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Quasi-Focal Curves of Spacelike Adjoint Curves in 3D Minkowski Space

Talat KÖRPINAR*[1], Ahmet SAZAK[1]

Abstract

Associated curves bring meaningful geometric expression to the physics and mathematics fields in the characterization of curves and surfaces created by those curves, their behavior, and the study of particle motion in space-time. In this work, we examine the relationship between adjoint curves and focal curves, which are two important examples of associated curves. We do this review for spacelike curves under quasi(q)-frame in 3D Minkowski space. To put it more clearly, we obtain some new characterizations by defining the focal curve of the adjoint curve of a spacelike curve in this space.

Keywords: Quasi-frame, spacelike curve, focal curve, adjoint curve.

1. INTRODUCTION

Associated curves provide meaningful mathematical expressions in the study of the characterization of curves and surfaces, their behavior, and the particle motion defined by such curves. Fundamentally, it allows a second curve geometrically related to a curve to be defined with the help of the main curve. One of the most obvious examples of associated curves is integral curves, which is a superscript of adjoint curves [1-6]. Integral curves are important in terms of the possibilities they provide for the solution of some differential equations that we encounter in geometric problems. The adjoint curves that we will consider in our study are the curves determined by the integral of the binormal vector field of a curve. For more information on such curves, see [7-13].

Another curve that we will consider in our study is the focal curve. The focal curve is determined by the center points of the imaginary spheres that oscillate tangentially along a selected curve with arc length parameters. Let this chosen curve be denoted by ϑ. In this case, the focal curve of ϑ is given as

$$\vartheta_F(s) = (\vartheta + \omega_1 \boldsymbol{N} + \omega_2 \boldsymbol{B})(s), \tag{1}$$

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where $\boldsymbol{B}$ and $\boldsymbol{N}$ are the vector fields that are binormal and normal elements for the Frenet-Serret (F-S) frame, s is the arc-length parameter and, ω_1 and ω_2 are focal curvatures of the curve ψ [7].

Based on these considerations, the goal of our article is to describe and characterize a new curve created by the center points of an imaginary sphere that oscillates tangential to a curve adjoining the principal curve. In this study, we are doing this analysis for a spacelike curve that we have chosen under the quasi(q)-frame in M_1^3 space. First of all, we define the adjoint curve of the curve we have chosen and calculate the expressions of the q-frame elements of these two curves in terms of each other. Then, we define the focal curve of the adjoint curve by obtaining the focal curvatures of the focal curve. Finally, with the help of this definition and other obtained equations, we obtain some characterizations related to curvatures and curves.

2. PRELIMINARIES

In this part, we remind the q-frame equations and some related fundamental information for the spacelike curves we will choose in the 3-dimensional Minkowski (M_1^3) space, which we will base our study on.

The F-S frame formulas for an arbitrary curve ψ with arc length parameters are given as follows:

$$\begin{bmatrix}\boldsymbol{T}^s\\ \boldsymbol{N}^s\\ \boldsymbol{B}^s\end{bmatrix}=\begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{T}\\ \boldsymbol{N}\\ \boldsymbol{B}\end{bmatrix}. \quad (2)$$

Also, the F-S frame elements are given by these equations

$$\boldsymbol{T}=\partial\psi/\partial s,\quad \boldsymbol{N}=\frac{\partial^2\psi/\partial s^2}{\|\partial^2\psi/\partial s^2\|},\quad \boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}. \quad (3)$$

Here, τ and κ are torsion and curvature of ψ, and the vector fields $\boldsymbol{T},\boldsymbol{N},\boldsymbol{B}$ are tangent, unit normal, and binormal vector fields of the F-S frame [12].

In addition, the expressions of q-frame elements in terms of F-S frame elements are written as

$$\begin{aligned}\boldsymbol{T}_{q_\psi}&=\boldsymbol{T},\\ \boldsymbol{N}_{q_\psi}&=(\boldsymbol{T}\times\rho)/\|\boldsymbol{T}\times\rho\|,\\ \boldsymbol{B}_{q_\psi}&=\boldsymbol{T}_{q_\psi}\times\boldsymbol{N}_{q_\psi}.\end{aligned} \quad (4)$$

Here $\rho=(0,1,0)$ is the projection vector. Let ψ be a spacelike curve and quasi(q)-normal vector field $\boldsymbol{N}_{q_\psi}$ be timelike. Then, the F-S frame formulas in M_1^3 are given as

$$\begin{bmatrix}\boldsymbol{T}_\psi^s\\ \boldsymbol{N}_\psi^s\\ \boldsymbol{B}_\psi^s\end{bmatrix}=\begin{bmatrix}0 & \kappa_\psi & 0\\ \kappa_\psi & 0 & \tau_\psi\\ 0 & \tau_\psi & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{T}_\psi\\ \boldsymbol{N}_\psi\\ \boldsymbol{B}_\psi\end{bmatrix}, \quad (5)$$

