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Fourier Transform of Orthogonal Polynomials over the Triangle with Four Parameters

Esra Güldoğan Lekesiz

Faculty of Engineering, Ostim Teknik University, Ostim 06374, Ankara, Turkey.

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Abstract. In this paper, some new families of orthogonal functions in two variables produced by using Fourier transform of bivariate orthogonal polynomials and their orthogonality relations obtained from Parseval identity are introduced.

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1. Introduction

It is known that Gamma and the Beta functions have the following definitions [7]

$$\Gamma(w) = \int_0^\infty z^{w-1} e^{-z} dz \ , \ \Re(w) > 0,$$

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \ , \ \Re(p), \Re(q) > 0,$$

respectively. With the help of the definition of Gamma function, we have [24]

$$\Gamma(w+s) = (w)_s \Gamma(w) \text{ and } \Gamma(w-s) = \frac{(-1)^s \Gamma(w)}{(1-w)_s}.$$

Also, the (p,q) generalized hypergeometric function are defined by

$$_pF_q\left(a_1, a_2, ..., a_p; b_1, b_2, ..., b_q; x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots \left(a_p\right)_k}{(b_1)_k (b_2)_k \dots \left(b_q\right)_k} \frac{x^k}{k!}, \tag{1.1}$$

where $(a)_k = \prod_{r=0}^{k-1}(a+r)$, $(a)_0 = 1$, $k \in \mathbb{N}_0$ denotes the Pochhammer symbol.

Let Λ_r represent the space of orthogonal polynomials of degree r with respect to $\langle .,. \rangle$. Several univariate orthogonal polynomials systems are mapped onto each other by the Fourier transform or other integral transforms [8, 20]. For example, Koelink [11] showed that Jacobi polynomials are mapped onto the continuous Hahn polynomials by the

Email address: esragldgn@gmail.com (E. Güldoğan Lekesiz)

Fourier transform and similarly Hermite functions are eigenfunctions of Fourier transform [11, 12, 14, 15]. By getting inspired of Koelink's paper, Masjed-Jamei et al. [19] set two new families of orthogonal functions by using Fourier transforms of the generalized ultraspherical polynomials [21, p.147] and the generalized Hermite polynomials, and by applying the Parseval identity. Also, the Fourier transform of Routh-Romanovski polynomials $J_r^{(\zeta,\eta)}(x;\xi_1,\xi_2,\xi_3,\xi_4)$ is given [18].

As known, the univariate Jacobi polynomial $P_r^{(\zeta,\eta)}(x)$ is one of the classical orthogonal polynomials. The series representation for the polynomial set is as follows

$$P_r^{(\zeta,\eta)}(x) = 2^{-r}\sum_{l=0}^{r}\binom{r+\zeta}{l}\binom{r+\eta}{r-l}(x+1)^l(x-1)^{r-l}.$$

They are orthogonal on the interval $(-1,1)$ with respect to the weight function $\omega(x) = (1-x)^\zeta(1+x)^\eta$. More precisely, the orthogonality relation is

$$\int_{-1}^{1}(1-x)^\zeta(1+x)^\eta P_r^{(\zeta,\eta)}(x)P_m^{(\zeta,\eta)}(x)\,dx = \frac{2^{\zeta+\eta+1}\Gamma(\zeta+r+1)\Gamma(\eta+r+1)\delta_{r,m}}{r!(\zeta+\eta+2r+1)\Gamma(\zeta+\eta+r+1)},$$

where $\min\{\Re(\zeta), \Re(\eta)\} > -1$, $r,m \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$, and $\delta_{r,m}$ is the Kronecker delta [24–26].

