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CONSTRUCTING DIRECTED STRONGLY REGULAR GRAPHS BY USING SEMIDIRECT PRODUCTS AND SEMIDIHEDRAL GROUPS

Semih PİRİ[1], Oktay ÖLMEZ[2] and Murat ŞAHİN[3]

[1]Mathematics Department, Kastamonu University, Kastamonu, TÜRKİYE
[2]Afiniti, Washington, D.C., USA
[3]Department of Mathematics, Ankara University, 06100 Ankara, TÜRKİYE

Abstract. In this paper, directed strongly regular graphs (DSRGs) are constructed by using semidirect products. The orbit condition in [3] has been weakened and this gives rise to the construction of DSRGs. Moreover, a different construction is given for DSRG by using semidihedral groups.

1. Introduction

Directed strongly regular graphs have attracted the attention of many mathematicians and many studies have been done on them. It was first discussed by Duval as the directed form of strongly regular graphs [2]. Duval also presented several construction methods in his work. The main problem today is to construct unknown ones by their parameters. For this purpose, many mathematical structures have been used. Some of these are designs [[5], [11]], coherent algebras [[5], [7], [10]], finite geometries [[4], [5], [6]], matrices [[2], [4], [6], [8]] and dihedral groups [10]. Some non-existence results are given by Jorgensen [9]. Duval [3] constructed directed strongly regular graphs by using semidirect products with an orbit condition. We change this condition with a weaker condition and give a construction of the directed strongly regular graphs. We also provide give a construction by using semidihedral groups. Our construction methods using semidirect product and semidihedral groups are new, however they do not give new parameters for small

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[1] spiri@kastamonu.edu.tr-Corresponding author; 0000-0002-3978-0679
[2] oktayolmez@gmail.com.tr; 0000-0002-9130-0038
[3] murat.sahin@ankara.edu.tr; 0000-0002-9480-0433.

examples. Also, they are simple to use for finding larger parameters. Uniqueness and enumeration studies can be found in [1].

This paper is designed as follows. In Section 2, necessary background information on the graph is given and the notations we will use are introduced, in Section 3 the semidirect construction of DSRG of Cayley graphs are given, and finally, in Section 4, DSRG is constructed from semidihedral groups which is an example of semidirect products.

2. Preliminaries

A *directed graph* $\Gamma = (V, E)$ consists of a vertex set V and an edge set E, where an edge is an ordered pair of distinct vertices of Γ. Writing $(x, y) \in E$ means that there is a *directed edge* from x to y and that is shown by $x \to y$. Throughout the paper, the edges of the form (y, y) for some $y \in V$, i.e., loops, are not allowed. However, we allow *bidirected edge*, that is having edges $x \to y$ and $y \to x$ for the vertices x and y, simultaneously. The *indegree (outdegree)* of a vertex y in a directed graph Γ is the number of vertices x such that $x \to y$ $(y \to x)$, respectively. A graph Γ is called *k-regular* if every vertex in Γ has indegree and outdegree k. A *path* of length l from x to y is a sequence of $l+1$ distinct vertices starting with x and ending with y such that consecutive vertices are adjacent. A directed graph Γ is called *directed strongly regular* with parameters (n, k, t, λ, μ) if it is k-regular and satisfies the following condition on the number of paths of length 2. The number of directed paths of length 2 between two vertices, say from x to y, of the graph Γ is λ if there is an edge from x to y, μ if there is not and t if $x = y$. Let G be a group and $S \subseteq G$ be a subset of G without the identity element. *Directed Cayley graph $Cay(G, S)$* is a directed graph whose vertex set is G and for any two vertices x, y, there is a directed edge from x to y if $xy^{-1} \in S$.

Example 1. Let G be a symmetric group of order six with elements $\{e, a, a^2, b, ab, a^2b\}$ and the subset $S \subseteq G$ be the set $\{a^2, a^2b\}$. Then the directed graph $Cay(G, S)$ is shown as in Figure 1. The Cayley table of the elements of symmetric group of order 6 is shown as in Table 1.