and the relationship between the F-S frame and the q-frame is given as

$$\begin{bmatrix}\boldsymbol{T}_{q_\psi}\\ \boldsymbol{N}_{q_\psi}\\ \boldsymbol{B}_{q_\psi}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & ch\theta & sh\theta\\ 0 & sh\theta & ch\theta\end{bmatrix}\begin{bmatrix}\boldsymbol{T}_\psi\\ \boldsymbol{N}_\psi\\ \boldsymbol{B}_\psi\end{bmatrix}, \quad (6)$$

$$\psi_1=-\kappa_\psi ch\theta,\qquad \psi_2=-\kappa_\psi sh\theta,\qquad \psi_3=d\theta+\tau_\psi.$$

where ψ_1,ψ_2,ψ_3 are curvatures of ψ, and θ is the hyperbolic angle between $\boldsymbol{N}_{q_\psi}$ and $\boldsymbol{N}_\psi$. Then, the q-frame formulas in M_1^3 are given by

$$\begin{bmatrix}\boldsymbol{T}_{q_\psi}^s\\ \boldsymbol{N}_{q_\psi}^s\\ \boldsymbol{B}_{q_\psi}^s\end{bmatrix}=\begin{bmatrix}0 & -\psi_1 & \psi_2\\ -\psi_1 & 0 & \psi_3\\ -\psi_2 & \psi_3 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{T}_{q_\psi}\\ \boldsymbol{N}_{q_\psi}\\ \boldsymbol{B}_{q_\psi}\end{bmatrix}, \quad (7)$$

and vector products of q-frame elements with each other are written as

$$\begin{aligned}\boldsymbol{T}_{q_\psi}\times\boldsymbol{N}_{q_\psi}&=-\boldsymbol{B}_{q_\psi},\\ \boldsymbol{N}_{q_\psi}\times\boldsymbol{B}_{q_\psi}&=-\boldsymbol{T}_{q_\psi},\\ \boldsymbol{B}_{q_\psi}\times\boldsymbol{T}_{q_\psi}&=\boldsymbol{N}_{q_\psi},\end{aligned} \quad (8)$$

Let ψ be a spacelike curve and q-binormal vector field $\boldsymbol{B}_{q_\psi}$ be timelike. Then, the F-S frame formulas in M_1^3 are given as

$$\begin{bmatrix} \boldsymbol{T}_\psi^s \\ \boldsymbol{N}_\psi^s \\ \boldsymbol{B}_\psi^s \end{bmatrix} = \begin{bmatrix} 0 & \kappa_\psi & 0 \\ -\kappa_\psi & 0 & \tau_\psi \\ 0 & \tau_\psi & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}_\psi \\ \boldsymbol{N}_\psi \\ \boldsymbol{B}_\psi \end{bmatrix}, \tag{9}$$

and the relationship between the F-S frame and the q-frame is given as

$$\begin{bmatrix} \boldsymbol{T}_{q_\psi} \\ \boldsymbol{N}_{q_\psi} \\ \boldsymbol{B}_{q_\psi} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & sh\theta & ch\theta \\ 0 & -ch\theta & -sh\theta \end{bmatrix} \begin{bmatrix} \boldsymbol{T}_\psi \\ \boldsymbol{N}_\psi \\ \boldsymbol{B}_\psi \end{bmatrix}, \tag{10}$$

$\psi_1 = \kappa_\psi sh\theta,\ \ \psi_2 = -\kappa_\psi ch\theta,\ \ \psi_3 = -d\theta - \tau_\psi.$

Then, the q-frame formulas in M_1^3 are given by

$$\begin{bmatrix} \boldsymbol{T}_{q_\psi}^s \\ \boldsymbol{N}_{q_\psi}^s \\ \boldsymbol{B}_{q_\psi}^s \end{bmatrix} = \begin{bmatrix} 0 & -\psi_1 & \psi_2 \\ -\psi_1 & 0 & \psi_3 \\ -\psi_2 & \psi_3 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}_{q_\psi} \\ \boldsymbol{N}_{q_\psi} \\ \boldsymbol{B}_{q_\psi} \end{bmatrix}, \tag{11}$$

and vector products of q-frame elements with each other are written as [13]

$$\begin{aligned} \boldsymbol{T}_{q_\psi} \times \boldsymbol{N}_{q_\psi} &= \boldsymbol{B}_{q_\psi}, \\ \boldsymbol{N}_{q_\psi} \times \boldsymbol{B}_{q_\psi} &= -\boldsymbol{T}_{q_\psi}, \\ \boldsymbol{B}_{q_\psi} \times \boldsymbol{T}_{q_\psi} &= -\boldsymbol{N}_{q_\psi}. \end{aligned} \tag{12}$$