The Jacobi polynomials on the interval $(0,1)$ which are actually shifted univariate Jacobi polynomials [16], also can be denoted as $\widetilde{P}_r^{(\zeta,\eta)}(x) := P_r^{(\zeta,\eta)}(2x-1)$,

$$\widetilde{P}_r^{(\zeta,\eta)}(x) = \frac{(-1)^r}{r!}(1-x)^{-\zeta}x^{-\eta}\frac{d^r}{dx^r}\left\{(1-x)^{r+\zeta}x^{r+\eta}\right\}, \quad r \geq 0$$

and are orthogonal with respect to the weight function $\widetilde{\omega}(x) = (1-x)^\zeta x^\eta$ for $\zeta,\eta > -1$. Thus, we get

$$\int_0^1 (1-x)^\zeta x^\eta \widetilde{P}_r^{(\zeta,\eta)}(x)\widetilde{P}_m^{(\zeta,\eta)}(x)\,dx = h_r^{(\zeta,\eta)}\delta_{r,m},$$

where

$$h_r^{(\zeta,\eta)} = \frac{\Gamma(r+\zeta+1)\Gamma(r+\eta+1)}{r!(2r+\zeta+\eta+1)\Gamma(r+\zeta+\eta+1)}.$$

We now remind orthogonal polynomials in two variables. Let Υ be the set of all two-variable polynomials and Υ_r represent the linear space of two-variable polynomials with total degree at most r.

If

$$\left\langle \upsilon,\tau\right\rangle := \int_\Omega \upsilon(x,y)\tau(x,y)\omega(x,y)dxdy = 0$$

holds for $\forall\tau \in \Upsilon_{r-1}$, $\upsilon \in \Upsilon_r$ is an orthogonal polynomial with respect to the weight function $\omega(x,y)$.

In 1975, by using an approach given by Agahanov [1], the method for obtaining two-variable generalizations of the one-variable orthogonal polynomials is given by Koornwinder [13]. By the application of his method, he presented some examples of two-variable orthogonal polynomials [13]. Fernandez et al. [9] have introduced some new families of Koornwinder polynomials, like Laguerre-Laguerre Koornwinder and Laguerre-Jacobi Koornwinder polynomials, by use of Koornwinder's method. Recently, there are many research on Koornwinder polynomials [2–5, 9, 10, 17, 22].

One of these polynomials is Jacobi polynomials over the triangle. The Jacobi polynomials over the triangle [13]

$$\begin{aligned} P_{r,k}^{(\mu,\gamma,\lambda)}(x,y) &= P_{r-k}^{(2k+\gamma+\lambda+1,\mu)}(2x-1)(1-x)^k P_k^{(\lambda,\gamma)}\left(\frac{2y}{1-x}-1\right) \\ &= \widetilde{P}_{r-k}^{(2k+\gamma+\lambda+1,\mu)}(x)(1-x)^k \widetilde{P}_k^{(\lambda,\gamma)}\left(\frac{y}{1-x}\right), (k=0,1,...,r, \quad r=0,1,...) \end{aligned} \tag{1.2}$$

were obtained from the corresponding univariate polynomials by using the method introduced by Koornwinder. These are orthogonal on the two-dimensional simplex

$$T^2 := \{(x,y) \in \mathbb{R} : x,y > 0, 1-x-y > 0\},$$

with respect to the weight function $\omega(x,y) = x^{\mu}y^{\gamma}(1-x-y)^{\lambda}$, where $\mu,\gamma,\lambda > -1$,

$$\iint\limits_{T^2} x^{\mu}y^{\gamma}(1-x-y)^{\lambda} P_{r,k}^{(\mu,\gamma,\lambda)}(x,y) P_{m,s}^{(\mu,\gamma,\lambda)}(x,y)\,dxdy$$

$$= \frac{\Gamma(r-k+\mu+1)\Gamma(r+k+\gamma+\lambda+2)\Gamma(k+\gamma+1)\Gamma(k+\lambda+1)\delta_{r,m}\delta_{k,s}}{(r-k)!k!(2r+\mu+\gamma+\lambda+2)(2k+\gamma+\lambda+1)\Gamma(r+k+\mu+\gamma+\lambda+2)\Gamma(k+\gamma+\lambda+1)}.$$