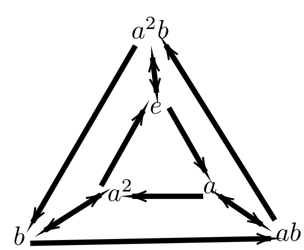

Figure 1. Cayley graph of symmetric group of order 6

$*$	e	a	a^2	b	ab	a^2b
e	e	a	a^2	b	ab	a^2b
a	a	a^2	e	ab	a^2b	b
a^2	a^2	e	a	a^2b	b	ab
b	b	a^2b	ab	e	a^2	a
ab	ab	b	a^2b	a	e	a^2
a^2b	a^2b	ab	b	a^2	a	e

TABLE 1. The Cayley table of the symmetric group of order 6

When studying directed strongly regular graphs adjacency matrix and group ring are advantageous tools. Let G be a finite group then the group ring $\mathbb{Z}[G]$ is a ring with identity element e and defined as the set of all formal sums of elements of G. The addition and multiplication are given by

$$\sum_{g\in G} a_g g + \sum_{g\in G} b_g g = \sum_{g\in G} (a_g + b_g) g$$

and

$$\left(\sum_{g\in G} a_g g\right)\left(\sum_{r\in G} b_r r\right) = \sum_{g,r\in G} a_g b_r (g+r)$$

Let G be a group and $\mathbb{Z}[G] = \{\sum_{g\in G} a_g g | a_g \in \mathbb{Z}\}$. If $S \subset G$, the group ring element $\underline{S}$ will then be defined using the abuse of notation as $\underline{S} = \sum_{s\in S} s$. Furthermore, the group ring elements $\underline{S}^{(-1)}$ and $\underline{G}$ will be defined as $\underline{S}^{(-1)} := \sum_{s\in S} s^{-1}$ and $\underline{G} := \sum_{g\in G} g$.

Let S be a subset of a group G. In [2] they showed that $Cay(G,S)$ corresponds to a DSRG with parameters (n,k,t,λ,μ) if and only if $|S| = k$, $|G| = n$ and it satisfies the following group ring equation:

$$\underline{S}^2 = te + \lambda\underline{S} + \mu(\underline{G} - e - \underline{S}).$$

Let Γ be a directed graph with n vertices, then the adjacency matrix M of Γ is an $n \times n$ matrix with entries a_{ij} where $a_{ij} = 1$ if $v_i \to v_j$. Otherwise $a_{ij} = 0$. Since we disallow loops, the diagonal entries of M are all 0. Let I and J denote the $n \times n$ identity matrix and the all-one matrix, respectively. Then Γ is a directed strongly regular graph if and only if

i) $MJ = JM = kJ$
ii) $M^2 = tI + \lambda M + \mu(J - I - M)$.

3. Semidirect Construction of Cayley DSRG

In this section, we give some definitions and lemmas related to the semidirect product of two groups. We will also proceed in a similar way to that of Duval and

Dmitri [3] by modifying the orbit setup they used. They proved that for a finite group A of order m and the cyclic group B of order q if some $\beta \in Aut(A)$ has the *q-orbit condition*, that is, each orbit of β contains only q elements, then the graph $Cay(A \ltimes_\theta B, A' \times B)$ is a DSRG with parameters

$$(mq, m-1, (m-1)/q, ((m-1)/q) - 1, (m-1)/q))$$

where $\theta : B \to Aut(A)$ by $\theta(b^r) = \beta^r$ and A' is the set of representatives of the nontrivial orbits of β.