3. THE QUASI-FOCAL CURVES OF SPACELIKE ADJOINT CURVES

Definition 1. Let ψ be an arc-length parametrised curve and $\boldsymbol{B}_\psi$ be the binormal (unit) vector field of ψ. Then, the adjoint of curve ψ is defined by the equation [9]

$$\vartheta(s) = \int_{s_0}^{s} \boldsymbol{B}_\psi(s) ds. \tag{13}$$

Theorem 2. Let ψ be a spacelike curve, $\boldsymbol{T}_{q_\psi}, \boldsymbol{N}_{q_\psi}, \boldsymbol{B}_{q_\psi}$ be q-frame elements of ψ, ϑ be adjoint of ψ and $\boldsymbol{T}_{q_\vartheta}, \boldsymbol{N}_{q_\vartheta}, \boldsymbol{B}_{q_\vartheta}$ be q-frame elements of ϑ. If the q-normal vector field $\boldsymbol{N}_{q_\psi}$ is timelike, then the equations related to the curve ψ of q-frame elements of ϑ are given in

$$\begin{aligned} \boldsymbol{T}_{q_\vartheta} &= \frac{\psi_2}{\kappa_\psi} \boldsymbol{N}_{q_\psi} - \frac{\psi_1}{\kappa_\psi} \boldsymbol{B}_\psi, \\ \boldsymbol{N}_{q_\vartheta} &= \frac{\vartheta_2}{\kappa_\vartheta} \boldsymbol{T}_{q_\psi} + \frac{\psi_1 \vartheta_1}{\kappa_\psi \kappa_\vartheta} \boldsymbol{N}_{q_\psi} - \frac{\psi_2 \vartheta_1}{\kappa_\psi \kappa_\vartheta} \boldsymbol{B}_{q_\psi}, \\ \boldsymbol{B}_{q_\vartheta} &= \frac{\vartheta_1}{\kappa_\vartheta} \boldsymbol{T}_{q_\psi} + \frac{\psi_1 \vartheta_2}{\kappa_\psi \kappa_\vartheta} \boldsymbol{N}_{q_\psi} - \frac{\psi_2 \vartheta_2}{\kappa_\psi \kappa_\vartheta} \boldsymbol{B}_{q_\psi}, \end{aligned} \tag{14}$$

and, If the q-binormal vector field $\boldsymbol{B}_{q_\psi}$ is timelike, then the equations related to the curve ψ of q-frame elements of ϑ are given in

$$\begin{aligned} \boldsymbol{T}_{q_\vartheta} &= -\frac{\psi_2}{\kappa_\psi} \boldsymbol{N}_{q_\psi} + \frac{\psi_1}{\kappa_\psi} \boldsymbol{B}_\psi, \\ \boldsymbol{N}_{q_\vartheta} &= -\frac{\vartheta_2}{\kappa_\vartheta} \boldsymbol{T}_{q_\psi} - \frac{\psi_1 \vartheta_1}{\kappa_\psi \kappa_\vartheta} \boldsymbol{N}_{q_\psi} + \frac{\psi_2 \vartheta_1}{\kappa_\psi \kappa_\vartheta} \boldsymbol{B}_{q_\psi}, \\ \boldsymbol{B}_{q_\vartheta} &= -\frac{\vartheta_1}{\kappa_\vartheta} \boldsymbol{T}_{q_\psi} - \frac{\psi_1 \vartheta_2}{\kappa_\psi \kappa_\vartheta} \boldsymbol{N}_{q_\psi} + \frac{\psi_2 \vartheta_2}{\kappa_\psi \kappa_\vartheta} \boldsymbol{B}_{q_\psi}, \end{aligned} \tag{15}$$

Proof. From (13), we can easily obtain the equations $\boldsymbol{T}_\vartheta = \boldsymbol{B}_\psi$. In the case of $\boldsymbol{N}_{q_\psi}$ is timelike, from (3) and (5), we obtain

$$\boldsymbol{N}_\vartheta = \frac{\partial^2 \vartheta / \partial s^2}{\| \partial^2 \vartheta / \partial s^2 \|} = \frac{\boldsymbol{B}_\psi^s}{\left\| \boldsymbol{B}_\psi^s \right\|} = \boldsymbol{N}_\psi, \tag{16}$$

$$\boldsymbol{B}_\vartheta = -\boldsymbol{T}_\vartheta \times \boldsymbol{N}_\vartheta = -\boldsymbol{B}_\psi \times \boldsymbol{N}_\psi = -\boldsymbol{T}_\psi. \tag{17}$$