In [23], Olver et al. presented a bivariate four-parameter variant of the Koornwinder polynomials on the triangle defined by

$$P_{r,k}^{(\mu,\gamma,\lambda,\theta)}(x,y) = \widetilde{P}_{r-k}^{(2k+\gamma+\lambda+\theta+1,\mu)}(x)(1-x)^k \widetilde{P}_k^{(\lambda,\gamma)}\left(\frac{y}{1-x}\right), \tag{1.3}$$

where $\mu,\gamma,\lambda,\theta > -1$, r and k are integers satisfying $0 \le k \le r$. These polynomials are orthogonal with respect to the weight function $\omega(x,y) = x^{\mu}y^{\gamma}(1-x-y)^{\lambda}(1-x)^{\theta}$ on the two-dimensional simplex T^2. Consequently,

$$\iint\limits_{T^2} x^{\mu}y^{\gamma}(1-x-y)^{\lambda}(1-x)^{\theta} P_{r,k}^{(\mu,\gamma,\lambda,\theta)}(x,y) P_{m,s}^{(\mu,\gamma,\lambda,\theta)}(x,y)\,dxdy$$

$$= \frac{\Gamma(r-k+\mu+1)\Gamma(r+k+\gamma+\lambda+\theta+2)}{(r-k)!k!(2r+\mu+\gamma+\lambda+\theta+2)(2k+\gamma+\lambda+1)} \frac{\Gamma(k+\gamma+1)\Gamma(k+\lambda+1)\delta_{r,m}\delta_{k,s}}{\Gamma(r+k+\mu+\gamma+\lambda+\theta+2)\Gamma(k+\gamma+\lambda+1)}.$$

It should be noted that, when $\theta = 0$ in (1.3), it gives the bivariate Koornwinder polynomials over the triangle defined in (1.2).

Remark 1.1. By substituting $x \to 1-x$, $y \to x-y$, the polynomial $P_{r,k}^{(\mu,\gamma,\lambda,\theta)}(x,y)$ can be written as

$$^{*}P_{r,k}^{(\mu,\gamma,\lambda,\theta)}(x,y) = P_{r,k}^{(\mu,\gamma,\lambda,\theta)}(1-x,x-y) := (-1)^r \widetilde{P}_{r-k}^{(\mu,2k+\gamma+\lambda+\theta+1)}(x)\, x^k \widetilde{P}_k^{(\gamma,\lambda)}\left(\frac{y}{x}\right). \tag{1.4}$$

These polynomials are orthogonal with respect to the weight function $\omega(x,y) = (1-x)^{\mu}(x-y)^{\gamma}y^{\lambda}x^{\theta}$ over the triangle $T^* := \{(x,y) \in \mathbb{R} : 0 < y < x < 1\}$. Moreover, it has the orthogonality relation

$$\iint\limits_{T^*} x^{\theta}(1-x)^{\mu}y^{\lambda}(x-y)^{\gamma}\ {}^{*}P_{r,k}^{(\mu,\gamma,\lambda,\theta)}(x,y)\ {}^{*}P_{m,s}^{(\mu,\gamma,\lambda,\theta)}(x,y)\,dxdy \tag{1.5}$$

$$= \frac{\Gamma(r-k+\mu+1)\Gamma(r+k+\gamma+\lambda+\theta+2)}{(r-k)!k!(2r+\mu+\gamma+\lambda+\theta+2)(2k+\gamma+\lambda+1)} \frac{\Gamma(k+\gamma+1)\Gamma(k+\lambda+1)\delta_{r,m}\delta_{k,s}}{\Gamma(r+k+\mu+\gamma+\lambda+\theta+2)\Gamma(k+\gamma+\lambda+1)}.$$

In [10], Güldoğan et al. had given the special functions derived from Fourier transform of bivariate orthogonal polynomials on the triangle with three parameters.

The main object of this study is to take into account new families of two-variable orthogonal functions by using the Fourier transforms of the two-variable orthogonal polynomials over the triangle with four parameters denoted by $^{*}P_{r,k}^{(\mu,\gamma,\lambda,\theta)}(x,y)$. By applying the Parseval identity their orthogonality relations are attained.