Definition 1. (see [3]) Let A and B be two groups and $\theta : B \to Aut(A)$ be an action of B on A. Then the semidirect product $A \ltimes_\theta B$ for the set $\{(a,b) : a \in A \text{ and } b \in B\}$ is defined as follows:

$$(a,b)(a'b') = (a[\theta_b(a')], bb').$$

For groups A and B, $A \ltimes_\theta B$ forms a group of order $|A||B|$ with the identity element (e_A, e_B) and inverse $(a,b)^{-1} = (\theta_{b^{-1}}(a^{-1}), b^{-1})$.

Let A and B be the additive groups of finite fields $\mathbb{F}_{p^2}$ and $\mathbb{F}_2$ respectively, where p is a prime number. The Frobenius automorphism is defined as follows:

$$\beta : \mathbb{F}_{p^2} \to \mathbb{F}_{p^2}$$
$$\beta(x) = x^p$$

We will use the following notation in the rest of the paper: P is the set of elements of $\mathbb{F}_p$, R is the set of representatives of orbits of β and R^p is the set as $\{x^p : x \in R\}$.

The orbits of the action β on $\mathbb{F}_{p^2}$ consists of p orbits of size one and $\frac{p^2-p}{2}$ orbits of size two.

Let $A \times B$ be the direct product of the sets A and B and define the operation $\ltimes$ as the product of two elements as follows:

$$(a_1, b_1) \ltimes (a_2, b_2) = \begin{cases} (a_1 + a_2, b_2), & \text{if } b_1 = 0, \\ (a_1 + a_2^p, b_2 + 1), & \text{if } b_1 = 1. \end{cases}$$

Lemma 1. *$(G, \ltimes)$ forms a group of order $2p^2$ where $G = A \times B$.*

Proof. Let us start the proof by showing that G is closed under the operation $\ltimes$. For any $(a_1, b_1), (a_2, b_2) \in G$,

$$(a_1, b_1) \ltimes (a_2, b_2) = \begin{cases} (a_1 + a_2, b_2) \in G, & \text{if } b_1 = 0, \\ (a_1 + a_2^p, b_2 + 1) \in G, & \text{if } b_1 = 1. \end{cases}$$

Hence, G is closed under $\ltimes$. It is easy to see that $(0,0)$ is the identity element of the group. Indeed for any element (a,b) the following is true,

$$(a,b) \ltimes (0,0) = (0,0) \ltimes (a,b) = (a,b).$$

Next, the inverse of any element $(a,b) \in G$ is given by

$$(a,b)^{-1} = \begin{cases} (-a,-b), & \text{if } b_1 = 0, \\ (-a^p,-b), & \text{if } b_1 = 1. \end{cases}$$

Finally, we will show the associative property. For $(a_1,b_1),(a_2,b_2),(a_3,b_3) \in G$ we have the following:

$$((a_1,b_1) \ltimes (a_2,b_2)) \ltimes (a_3,b_3) = \begin{cases} (a_1+a_2+a_3,b_3), & \text{if } b_1=0, b_2=0, \\ (a_1+a_2+a_3^p,b_3+1), & \text{if } b_1=0, b_2=1, \\ (a_1+a_2^p+a_3^p,b_3+1), & \text{if } b_1=1, b_2=0, \\ (a_1+a_2^p+a_3,b_3), & \text{if } b_1=1, b_2=1. \end{cases}$$

$$= (a_1,b_1) \ltimes ((a_2,b_2) \ltimes (a_3,b_3))$$

and we are done. □

We say that a group automorphism β has the *q-orbit condition* if each of its orbits contains either q elements or one element (including the trivial orbit that contains only identity element). We change (weakened) the q-orbit condition that is defined in [3]. Before giving our main theorem, we need the following lemma.