Let φ be the hyperbolic angle between $\boldsymbol{N}_{q_\vartheta}$ and $\boldsymbol{N}_\vartheta$. Then, from (6), we get

$$\begin{aligned} \boldsymbol{T}_{q_\vartheta} = \boldsymbol{T}_\vartheta = \boldsymbol{B}_\psi &= -sh\theta \boldsymbol{N}_{q_\psi} + ch\theta \boldsymbol{B}_{q_\psi} \\ &= \frac{\psi_2}{\kappa_\psi} \boldsymbol{N}_{q_\psi} - \frac{\psi_1}{\kappa_\psi} \boldsymbol{B}_{q_\psi}, \end{aligned}$$

$$\begin{aligned} \boldsymbol{N}_{q_\vartheta} &= ch\varphi \boldsymbol{N}_\vartheta + sh\varphi \boldsymbol{B}_\vartheta = ch\varphi \boldsymbol{N}_\psi - sh\varphi \boldsymbol{T}_\psi \\ &= ch\varphi \left(ch\theta \boldsymbol{N}_{q_\psi} - sh\theta \boldsymbol{B}_{q_\psi} \right) - sh\varphi \boldsymbol{T}_{q_\psi} \\ &= \frac{\vartheta_2}{\kappa_\vartheta} \boldsymbol{T}_{q_\psi} + \frac{\psi_1 \vartheta_1}{\kappa_\psi \kappa_\vartheta} \boldsymbol{N}_{q_\psi} - \frac{\psi_2 \vartheta_1}{\kappa_\psi \kappa_\vartheta} \boldsymbol{B}_{q_\psi}, \end{aligned}$$

$$\begin{aligned} \boldsymbol{B}_{q_\vartheta} &= sh\varphi \boldsymbol{N}_\vartheta + ch\varphi \boldsymbol{B}_\vartheta = sh\varphi \boldsymbol{N}_\psi - ch\varphi \boldsymbol{T}_\psi \\ &= sh\varphi (ch\theta \boldsymbol{N}_{q_\psi} - sh\theta \boldsymbol{B}_{q_\psi}) - ch\varphi \boldsymbol{T}_{q_\psi} \\ &= \frac{\vartheta_1}{\kappa_\vartheta} \boldsymbol{T}_{q_\psi} + \frac{\psi_1 \vartheta_2}{\kappa_\psi \kappa_\vartheta} \boldsymbol{N}_{q_\psi} - \frac{\psi_2 \vartheta_2}{\kappa_\psi \kappa_\vartheta} \boldsymbol{B}_{q_\psi}, \end{aligned}$$

In case of $\boldsymbol{B}_{q_\psi}$ is timelike, we similarly obtain the other equations.

Definition 3. Let $\boldsymbol{T}_{q_\vartheta}, \boldsymbol{N}_{q_\vartheta}, \boldsymbol{B}_{q_\vartheta}$ be q-frame elements of a regular space curve ϑ. The focal curve of ϑ is expressed as

$$\vartheta_F = \vartheta + \omega_1 \boldsymbol{N}_{q_\vartheta} + \omega_2 \boldsymbol{B}_{q_\vartheta}. \tag{18}$$

Here ω_1, ω_2 are the coefficients describing focal curvatures of ϑ [10].

Theorem 4. Let ψ be a spacelike curve, ϑ be adjoint curve of ψ, ϑ_F its focal curve in M_1^3 and ϑ_i and ψ_i $(i = 1,2,3)$ be, respectively, the curvatures of ϑ and ψ according to q-frame. If the q-normal vector field $\boldsymbol{N}_{q_\psi}$ is timelike, the focal curve of ϑ is

$$\begin{aligned}\vartheta_F = \vartheta - e^{\int \frac{\vartheta_1\vartheta_3}{\vartheta_2}ds}\left(\int e^{-\int \frac{\vartheta_1\vartheta_3}{\vartheta_2}ds}\frac{\vartheta_3}{\vartheta_2}ds - C\right)\left(\frac{\vartheta_2}{\kappa_\vartheta}\boldsymbol{T}_{q_\psi}\right. \\ \left.+\frac{\psi_1\vartheta_1}{\kappa_\psi\kappa_\vartheta}\boldsymbol{N}_{q_\psi} - \frac{\psi_2\vartheta_1}{\kappa_\psi\kappa_\vartheta}\boldsymbol{B}_{q_\psi}\right) + \left(\frac{1}{\vartheta_2} - \frac{\vartheta_1}{\vartheta_2}e^{\int \frac{\vartheta_1\vartheta_3}{\vartheta_2}ds}\left(C - \right.\right. \\ \left.\left.\int e^{-\int \frac{\vartheta_1\vartheta_3}{\vartheta_2}ds}\frac{\vartheta_3}{\vartheta_2}ds\right)\right)\left(\frac{\vartheta_1}{\kappa_\vartheta}\boldsymbol{T}_{q_\psi} + \frac{\psi_1\vartheta_2}{\kappa_\psi\kappa_\vartheta}\boldsymbol{N}_{q_\psi} - \frac{\psi_2\vartheta_2}{\kappa_\psi\kappa_\vartheta}\boldsymbol{B}_{q_\psi}\right).\end{aligned} \tag{19}$$

and, if the q-binormal vector field $\boldsymbol{B}_{q_\psi}$ is timelike, the focal curve of ϑ is