2. A Family of Functions Obtained from Orthogonal Polynomials over the Triangle with Four Parameters

The next section includes the principal results of this study. Here new families of bivariate orthogonal functions by applying the Fourier transforms of the bivariate orthogonal polynomials with four parameters $^{*}P_{r,k}^{(\mu,\gamma,\lambda,\theta)}(x,y)$ and the orthogonality relations are then obtained by using the Parseval identity.

The Fourier transform of a bivariate function $\tau(x,y)$ is defined as [6]

$$\mathcal{F}(\tau(x,y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\rho_1 x+\rho_2 y)}\tau(x,y)\,dxdy$$

and the Parseval identity of Fourier theory is given by the expression

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tau(x,y)\overline{\upsilon(x,y)}dydx = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{F}(\tau(x,y))\overline{\mathcal{F}(\upsilon(x,y))}d\rho_1 d\rho_2. \tag{2.1}$$

Now, to be able to use the orthogonality relation (1.5) let us consider the function

$$\tau_{r,k}(x,y;\xi_1,\xi_2,\xi_3,\xi_4,\mu,\gamma,\lambda,\theta) = (1-\tanh x)^{\xi_1}(1+\tanh x)^{\xi_2+\xi_3+\xi_4+1/2}$$
$$\times(1-\tanh y)^{\xi_2}(1+\tanh y)^{\xi_3}\frac{e^{x+y}}{(e^{2x}+1)(e^{2y}+1)}\,{}^{*}P_{r,k}^{(\mu,\gamma,\lambda,\theta)}\left(\frac{1+\tanh x}{2},\frac{(1+\tanh x)(1+\tanh y)}{4}\right),$$

which is in terms of polynomials ${}^{*}P_{r,k}^{(\mu,\gamma,\lambda,\theta)}$ defined in (1.4), where $\xi_1,\xi_2,\xi_3,\xi_4,\mu,\gamma,\lambda,\theta$ are real parameters.

By applying the suitable substitutions we calculate the Fourier transform of the function as follows

