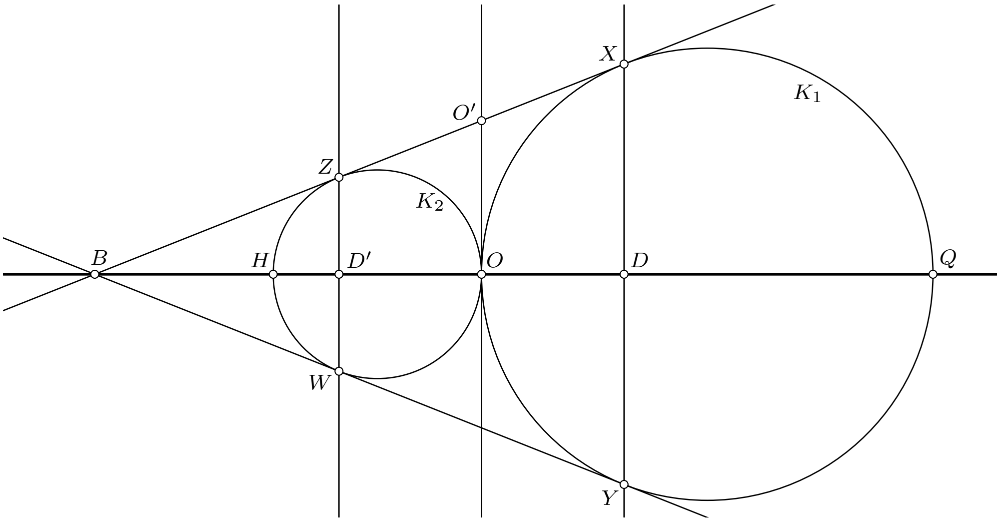

Figure 10. Construction of the points B and D.

Theorem 8. *Let $ABCDEF$ be a convex complete quadrangle with the notation used before. A necessary and sufficient condition for the quadrilateral $ABCD$ to be the level set of the sum of the distances to three lines is that the Thales circle over EF passes through O and that one of the following (and consequently both) conditions hold:*

(i) The Thales circle over AC on ℓ_1 passes through Q.

(ii) The Thales circle over BD on ℓ_2 passes through P.

If $ABCD$ is a trapezoid, E or F lie on the ideal line. In this case, the Thales circle over EF in Theorem 8 has to be interpreted as in Remark 4.

Recall that condition (18) is equivalent to the fact that the Thales circle over EF passes through O. We also note that according to the Bodenmiller-Steiner Theorem, the three Thales circles over the diagonals AC, BD and EF of a complete quadrangle meet in two points, and hence, their centers are collinear (see [3], [4], and Figure 11).

The following calculation shows that the conditions (i) and (ii) in Theorem 8 *together* also imply (18): We have

$$P = 2 \cdot \frac{ac}{a-c} M n_1, \qquad Q = 2 \cdot \frac{bd}{d-b} M n_2.$$

Eliminating $\langle n_1, n_2 \rangle$ from the equations

$$\langle A - Q,\ C - Q \rangle = 0, \qquad \langle B - P,\ D - P \rangle = 0,$$

yields again

$$(abc - abd + acd - bcd)(abc + abd + acd + bcd) = 0$$

which is equivalent to (18). This gives a further possibility to reformulate the result:

Theorem 9. *Let $ABCDEF$ a convex complete quadrangle with the notation used before. A necessary and sufficient condition for the quadrilateral $ABCD$ to be the level set of the sum of the distances to three lines is that the Thales circle over AC passes through Q, and the Thales circle over BD passes through P.*

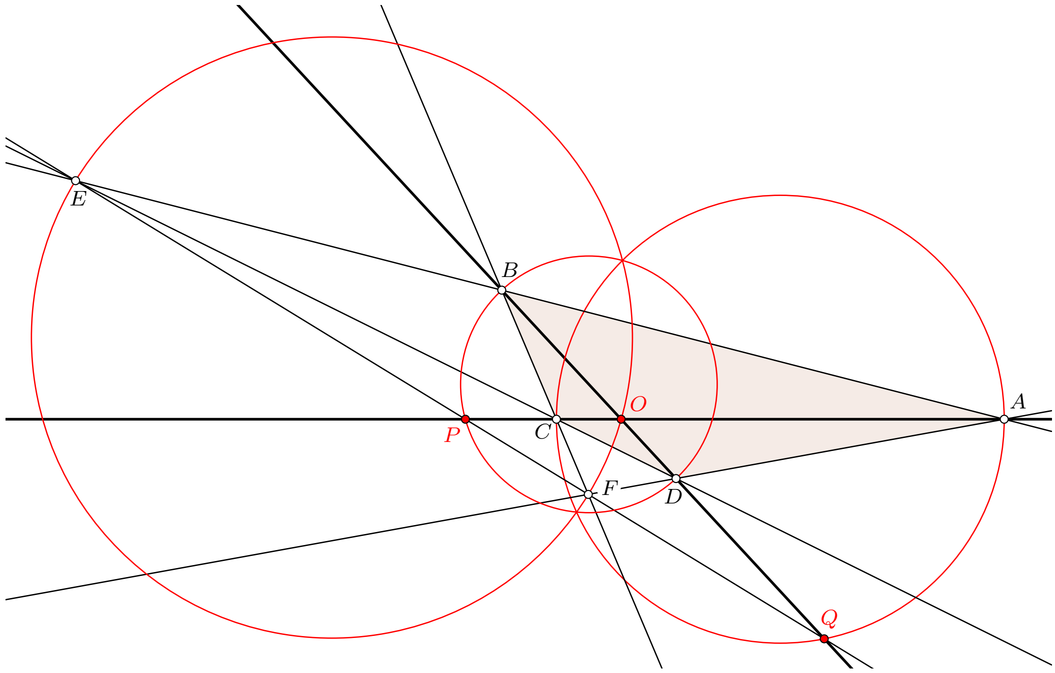

Figure 11. Necessary and sufficient conditions for a quadrilateral to be the level set of the sum of the distances to three lines.

3.4. *The case of four or more lines*

One can readily verify that actually all 2×2 minors of the matrix (17) vanish for

$$\alpha = \frac{a(b+d)}{2d}, \quad \beta = \frac{c(b+d)}{2d}, \quad \gamma = \frac{c(b+d)}{2b}, \quad \delta = \frac{a(b+d)}{2b},$$

if the condition (18) holds. For this choice the equations (13)–(16) coincide and yield

$$n_0 = n_1 \frac{d-b}{d+b} + n_2 \frac{c-a}{c+a}.$$

Clearly, we have $\|n_0\| < 1$. Therefore, it is possible to choose unit vectors $n_3, \ldots, n_m$ (in particular $m = 4$) such that

$$n_0 = n_3 + \ldots + n_m.$$

Then we can take lines $\ell_3, \ldots, \ell_m$ with corresponding unit normale vectors $n_3, \ldots, n_m$ such that the quadrilateral lies in the half planes into which these vectors point and we obtain the following result:

Theorem 10. *Every convex quadrilateral which satisfies condition (18) is the level set of the sum of the distances to 4 (or any number greater than 4) lines.*

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