$$\begin{aligned}\vartheta_F = \vartheta + e^{-\int \frac{\vartheta_1\vartheta_3}{\vartheta_2}ds}\left(\int e^{\int \frac{\vartheta_1\vartheta_3}{\vartheta_2}ds}\frac{\vartheta_3}{\vartheta_2}ds + C\right)\left(\frac{\vartheta_2}{\kappa_\vartheta}\boldsymbol{T}_{q_\psi}\right. \\ \left.+\frac{\psi_1\vartheta_1}{\kappa_\psi\kappa_\vartheta}\boldsymbol{N}_{q_\psi} - \frac{\psi_2\vartheta_1}{\kappa_\psi\kappa_\vartheta}\boldsymbol{B}_{q_\psi}\right) + \left(\frac{1}{\vartheta_2} - \frac{\vartheta_1}{\vartheta_2}e^{-\int \frac{\vartheta_1\vartheta_3}{\vartheta_2}ds}\left(C + \right.\right. \\ \left.\left.\int e^{\int \frac{\vartheta_1\vartheta_3}{\vartheta_2}ds}\frac{\vartheta_3}{\vartheta_2}ds\right)\right)\left(\frac{\vartheta_1}{\kappa_\vartheta}\boldsymbol{T}_{q_\psi} + \frac{\psi_1\vartheta_2}{\kappa_\psi\kappa_\vartheta}\boldsymbol{N}_{q_\psi} - \frac{\psi_2\vartheta_2}{\kappa_\psi\kappa_\vartheta}\boldsymbol{B}_{q_\psi}\right).\end{aligned} \tag{20}$$

Proof. Let ψ be a spacelike curve, ϑ be adjoint curve of ψ, ϑ_F its focal curve in M_1^3 and ϑ_i and ψ_i $(i = 1,2,3)$ be respectively curvatures of ϑ and ψ according to q-frame. Let q-normal vector field $\boldsymbol{N}_{q_\psi}$ be timelike. Applying derivative to the equation (18), we get

$$\begin{aligned}\vartheta_F^s = (1 - \vartheta_1\omega_1 - \vartheta_2\omega_2)\boldsymbol{T}_{q_\vartheta} + (\vartheta_3\omega_2 \\ +\omega_1^s)\boldsymbol{N}_{q_\vartheta} + (\omega_2^s + \vartheta_3\omega_1)\boldsymbol{B}_{q_\vartheta}.\end{aligned}$$

Since the focal curve expresses the centers of tangential oscillating spheres, the components of $\boldsymbol{T}_{q_\vartheta}$ and $\boldsymbol{N}_{q_\vartheta}$ vanish when applying the derivative based on the spring parameter to the curve. Hence, it's obtained

$$1 - \vartheta_1\omega_1 - \vartheta_2\omega_2 = 0, \tag{21}$$

$$\omega_1^s + \vartheta_3\omega_2 = 0. \tag{22}$$

From these equations, for $\vartheta_2 \neq 0$, we get

$$\omega_2 = \frac{\omega_1\vartheta_1 - 1}{\vartheta_2},$$

$$\omega_1^s - \frac{\vartheta_1\vartheta_3}{\vartheta_2}\omega_1 = -\frac{\vartheta_3}{\vartheta_2}.$$

By solving the above differential equation, it's found

$$\omega_1 = e^{\int \frac{\vartheta_1\vartheta_3}{\vartheta_2}ds}\left(-\int e^{-\int \frac{\vartheta_1\vartheta_3}{\vartheta_2}ds}\frac{\vartheta_3}{\vartheta_2}ds + C\right),$$

$$\omega_2 = \frac{1}{\vartheta_2} - \frac{\vartheta_1}{\vartheta_2}e^{\int \frac{\vartheta_1\vartheta_3}{\vartheta_2}ds}\left(-\int e^{-\int \frac{\vartheta_1\vartheta_3}{\vartheta_2}ds}\frac{\vartheta_3}{\vartheta_2}ds + C\right).$$

Also, from (14), we obtain

$$\begin{aligned}\vartheta_F = \vartheta - \omega_1\left(\frac{\vartheta_2}{\kappa_\vartheta}\boldsymbol{T}_{q_\psi} + \frac{\psi_1\vartheta_1}{\kappa_\psi\kappa_\vartheta}\boldsymbol{N}_{q_\psi} - \frac{\psi_2\vartheta_1}{\kappa_\psi\kappa_\vartheta}\boldsymbol{B}_{q_\psi}\right) \\ -\omega_2\left(\frac{\vartheta_1}{\kappa_\vartheta}\boldsymbol{T}_{q_\psi} + \frac{\psi_1\vartheta_2}{\kappa_\psi\kappa_\vartheta}\boldsymbol{N}_{q_\psi} - \frac{\psi_2\vartheta_2}{\kappa_\psi\kappa_\vartheta}\boldsymbol{B}_{q_\psi}\right).\end{aligned}$$