Lemma 2. *The following equations hold in the group ring $\mathbb{Z}[G]$.*

(a) $(\underline{P \times \{1\}})^2 = |P|(\underline{P \times \{0\}})$
(b) $(\underline{R \times B})(\underline{P \times \{1\}}) = |P|(\underline{R \times B})$
(c) $(\underline{P \times \{1\}})(\underline{R \times B}) = |P|(\underline{R^p \times B})$
(d) $(\underline{R \times B})^2 = \frac{p^2-3p}{2}(\underline{R \times B}) + \frac{p^2-p}{2}(\underline{R^p \times B}) + \frac{p^2-p}{2}(\underline{P \times B})$

Proof. We will only prove (b). We know that P is the set of elements of the obvious orbits of β which are in $\mathbb{F}_p$ and R is the set of representatives of orbits of β. We also know that $B = \mathbb{F}_2$. Then we have the following:

$$\begin{aligned} (\underline{R \times B})(\underline{P \times \{1\}}) &= ((\underline{R \times \{0\}}) + (\underline{R \times \{1\}}))(\underline{P \times \{1\}}) \\ &= (\underline{R \times \{0\}})(\underline{P \times \{1\}}) + (\underline{R \times \{1\}})(\underline{P \times \{1\}}) \\ &= (\underline{\{(\sigma+\gamma,1) : \sigma \in R \text{ and } \gamma \in P\}}) \\ &+ (\underline{\{(\sigma+\gamma^p,1) : \sigma \in R \text{ and } \gamma \in P\}}) \\ &= |P|(\underline{R \times \{0\}}) + |P|(\underline{R \times \{1\}}) \\ &= |P|(\underline{R \times B}). \end{aligned}$$

The proof of (a), (c) and (d) are similar. □

Theorem 1. *Let $A = \mathbb{F}_{p^2}$ and $B = \mathbb{F}_2$ be two additive finite fields where p is an odd prime. If some $\beta \in Aut(A)$ has the q-orbit condition (for instance, Frobenius automorphism), then we may construct a directed strongly regular graph with* *parameters*

$$(n = 2p^2, k = p^2, t = (p^2 + p)/2, \lambda = (p^2 - p)/2, \mu = (p^2 + p)/2)$$

as follows: Let us define $\theta : B \to Aut(A)$ *with* $\theta_0 = Id$ *and* $\theta_1 = \beta(x) = x^p$ *for the additive group* $B = \mathbb{F}_2$*. Let* R *be the set representatives of orbits with two elements and* P *be the set of orbits with one element (only base field elements). Note that* $R \cap -R = \emptyset$ *where* $-R = \{-r : r \in R\}$*. Then, for the set* $S = (R \times B) \cup (P \times \{1\})$ *the Cayley graph*

$$Cay(A \times_\theta B, S)$$

is a DSRG with parameters above.

Proof. Let the set S be $(R \times B) \cup (P \times \{1\})$. Then $|S| = k = 2|R| + |P| = 2 \cdot [(p^2 - p)/2] + p = p^2$. Our goal is to show that the graph $Cay(G, S)$ is a DSRG with parameters (n, k, t, λ, μ). So, we need to show that the summation $\underline{S} = \sum_{s \in S} s$ is valid in the following equation in $\mathbb{Z}[G]$,

$$\underline{S}^2 = te + \lambda \underline{S} + \mu(\underline{G} - e - \underline{S}).$$

To do that it will be enough to show that $\underline{S}$ satisfies the equation

$$\underline{S}^2 + |P|\underline{S} = \mu \underline{G}.$$

By Lemma 1 and Lemma 2 we get,

$$\begin{aligned}
\underline{S}^2 + |P|\underline{S} &= (\underline{(R \times B) \cup (P \times \{1\})})^2 + |P|(\underline{(R \times B) \cup (P \times \{1\})}) \\
&= \underline{(R \times B) \times_\theta (R \times B)} + \underline{(P \times \{1\}) \times_\theta (P \times \{1\})} + \underline{(R \times B) \times_\theta (P \times \{1\})} + \\
&\quad \underline{(P \times \{1\}) \times_\theta (R \times B)} + |P|\underline{(R \times B)} + |P|\underline{(P \times \{1\})} \\
&= ((p^2 - 3p)/2)\underline{(R \times B)} + ((p^2 - p)/2)\underline{(R^p \times B)} + ((p^2 - p)/2)\underline{(P \times B)} + \\
&\quad p\underline{(P \times \{0\})} + p\underline{(R \times B)} + p\underline{(R^p \times B)} + p\underline{(R \times B)} + p\underline{(P \times \{1\})} \\
&= p\underline{G} + ((p^2 - p)/2)\underline{G} \\
&= ((p^2 + p)/2)\underline{G} = \mu \underline{G}
\end{aligned}$$