$$\begin{aligned}
&\mathcal{F}\left(\tau_{r,k}(x,y;\xi_1,\xi_2,\xi_3,\xi_4,\mu,\gamma,\lambda,\theta)\right)\\
=&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\rho_1x+\rho_2y)}\tau_{r,k}(x,y;\xi_1,\xi_2,\xi_3,\xi_4,\mu,\gamma,\lambda,\theta)\,dxdy\\
=&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1-\tanh x)^{\xi_1}(1+\tanh x)^{\xi_2+\xi_3+\xi_4+1/2}(1-\tanh y)^{\xi_2}(1+\tanh y)^{\xi_3}\\
&\times\frac{e^{x+y-i(\rho_1x+\rho_2y)}}{(e^{2x}+1)(e^{2y}+1)}P_{r,k}^{(\mu,\gamma,\lambda,\theta)}\left(\frac{1+\tanh x}{2},\frac{(1-\tanh x)(1+\tanh y)}{4}\right)dxdy\\
=&\frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}(1-u)^{\xi_1+\frac{i\rho_1-1}{2}}(1+u)^{\xi_2+\xi_3+\xi_4-\frac{i\rho_1}{2}}(1-v)^{\xi_2+\frac{i\rho_2-1}{2}}(1+v)^{\xi_3-\frac{i\rho_2+1}{2}}\,{}^{*}P_{r,k}^{(\mu,\gamma,\lambda,\theta)}\left(\frac{1+u}{2},\frac{(1+u)(1+v)}{4}\right)dudv\\
=&(-1)^n\,2^{\xi_1+2(\xi_2+\xi_3)+\xi_4-3/2}\int_0^1(1-t)^{\xi_1+\frac{i\rho_1-1}{2}}\,t^{k+\xi_2+\xi_3+\xi_4-\frac{i\rho_1}{2}}\\
&\times P_{r-k}^{(\mu,2k+\gamma+\lambda+\theta+1)}(2t-1)\,dt\int_0^1(1-w)^{\xi_2+\frac{i\rho_2-1}{2}}\,w^{\xi_3-\frac{i\rho_2+1}{2}}P_k^{(\gamma,\lambda)}(2w-1)\,dw\\
=&2^{\xi_1+2(\xi_2+\xi_3)+\xi_4-3/2}\sum_{l_1=0}^{r-k}(-1)^{l_1}\binom{r-k+\mu}{l_1}\binom{r+k+\gamma+\lambda+\theta+1}{r-k-l_1}\int_0^1(1-t)^{r-k+\xi_1+\frac{i\rho_1-1}{2}-l_1}\,t^{k+\xi_2+\xi_3+\xi_4-\frac{i\rho_1}{2}+l_1}dt\\
&\times\sum_{l_2=0}^{k}(-1)^{l_2}\binom{k+\gamma}{l_2}\binom{k+\lambda}{k-l_2}\int_0^1(1-w)^{k+\xi_2+\frac{i\rho_2-1}{2}-l_2}\,w^{\xi_3-\frac{i\rho_2+1}{2}+l_2}dw\\
=&2^{\xi_1+2(\xi_2+\xi_3)+\xi_4-3/2}\frac{\Gamma(r+k+\gamma+\lambda+\theta+2)(\lambda+1)_k}{(r-k)!k!\Gamma(2k+\gamma+\lambda+\theta+2)}\\
&\times\sum_{l_1=0}^{r-k}\frac{(-(r-k))_{l_1}(-(r-k+\mu))_{l_1}}{(2k+\gamma+\lambda+\theta+2)_{l_1}\,l_1!}\frac{\Gamma\left(k+\xi_2+\xi_3+\xi_4-\frac{i\rho_1}{2}+1+l_1\right)\Gamma\left(r-k+\xi_1+\frac{i\rho_1+1}{2}-l_1\right)}{(-1)^{l_1}\Gamma(r+\xi_1+\xi_2+\xi_3+\xi_4+3/2)}\\
&\times\sum_{l_2=0}^{k}\frac{(-k)_{l_2}(-(k+\gamma))_{l_2}}{(-1)^{l_2}(\lambda+1)_{l_2}\,l_2!}\frac{\Gamma\left(k+\xi_2+\frac{i\rho_2+1}{2}-l_2\right)\Gamma\left(\xi_3+\frac{1-i\rho_2}{2}+l_2\right)}{\Gamma(k+\xi_2+\xi_3+1)}\\
=&\frac{2^{\xi_1+2(\xi_2+\xi_3)+\xi_4-3/2}\Gamma(r+k+\gamma+\lambda+\theta+2)(\lambda+1)_k}{(r-k)!k!\Gamma(2k+\gamma+\lambda+\theta+2)}\\
&\times\Psi(r,k,\xi_1,\xi_2,\xi_3,\xi_4,\rho_1,\rho_2)\,\Phi(r,k,\mu,\gamma,\lambda,\theta,\xi_1,\xi_2,\xi_3,\xi_4,\rho_1,\rho_2),
\end{aligned}$$

where

$$\begin{aligned}
\Psi(r,k,\xi_1,\xi_2,\xi_3,\xi_4,\rho_1,\rho_2) &= B\left(k+\xi_2+\frac{1+i\rho_2}{2},\ \xi_3+\frac{1-i\rho_2}{2}\right)\\
&\times B\left(r-k+\xi_1+\frac{1+i\rho_1}{2},\ k+\xi_2+\xi_3+\xi_4+1-\frac{i\rho_1}{2}\right)
\end{aligned}$$

and

$$\Phi(r,k,\mu,\gamma,\lambda,\theta,\xi_1,\xi_2,\xi_3,\xi_4,\rho_1,\rho_2) ={}_3F_2\left(-k,-(k+\gamma),\xi_3+\frac{1-i\rho_2}{2};-k-\xi_2-\frac{i\rho_2-1}{2},\lambda+1;1\right)$$
$$\times{}_3F_2\left(-(r-k),-(r-k+\mu),k+\xi_2+\xi_3+\xi_4+1-\frac{i\rho_1}{2};-r+k-\xi_1-\frac{i\rho_1-1}{2},2k+\gamma+\lambda+\theta+2;1\right),$$