Hence, the first statement of the theorem is obtained as

$$\begin{aligned}\vartheta_F = \vartheta - e^{\int \frac{\vartheta_1\vartheta_3}{\vartheta_2}ds}\left(\int e^{-\int \frac{\vartheta_1\vartheta_3}{\vartheta_2}ds}\frac{\vartheta_3}{\vartheta_2}ds - C\right)\left(\frac{\vartheta_2}{\kappa_\vartheta}\boldsymbol{T}_{q_\psi}\right. \\ \left.+\frac{\psi_1\vartheta_1}{\kappa_\psi\kappa_\vartheta}\boldsymbol{N}_{q_\psi} - \frac{\psi_2\vartheta_1}{\kappa_\psi\kappa_\vartheta}\boldsymbol{B}_{q_\psi}\right) + \left(\frac{1}{\vartheta_2} - \frac{\vartheta_1}{\vartheta_2}e^{\int \frac{\vartheta_1\vartheta_3}{\vartheta_2}ds}\left(C - \right.\right. \\ \left.\left.\int e^{-\int \frac{\vartheta_1\vartheta_3}{\vartheta_2}ds}\frac{\vartheta_3}{\vartheta_2}ds\right)\right)\left(\frac{\vartheta_1}{\kappa_\vartheta}\boldsymbol{T}_{q_\psi} + \frac{\psi_1\vartheta_2}{\kappa_\psi\kappa_\vartheta}\boldsymbol{N}_{q_\psi} - \frac{\psi_2\vartheta_2}{\kappa_\psi\kappa_\vartheta}\boldsymbol{B}_{q_\psi}\right).\end{aligned}$$

On the other hand, let the q-binormal vector field $\boldsymbol{B}_{q_\psi}$ be timelike. Similarly, the focal curvatures of ϑ_F are computed as

$$\omega_1 = e^{-\int \frac{\vartheta_1 \vartheta_3}{\vartheta_2} ds} \left(\int e^{\int \frac{\vartheta_1 \vartheta_3}{\vartheta_2} ds} \frac{\vartheta_3}{\vartheta_2} ds + C \right),$$

$$\omega_2 = \frac{1}{\vartheta_2} - \frac{\vartheta_1}{\vartheta_2} e^{-\int \frac{\vartheta_1 \vartheta_3}{\vartheta_2} ds} \left(\int e^{\int \frac{\vartheta_1 \vartheta_3}{\vartheta_2} ds} \frac{\vartheta_3}{\vartheta_2} ds + C \right).$$

Therefore, from (15), we obtain the second statement of the theorem.

The following result is a consequence of Theorem 4.

Corollary 5. Let ψ be a spacelike curve, ϑ be adjoint curve of ψ, ϑ_F its focal curve in M_1^3 and ϑ_i and ψ_i $(i = 1,2,3)$ be respectively curvatures of ϑ and ψ according to q-frame. In the case of q-normal vector field $\boldsymbol{N}_{q_\psi}$ is timelike, the focal curvatures of curve ϑ are

$$\omega_1 = e^{\int \frac{\vartheta_1 \vartheta_3}{\vartheta_2} ds} \left(-\int e^{-\int \frac{\vartheta_1 \vartheta_3}{\vartheta_2} ds} \frac{\vartheta_3}{\vartheta_2} ds + C \right), \tag{23}$$

$$\omega_2 = \frac{1}{\vartheta_2} + \frac{\vartheta_1}{\vartheta_2} e^{\int \frac{\vartheta_1 \vartheta_3}{\vartheta_2} ds} \left(\int e^{-\int \frac{\vartheta_1 \vartheta_3}{\vartheta_2} ds} \frac{\vartheta_3}{\vartheta_2} ds - C \right), \tag{24}$$

and, in the case of q -binormal vector field $\boldsymbol{B}_{q_\psi}$ is timelike, the focal curvatures of curve ϑ are

$$\omega_1 = e^{-\int \frac{\vartheta_1 \vartheta_3}{\vartheta_2} ds} \left(\int e^{\int \frac{\vartheta_1 \vartheta_3}{\vartheta_2} ds} \frac{\vartheta_3}{\vartheta_2} ds + C \right), \tag{25}$$

$$\omega_2 = \frac{1}{\vartheta_2} - \frac{\vartheta_1}{\vartheta_2} e^{-\int \frac{\vartheta_1 \vartheta_3}{\vartheta_2} ds} \left(\int e^{\int \frac{\vartheta_1 \vartheta_3}{\vartheta_2} ds} \frac{\vartheta_3}{\vartheta_2} ds + C \right). \tag{26}$$

As a result of Theorem 4 and Corollary 5, we obtain Corollary 6.