as required. ∎

Example 2. Let $p = 3$, $A = \mathbb{F}_{p^2}$, $B = \mathbb{F}_2$. Consider the Frobenius automorphism

$$\beta : \mathbb{F}_{p^2} \to \mathbb{F}_{p^2}$$
$$\beta(x) = x^p.$$

For $G = A \times B$, $(G, \ltimes)$ forms a group of order $2p^2$. The product of (a_1, b_1) and (a_2, b_2) is given by

$$(a_1, b_1) \ltimes (a_2, b_2) = \begin{cases} (a_1 + a_2, b_2), & \text{if } b_1 = 0, \\ (a_1 + a_2^p, b_2 + 1), & \text{if } b_1 = 1. \end{cases}$$

Similarly, the inverse of (a, b) is given by

$$(a,b)^{-1} = \begin{cases} (-a,-b), & \text{if } b=0, \\ ((-a)^p,-b), & \text{if } b=1. \end{cases}$$

Thus the orbits of β are $\{0\}, \{a, 2a+1\}, \{a+1, 2a+2\}, \{2\}, \{a+2, 2a\}, \{1\}$.

From Theorem 1, multiplying one-element orbits by $\{1\}$ and two-element orbits by the set B, we construct the set $S = \{(a,0),(a,1),(a+1,0),(a+1,1),(a+2,0),(a+2,1),(0,1),(1,1),(2,1)\}$. Then the Cayley graph $Cay(A\times B, S)$ is a directed strongly regular graph with parameters $(18,9,6,3,6)$.

4. Semidihedral Construction of Cayley DSRG

In this section, we will construct directed strongly regular graphs from semidihedral groups by using Cayley graphs. The method of producing DSRG's using semidihedral groups in this section is different from the semidirect method given in Section 3. The choice of our generator set S here is independent of the q-orbit condition. A semidihedral group $SD(m)$ is also an example of the semidirect product of cyclic group C_2 with the dihedral group. But in this construction C_2 acts on $C_{2^{m-1}}$ by $x \rightarrowtail x^{2^{m-2}-1}$ instead of $x \rightarrowtail x^{-1}$. Before we give the main theorem, we need the following lemma.

Lemma 3. *Let $G = SD(m) = \langle a, x | a^{2^{m-1}} = x^2 = e, xax = a^{2^{m-2}-1}\rangle$ be the semidihedral group of order $m \geq 4$. Let $P = P_1 \cup P_2$ where $P_i = \{a^{i+4k} : k = 0,1,...,2^{m-3}-1\}$. Then*

$$xP = P'x \text{ where } P' = P_2 \cup P_3.$$

Proof. Let $P = P_1 \cup P_2$. By multiplying both sides of this equality by x, we get

$$\begin{aligned} xP &= xP_1 \cup xP_2 \\ &= \{xa^{1+4k} : k = 0,1,...,2^{m-3}-1\} \cup \{xa^{2+4k} : k = 0,1,...,2^{m-3}-1\} \\ &= \{a^{(1+4k)\cdot(2^{m-2}-1)}x : k = 0,1,...,2^{m-3}-1\} \\ &\cup \{a^{(2+4k)\cdot(2^{m-2}-1)}x : k = 0,1,...,2^{m-3}-1\}. \end{aligned} \tag{1}$$