${}_3F_2$ is a special case of the generalized hypergeometric function defined by (1.1). By using the Parseval identity (2.1) it produces

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tau_{r,k}(x,y;\xi_1,\xi_2,\xi_3,\xi_4,\mu,\gamma,\lambda,\theta)\,\tau_{m,s}(x,y;\chi_1,\chi_2,\chi_3,\chi_4,\beta,\eta,\alpha,\delta)\,dxdy\\
=&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1-\tanh x)^{\xi_1+\chi_1}(1+\tanh x)^{\xi_2+\chi_2+\xi_3+\chi_3+\xi_4+\chi_4+1}(1-\tanh y)^{\xi_2+\chi_2}(1+\tanh y)^{\xi_3+\chi_3}\frac{e^{2(x+y)}}{(e^{2x}+1)^2(e^{2y}+1)^2}\\
&\times{}^*P_{r,k}^{(\mu,\gamma,\lambda,\theta)}\left(\frac{1+\tanh x}{2},\frac{(1+\tanh x)(1+\tanh y)}{4}\right){}^*P_{m,s}^{(\beta,\eta,\alpha,\delta)}\left(\frac{1+\tanh x}{2},\frac{(1+\tanh x)(1+\tanh y)}{4}\right)dydx\\
=&\frac{1}{4}\int_{-1}^{1}\int_{-\infty}^{\infty}(1-u)^{\xi_1+\chi_1}(1+u)^{\xi_2+\chi_2+\xi_3+\chi_3+\xi_4+\chi_4+1}(1-\tanh y)^{\xi_2+\chi_2}(1+\tanh y)^{\xi_3+\chi_3}\\
&\times\frac{e^{2y}}{(e^{2y}+1)^2}{}^*P_{r,k}^{(\mu,\gamma,\lambda,\theta)}\left(\frac{1+u}{2},\frac{(1+u)(1+\tanh y)}{4}\right){}^*P_{m,s}^{(\beta,\eta,\alpha,\delta)}\left(\frac{1+u}{2},\frac{(1+u)(1+\tanh y)}{4}\right)dydu\\
=&2^{\xi_1+\chi_1+\xi_2+\chi_2+\xi_3+\chi_3+\xi_4+\chi_4}\int_{0}^{1}\int_{-\infty}^{\infty}(1-w)^{\xi_1+\chi_1}w^{\xi_2+\chi_2+\xi_3+\chi_3+\xi_4+\chi_4+1}\\
&\times(1-\tanh y)^{\xi_2+\chi_2}(1+\tanh y)^{\xi_3+\chi_3}\frac{e^{2y}}{(e^{2y}+1)^2}{}^*P_{r,k}^{(\mu,\gamma,\lambda,\theta)}\left(w,\frac{w(1+\tanh y)}{2}\right){}^*P_{m,s}^{(\beta,\eta,\alpha,\delta)}\left(w,\frac{w(1+\tanh y)}{2}\right)dydw\\
=&2^{\xi_1+\chi_1+\xi_2+\chi_2+\xi_3+\chi_3+\xi_4+\chi_4-2}\int_{0}^{1}\int_{-w}^{w}(1-w)^{\xi_1+\chi_1}w^{\xi_4+\chi_4+1}(w-z)^{\xi_2+\chi_2}\\
&\times(w+z)^{\xi_3+\chi_3}\,{}^*P_{r,k}^{(\mu,\gamma,\lambda,\theta)}\left(w,\frac{w+z}{2}\right){}^*P_{m,s}^{(\beta,\eta,\alpha,\delta)}\left(w,\frac{w+z}{2}\right)dzdw\\
=&2^{\xi_1+\chi_1+2(\xi_2+\chi_2+\xi_3+\chi_3)+\xi_4+\chi_4-1}\int_{0}^{1}\int_{0}^{w}(1-w)^{\xi_1+\chi_1}(w-t)^{\xi_2+\chi_2}t^{\xi_3+\chi_3}w^{\xi_4+\chi_4+1}\,{}^*P_{r,k}^{(\mu,\gamma,\lambda,\theta)}(w,t)\,{}^*P_{m,s}^{(\beta,\eta,\alpha,\delta)}(w,t)\,dtdw,
\end{aligned}$$