Corollary 6. Let ψ be a spacelike curve, ϑ be adjoint curve of ψ, ϑ_F its focal curve in M_1^3 and ϑ_i and ψ_i $(i = 1,2,3)$ be respectively curvatures of ϑ and ψ according to q-frame. Suppose curvatures of ϑ are constant. In this case, if q-normal vector field is timelike, the focal curve ϑ_F is obtained as

$$\begin{aligned} \vartheta_F = \vartheta &+ \frac{\vartheta_1}{\vartheta_2} e^{\frac{\vartheta_1 \vartheta_3}{\vartheta_2} s} C \left(\frac{\vartheta_1}{\kappa_\vartheta} \boldsymbol{T}_{q_\psi} + \frac{\psi_1 \vartheta_2}{\kappa_\psi \kappa_\vartheta} \boldsymbol{N}_{q_\psi} - \frac{\psi_2 \vartheta_2}{\kappa_\psi \kappa_\vartheta} \boldsymbol{B}_{q_\psi} \right) \\ &+ \left(\frac{1}{\vartheta_1} + e^{\frac{\vartheta_1 \vartheta_3}{\vartheta_2} s} C \right) \left(\frac{\vartheta_2}{\kappa_\vartheta} \boldsymbol{T}_{q_\psi} + \frac{\psi_1 \vartheta_1}{\kappa_\psi \kappa_\vartheta} \boldsymbol{N}_{q_\psi} - \frac{\psi_2 \vartheta_1}{\kappa_\psi \kappa_\vartheta} \boldsymbol{B}_{q_\psi} \right). \end{aligned} \tag{27}$$

and, if q-binormal vector field is timelike, the focal curve ϑ_F is obtained as

$$\begin{aligned} \vartheta_F = \vartheta &+ \frac{\vartheta_1}{\vartheta_2} e^{-\frac{\vartheta_1 \vartheta_3}{\vartheta_2} s} C \left(\frac{\vartheta_1}{\kappa_\vartheta} \boldsymbol{T}_{q_\psi} + \frac{\psi_1 \vartheta_2}{\kappa_\psi \kappa_\vartheta} \boldsymbol{N}_{q_\psi} - \frac{\psi_2 \vartheta_2}{\kappa_\psi \kappa_\vartheta} \boldsymbol{B}_{q_\psi} \right) \\ &+ \left(\frac{1}{\vartheta_1} + e^{-\frac{\vartheta_1 \vartheta_3}{\vartheta_2} s} C \right) \left(\frac{\vartheta_2}{\kappa_\vartheta} \boldsymbol{T}_{q_\psi} + \frac{\psi_1 \vartheta_1}{\kappa_\psi \kappa_\vartheta} \boldsymbol{N}_{q_\psi} - \frac{\psi_2 \vartheta_1}{\kappa_\psi \kappa_\vartheta} \boldsymbol{B}_{q_\psi} \right). \end{aligned} \tag{28}$$

Example. Let us consider the unit speed spacelike curve $\psi(s) = \left(\frac{1}{\sqrt{2}} sh(s), \frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}} ch(s) \right)$.

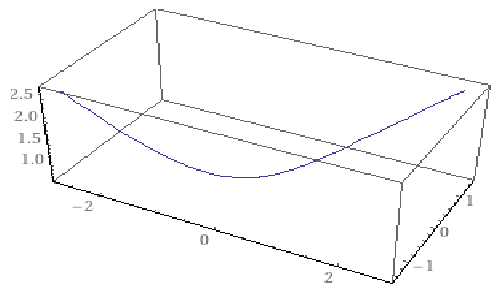

Figure 1 The unit speed spacelike curve ψ

Then, the F-S vectors of ψ are obtained as

$$\boldsymbol{T}_\psi = \frac{\partial \psi}{\partial s} = \left(\frac{1}{\sqrt{2}} ch(s), \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} sh(s) \right),$$

$$\boldsymbol{N}_\psi = \frac{\partial^2 \psi / \partial s^2}{\| \partial^2 \psi / \partial s^2 \|} = (sh(s), 0, ch(s)),$$

$$\boldsymbol{B}_\psi = -\boldsymbol{T}_\psi \times \boldsymbol{N}_\psi = \left(\frac{-1}{\sqrt{2}} ch(s), \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} sh(s) \right).$$

Here, it can be easily seen that $\boldsymbol{N}_\psi$ is timelike and $\boldsymbol{B}_\psi$ is spacelike. Let ϑ be the adjoint curve of ψ. From (13), we obtain

$$\vartheta(s) = \int \boldsymbol{B}_\psi ds = \left(\frac{-1}{\sqrt{2}} sh(s), \frac{-s}{\sqrt{2}}, \frac{-1}{\sqrt{2}} ch(s) \right) + c.$$

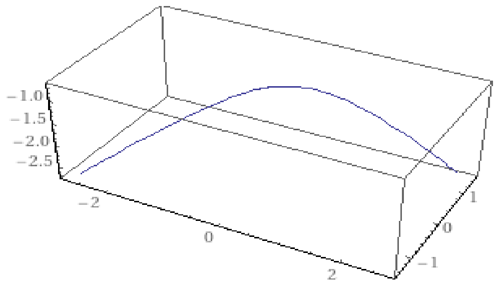

Figure 2 For $c=0$, the curve ϑ, the adjoint of ψ

Here c is the integration constant. The F-S vectors of ϑ are obtained as

$$\boldsymbol{T}_\vartheta = \frac{\partial\vartheta}{\partial s} = \left(\frac{-1}{\sqrt{2}}ch(s), \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}sh(s)\right),$$

$$\boldsymbol{N}_\vartheta = \frac{\partial^2\vartheta/\partial s^2}{\|\partial^2\vartheta/\partial s^2\|} = (-sh(s), 0, -ch(s)),$$

$$\boldsymbol{B}_\vartheta = -\boldsymbol{T}_\vartheta \times \boldsymbol{N}_\vartheta = \left(\frac{-1}{\sqrt{2}}ch(s), \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}sh(s)\right).$$