Since the power of a in P_1 and P_2 is $1 \mod 4$, $2 \mod 4$ respectively and $m \geq 4$, if we multiply the powers of a by $2^{m-2}-1$ we will have

$$\begin{aligned} 1+4k &\equiv 1(\mod 4) \\ (1+4k)\cdot(2^{m-2}-1) &\equiv 2^{m-2}-1(\mod 4) \\ &\equiv -1(\mod 4) \\ &\equiv 3(\mod 4) \end{aligned} \tag{2}$$

and

$$\begin{aligned} 2+4k &\equiv 2(\mod 4) \\ (2+4k)\cdot(2^{m-2}-1) &\equiv 2^{m-1}-2(\mod 4) \\ &\equiv -2(\mod 4) \\ &\equiv 2(\mod 4). \end{aligned} \tag{3}$$

Therefore, using Equations (2) and (3) in Equation (1), we will have the following

$$\begin{aligned} &\{a^{(1+4k)\cdot(2^{m-2}-1)}x : k=0,1,...,2^{m-3}-1\} \cup \{a^{(2+4k)\cdot(2^{m-2}-1)}x : k=0,1,...,2^{m-3}-1\} \\ &= \{a^{3+4k}x : k=0,1,...,2^{m-3}-1\} \cup \{a^{2+4k}x : k=0,1,...,2^{m-3}-1\} \\ &= P_3x \cup P_2x = (P_2 \cup P_3)x = P'x. \end{aligned}$$

This completes the proof. □

Note that we also have the equations $xP_1 = P_3x$ ($P_1x = xP_3$) and $xP_2 = P_2x$.

Theorem 2. *Let $G = SD(m) = \langle a, x | a^{2^{m-1}} = x^2 = e, xax = a^{2^{m-2}-1}\rangle$ be the semidihedral group of order $m \geq 4$. Let $P = P_1 \cup P_2$ where $P_i = \{a^{i+4k} : k = 0,1,...,2^{m-3}-1\}$. Then $Cay(G, P \cup xP)$ is a DSRG with parameters*

$$(n = 2^m, k = 2^{m-1}, t = 3.2^{m-3}, \lambda = 2^{m-3}, \mu = 3.2^{m-3}).$$

Proof. Let $S = P \cup xP$. Then the parameter $k = |S| = 2|P| = 2\cdot 2^{m-2} = 2^{m-1}$. Our goal is to show that $Cay(G, S)$ is a DSRG with parameters (n, k, t, λ, μ). Thus the formal sum $\underline{S} = \sum_{s\in S} s$ should satisfy the equation

$$\underline{S}^2 = te + \lambda\underline{S} + \mu(\underline{G} - e - \underline{S})$$

in the group ring $\mathbb{Z}[G]$. Therefore, we need to show that the equation

$$\underline{S}^2 + 2^{m-2}\underline{S} = 3\cdot 2^{m-3}\underline{G}$$

holds. So,

$$\begin{aligned} \underline{S}^2 + 2^{m-2}\underline{S} &= (\underline{P} + \underline{xP})^2 + 2^{m-2}(\underline{P} + \underline{xP}) \\ &= \underline{P}^2 + \underline{P}\cdot\underline{xP} + \underline{xP}\cdot\underline{P} + \underline{xP}\cdot\underline{xP} + 2^{m-2}(\underline{P} + \underline{xP}) \\ &= \underline{P}^2 + \underline{xP'}\cdot\underline{P} + \underline{xP}\cdot\underline{P} + \underline{P'}\cdot\underline{P} + 2^{m-2}(\underline{P} + \underline{xP}) \end{aligned} \tag{4}$$

where $P' = P_2 \cup P_3$ by Lemma 3.