that is,

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tau_{r,k}(x,y;\xi_1,\xi_2,\xi_3,\xi_4,\mu,\gamma,\lambda,\theta)\,\tau_{m,s}(x,y;\chi_1,\chi_2,\chi_3,\chi_4,\beta,\eta,\alpha,\delta)\,dxdy \qquad (2.2)\\
=&\frac{2^{\xi_1+\chi_1+2(\xi_2+\chi_2+\xi_3+\chi_3)+\xi_4+\chi_4-5}(\lambda+1)_k(\alpha+1)_s\Gamma(r+k+\gamma+\lambda+\theta+2)}{\pi^2(r-k)!(m-s)!k!s!\Gamma(2k+\gamma+\lambda+\theta+2)}\\
&\times\frac{\Gamma(m+s+\eta+\alpha+\delta+2)}{\Gamma(2s+\eta+\alpha+\delta+2)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{\Phi(m,s,\beta,\eta,\alpha,\delta,\chi_1,\chi_2,\chi_3,\chi_4,\rho_1,\rho_2)}\\
&\times\overline{\Psi(m,s,\chi_1,\chi_2,\chi_3,\chi_4,\rho_1,\rho_2)}\Psi(r,k,\xi_1,\xi_2,\xi_3,\xi_4,\rho_1,\rho_2)\,\Phi(r,k,\mu,\gamma,\lambda,\theta,\xi_1,\xi_2,\xi_3,\xi_4,\rho_1,\rho_2)\,d\rho_1d\rho_2.
\end{aligned}$$

Now, by taking $\xi_1+\chi_1=\mu=\beta$, $\xi_2+\chi_2=\gamma=\eta$, $\xi_2+\chi_3=\lambda=\alpha$ and $\xi_4+\chi_4+1=\theta=\delta$ in (2.2), we use the orthogonality relation for the polynomials in the left-hand side of (2.2), to arrive

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\overline{\Psi(m,s,\chi_1,\chi_2,\chi_3,\chi_4,\rho_1,\rho_2)\,\Phi(m,s,\mu,\gamma,\lambda,\theta,\chi_1,\chi_2,\chi_3,\chi_4,\rho_1,\rho_2)}$$
$$\times\Psi(r,k,\xi_1,\xi_2,\xi_3,\xi_4,\rho_1,\rho_2)\,\Phi(r,k,\mu,\gamma,\lambda,\theta,\xi_1,\xi_2,\xi_3,\xi_4,\rho_1,\rho_2)\,d\rho_1 d\rho_2$$
$$=\frac{2^4\pi^2(r-k)!k!}{(2r+\xi_1+\chi_1+\xi_2+\chi_2+\xi_3+\chi_3+\xi_4+\chi_4+3)(2k+\xi_2+\chi_2+\xi_3+\chi_3+1)}$$
$$\times\frac{\Gamma^2(2k+\xi_2+\chi_2+\xi_3+\chi_3+\xi_4+\chi_4+3)\,\Gamma(r-k+\xi_1+\chi_1+1)}{\Gamma(r+k+\xi_2+\chi_2+\xi_3+\chi_3+\xi_4+\chi_4+3)\,\Gamma(k+\xi_2+\chi_2+\xi_3+\chi_3+1)}$$
$$\times\frac{\Gamma(k+\xi_2+\chi_2+1)\,\Gamma(k+\xi_3+\chi_3+1)\,\delta_{r,m}\delta_{k,s}}{(\xi_3+\chi_3+1)_k^2\,\Gamma(r+k+\xi_1+\chi_1+\xi_2+\chi_2+\xi_3+\chi_3+\xi_4+\chi_4+3)}.$$

In the wake of this, we give the following theorem.