For the spacelike projection vector $\rho = (0,1,0)$, from (4), The q-frame elements and the curvatures of ϑ are obtained as

$$\boldsymbol{T}_{q_\vartheta} = \boldsymbol{T}_\vartheta = \left(\frac{-1}{\sqrt{2}}ch(s), \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}sh(s)\right),$$

$$\boldsymbol{N}_{q_\vartheta} = \frac{\boldsymbol{T}_\vartheta \times \rho}{\|\boldsymbol{T}_\vartheta \times \rho\|} = \left(\frac{1}{\sqrt{2}}sh(s), 0, \frac{1}{\sqrt{2}}ch(s)\right),$$

$$\boldsymbol{B}_{q_\vartheta} = -\boldsymbol{T}_\vartheta \times \boldsymbol{N}_{q_\vartheta} = \left(\frac{1}{2}ch(s), \frac{1}{2}, \frac{1}{2}sh(s)\right),$$

$$\vartheta_1 = \langle \boldsymbol{T}_\vartheta^s, \boldsymbol{N}_{q_\vartheta} \rangle = \frac{1}{2},$$

$$\vartheta_2 = \langle \boldsymbol{T}_\vartheta^s, \boldsymbol{B}_{q_\vartheta} \rangle = 0,$$

$$\vartheta_3 = \langle \boldsymbol{N}_{q_\vartheta}^s, \boldsymbol{B}_{q_\vartheta} \rangle = \frac{1}{2\sqrt{2}}.$$

On the other hand, the q-focal curve of ϑ is given by

$$\vartheta_F = \vartheta + \omega_1 \boldsymbol{N}_{q_\vartheta} + \omega_2 \boldsymbol{B}_{q_\vartheta}.$$

By using (21) and (22), we get

$$1 - \frac{1}{2}\omega_1 = 0,$$

$$\omega_1^s + \frac{1}{2\sqrt{2}}\omega_2 = 0.$$

Then, we obtain $\omega_1 = 2$, $\omega_2 = 0$. Therefore, for $c=0$, the q-focal curve of ϑ is obtained as

$$\vartheta_F = \vartheta + 2\boldsymbol{N}_{q_\vartheta} = \left(\frac{1}{\sqrt{2}}sh(s), -s, \frac{1}{\sqrt{2}}ch(s)\right).$$

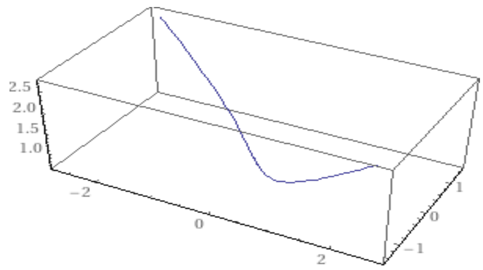

Figure 3 The q-focal curve of the adjoint curve ϑ

4. CONCLUSION

In this study, we aimed to obtain the equations characterizing the focal curve of the adjoint of a spacelike curve under the quasi frame in M_1^3. Here we obtained different equations for the two cases where the q-normal vector field is timelike, and the q-binormal vector field is timelike. Consequently, these equations showed that the q-focal curve of the adjoint curve of a main curve can be obtained depending on the curvatures of the main curve and the adjoint curve. Finally, as an example, we presented the q-focal curve of the adjoint curve of a spacelike curve with a timelike q-normal vector field and gave 3D plots of the main curve, the adjoint curve, and the q-focal curve.

The results we have obtained will help us to give characterizations of such curves and surfaces to be formed with these curves, their behavior and geometric expressions of particle motions that can be expressed by these curves. Therefore, we think that this study will bring a new perspective to the subject of associated, focal, and adjoint curves. In our next study, we are thinking of working on some special surfaces that such curves will create.

The Declaration of Conflict of Interest/ Common Interest

No conflict of interest or common interest has been declared by the authors.

Authors' Contribution

The authors contributed equally to the study.

The Declaration of Ethics Committee Approval

This study does not require ethics committee permission or any special permission.

The Declaration of Research and Publication Ethics

The authors of the paper declare that they comply with the scientific, ethical and quotation rules of SAUJS in all processes of the paper and that they do not make any falsification on the data collected. In addition, they declare that Sakarya University Journal of Science and its editorial board have no responsibility for any ethical violations that may be encountered, and that this study has not been evaluated in any academic publication environment other than Sakarya University Journal of Science.

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