In order to complete the proof let us compute P_iP_i and P_jP_j. Since P_0 is a subgroup of order 2^{m-3} and $P_1 = aP_0$, $P_2 = a^2P_0$ and $P_3 = a^3P_0$ are its cosets, we have

$$\begin{aligned} \underline{P_iP_i} &= \underline{a^{2i}P_0P_0} = |P_0|\underline{P_{2i}} \\ \underline{P_i}\underline{P_j} &= \underline{a^{i+j}P_0P_0} = |P_0|\underline{P_{i+j}}. \end{aligned}$$

It follows that

$$\begin{aligned}\underline{P}^2 &= \underline{P_1P_1} + \underline{P_1P_2} + \underline{P_2P_1} + \underline{P_2P_2}\\ &= |P_0|\underline{P_2} + |P_0|\underline{P_3} + |P_0|\underline{P_3} + |P_0|\underline{P_4}\\ &= |P_0|\underline{P_2} + 2\cdot|P_0|\underline{P_3} + |P_0|\underline{P_0},\end{aligned}$$

and

$$\begin{aligned}\underline{P'P} &= \underline{P_2P_1} + \underline{P_2P_2} + \underline{P_3P_1} + \underline{P_3P_2}\\ &= |P_0|\underline{P_3} + |P_0|\underline{P_0} + |P_0|\underline{P_0} + |P_0|\underline{P_1}\\ &= |P_0|\underline{P_3} + 2\cdot|P_0|\underline{P_0} + |P_0|\underline{P_1}.\end{aligned}$$

Now it only remains to write them in Equation (4) :

$$\begin{aligned}\underline{S}^2 + 2^{m-2}\underline{S} &= (\underline{P} + \underline{xP})^2 + 2^{m-2}(\underline{P} + \underline{xP})\\ &= \underline{P}^2 + \underline{PxP} + \underline{xPP} + \underline{xPxP} + 2^{m-2}(\underline{P} + \underline{xP})\\ &= \underline{P}^2 + \underline{xP'P} + \underline{xPP} + \underline{P'P} + 2^{m-2}(\underline{P} + \underline{xP})\\ &= \underline{P}^2 + \underline{P'P} + (2\cdot|P_0|)\underline{P} + x(\underline{P}^2 + \underline{P'P} + (2\cdot|P_0|)\underline{P})\\ &= |P_0|\underline{P_2} + 2\cdot|P_0|\underline{P_3} + |P_0|\underline{P_0} + |P_0|\underline{P_3} + 2\cdot|P_0|\underline{P_0} + |P_0|\underline{P_1}\\ &+ 2\cdot|P_0|\underline{P_1} + 2\cdot|P_0|\underline{P_2} + x(|P_0|\underline{P_2} + 2\cdot|P_0|\underline{P_3} + |P_0|\underline{P_0} + |P_0|\underline{P_3}\\ &+ 2\cdot|P_0|\underline{P_0} + |P_0|\underline{P_1} + 2\cdot|P_0|\underline{P_1} + 2\cdot|P_0|\underline{P_2})\\ &= 3\cdot|P_0|(\underline{P_0} + \underline{P_1} + \underline{P_2} + \underline{P_3} + x\underline{P_0} + x\underline{P_1} + x\underline{P_2} + x\underline{P_3})\\ &= 3\cdot(2^{m-3})\cdot\underline{G}.\end{aligned}$$

This completes the proof. □

Example 3. *Let $G = SD(4)$ be the semidihedral group of order 4 for $m = 4$ with elements $\{e, a, a^2, a^3, a^4, a^5, a^6, a^7, x, xa, xa^2, xa^3, xa^4, xa^5, xa^6, xa^7\}$. Construct the subset S according to Theorem 2 as $\{P \cup xP\}$ where $P = \{a, a^2, a^5, a^6\}$. Then $Cay(G, S)$ is a DSRG with parameters $(16, 8, 6, 2, 6)$.*

Remark 1. *The directed strongly regular graph constructed in the Example 3 has already been presented in [2] by Duval. The author constructed the DSRG with parameters $(16, 8, 2, 6, 2)$ from a DSRG with parameters $(8, 4, 1, 3, 1)$ known to exist. This construction is specified as $T10$ in [1].*

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