Theorem 2.1. *The family of functions*

$$E_{r,k}(x,y;\xi_1,\xi_2,\xi_3,\xi_4,\chi_4,\chi_3,\chi_2,\chi_1)=\left(\xi_1+\frac{1+x}{2}\right)_{r-k}\left(\xi_2+\xi_3+\xi_4+1-\frac{x}{2}\right)_k\left(\xi_2+\frac{1+y}{2}\right)_k$$
$$\times\Phi(r,k,\xi_1+\chi_1,\xi_2+\chi_2,\xi_3+\chi_3,\xi_4+\chi_4+1,\xi_1,\xi_2,\xi_3,\xi_4,-ix,-iy)$$

satisfies the orthogonality relation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Gamma\left(\xi_1+\frac{1+ix}{2}\right)\Gamma\left(\xi_2+\xi_3+\xi_4+1-\frac{ix}{2}\right)\Gamma\left(\xi_2+\frac{1+iy}{2}\right)\Gamma\left(\xi_3+\frac{1-iy}{2}\right)$$
$$\times\Gamma\left(\chi_1+\frac{1-ix}{2}\right)\Gamma\left(\chi_2+\chi_3+\chi_4+1+\frac{ix}{2}\right)\Gamma\left(\chi_2+\frac{1-iy}{2}\right)\Gamma\left(\chi_3+\frac{1+iy}{2}\right)$$
$$\times E_{r,k}(ix,iy;\xi_1,\xi_2,\xi_3,\xi_4,\chi_4,\chi_3,\chi_2,\chi_1)\,E_{m,s}(-ix,-iy;\chi_1,\chi_2,\chi_3,\chi_4,\xi_4,\xi_3,\xi_2,\xi_1)\,dxdy$$
$$=\frac{2^4\pi^2(r-k)!k!\Gamma^2(2k+\xi_2+\chi_2+\xi_3+\chi_3+\xi_4+\chi_4+3)}{(2r+\xi_1+\chi_1+\xi_2+\chi_2+\xi_3+\chi_3+\xi_4+\chi_4+3)}$$
$$\times\frac{\Gamma(r-k+\xi_1+\chi_1+1)\,\Gamma(k+\xi_2+\chi_2+1)\,\Gamma(k+\xi_3+\chi_3+1)}{(2k+\xi_2+\chi_2+\xi_3+\chi_3+1)\,\Gamma(k+\xi_2+\chi_2+\xi_3+\chi_3+1)}$$
$$\times\frac{\Gamma(k+\xi_2+\xi_3+1)\,\Gamma(r+\xi_1+\xi_2+\xi_3+\xi_4+3/2)}{(\xi_3+\chi_3+1)_k^2\,\Gamma(r+k+\xi_2+\chi_2+\xi_3+\chi_3+\xi_4+\chi_4+3)}$$
$$\times\frac{\Gamma(k+\chi_2+\chi_3+1)\,\Gamma(r+\chi_1+\chi_2+\chi_3+\chi_4+3/2)\,\delta_{r,m}\delta_{k,s}}{\Gamma(r+k+\xi_1+\chi_1+\xi_2+\chi_2+\xi_3+\chi_3+\xi_4+\chi_4+3)}$$

for $\xi_1,\chi_1,\xi_2,\chi_2,\xi_3,\chi_3>-1/2$, $\xi_4,\chi_4>0$ *and* $\xi_2+\chi_2+\xi_3+\chi_3>-(k+1)$.

Remark 2.2. The weight function of this orthogonality relation is positive for $\xi_1=\chi_1$, $\xi_2=\chi_2$, $\xi_3=\chi_3$, $\xi_4=\chi_4$.

Corollary 2.3. *In case $\theta=\delta=0$, the obtained results give the results which are presented in [10] for polynomial $P_{r,k}^{(\mu,\gamma,\lambda)}(x,y)$ in [13].*

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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