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Research Article

Numerical solutions of Troesch and Duffing equations by Taylor wavelets

Gökçe Özaltun Şimşek[1], Sevin Gümgüm*[1]

[1] *Department of Mathematics, Faculty of Arts and Science, Izmir University of Economics, Izmir, Türkiye*

Abstract

The aim of this study is to obtain accurate numerical results for the Troesch and Duffing equations by using Taylor wavelets. Important features of the method include easy implementation and simple calculation. The effectiveness and accuracy of the applied method is illustrated by solving these problems for several variables. One of the important variable is the resolution parameter which enables to use low degree polynomials and decrease the computational cost. Results show that the proposed method yields highly accurate solutions by using quite low degree polynomials.

Mathematics Subject Classification (2020). 34B15, 65L05, 40C05

Keywords. Taylor wavelets, Troesch equation, Duffing equation

1. Introduction

Nonlinear boundary and initial value problems are used to model many physical phenomena. One of them is the Troesch's problem which arises in the areas such as the theory of gas porous electrodes [33] and the confinement of a plasma column by radiation pressure [50]. Various numerical techniques have been applied to solve this problem, including Laguerre wavelet method [18], B-spline collocation approach [28], optimal Homotopy asymptotic method [26], Chebyshev wavelet method [37], an accurate asymptotic approximation [46], modified nonlinear Shooting method [2], the reproducing kernel and the Adomian decomposition techniques [17], decomposition method [11], homotopy perturbation method [34], Scott and the Kagiwada-Kalaba algorithms [41], Christov rational functions [40], Jacobi collocation method [12], hybrid heuristic computing [32], Laplace transform together with a modified decomposition method [27], finite difference method [45], high- order difference schemes [7], modified Homotopy perturbation method [15], variational iteration method [9,35], Newton-Raphson-Kantorovich approximation method [6], perturbation method and Laplace-Pade approximation [16], sinc-collocation method [13], Chebyshev collocation technique [14], sinc-Galerkin method [52] and differential transform method [10].

Another important problem is the Duffing equation which is used in the modeling of many physical phenomena including classical oscillator in chaotic systems, orbit extraction, the prediction of diseases and nonlinear vibration of beams and plates. Hence, Duffing

*Corresponding Author.
Email addresses: sevin.gumgum@ieu.edu.tr, gokce.ozaltun@ieu.edu.tr
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equation has been solved by using many numerical methods such as the Laplace decomposition algorithm [51], shifted Chebyshev polynomials [3], Runge-Kutta-Fehlenberg algorithm [23], Daftardar-Jafari method [1], Adomian decomposition method [49], differential transform method [44], the improved Taylor matrix technique [8], generalized differential quadrature method [30], Legendre wavelets [36], homotopy perturbation method [31], Lucas polynomial approach [22], cubination method [5] and iterative splitting method [29].

Wavelet methods require neither the estimation of the nonlinear term nor the discretization of the domain like most numerical methods. The implementation of the technique transforms the given equations to a system of nonlinear equations. Hence, these methods have drawn attention of researchers due to these advantages. Different types of discrete wavelets such as the Laguerre [18], Legendre [20,21] and Haar [43] wavelets are used commonly to solve several differential equations. The suggested technique has been used to solve the Bratu equation [25], Lane-Emden equation [19], fractional delay and integro-differential equations [24,47], parabolic and hyperbolic differential equations [38].

In Section 2, we define the Taylor wavelets, analyze the convergence and error bound. In Section 3, we give the implementation of the method to Troesch and Duffing equations with the given boundary and initial conditions. In Section 4, we solve these problems for various parameters and discuss the results obtained. We illustrate the efficacy of the technique by comparing these numerical results with the analytical and other numerical results.

2. Fundamentals of Taylor wavelets

Wavelets are generated by translation and dilation of the mother wavelet which are a family of continuous functions. If the dilation (i) and translation (j) parameters are continuous, then we have continuous wavelets of the form

$$\phi_{i,j}(x) = |i|^{-1/2}\phi\left(\frac{x-j}{i}\right),\ \ i,j \in \mathbf{R},\ \ i \neq 0.$$

If these parameters are restricted to $i = 2^{-(k-1)}$ and $j = (n-1)2^{-(k-1)}$, then we have discrete wavelets which are called as the Taylor wavelets [25]

$$\phi_{nm}(x) = \begin{cases} 2^{(k-1)/2}T_m(2^{k-1}x - n + 1), & \text{if } \dfrac{n-1}{2^{k-1}} \leq x < \dfrac{n}{2^{k-1}}, \\ 0, & \text{otherwise} \end{cases}$$

where k is a positive integer called the resolution parameter, $n = 1, 2, \ldots, 2^{k-1}$ and $m = 0, 1, 2, \ldots, M-1$. $T_m(x) = \sqrt{2m+1}x^m$, $\sqrt{2m+1}$ is for orthonormality and x^m is the m^{th} order Taylor polynomial.

Since Taylor wavelets form an orthonormal basis in $L^2(\mathbf{R})$, we can express a differentiable function by means of Taylor wavelets as

$$y(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} e_{nm}\phi_{nm}(x) \tag{2.1}$$

where $e_{nm} =< y(x), \phi_{nm}(x) >$ are the Taylor wavelet coefficients and $< \cdot, \cdot >$ is the inner product. Once we truncate the series in Eq. (2.1), we get

$$y(x) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} e_{nm}\phi_{nm}(x) = \mathbf{E}\mathbf{\Phi}(x) \tag{2.2}$$

where $\mathbf{E}$ and $\mathbf{\Phi}(x)$ are row and column vectors which have $2^{(k-1)}M$ elements

$$\mathbf{E} = [e_{10}, e_{11}, \ldots, e_{1(M-1)}, e_{20}, \ldots, e_{2(M-1)}, e_{(2^{k-1})0}, \ldots, e_{(2^{k-1})(M-1)}],$$
$$\mathbf{\Phi}(\mathbf{x}) = [\phi_{10}(x), \ldots, \phi_{1(M-1)}(x), \ldots, \phi_{(2^{k-1})0}(x), \ldots, \phi_{(2^{k-1})(M-1)}(x)]^T.$$

2.1. Convergence and error bound

The convergence and error bound are given by the following theorems.

Theorem 2.1. *The series expansion obtained by the Taylor wavelets in Eq. (2.2) converges to $y(x)$.*

Proof. The Taylor wavelets form a basis in $L^2[0,1]$ which is a Hilbert space. Since Hilbert is a complete space, every Cauchy sequence converges. So, in order to show the convergence of the series given in Eq. (2.1), we define a partial sum $S_{\alpha,\beta} = \sum_{n=1}^{\alpha}\sum_{m=0}^{\beta} e_{nm}\phi_{nm}(x)$, where $\alpha = 2^{k-1}$ and $\beta = M-1$. Assume that $\bar{\alpha} = 2^{l-1}$ and $\bar{\beta} = N-1$ with $\alpha > \bar{\alpha}$, $\beta > \bar{\beta}$. Consider

$$
\begin{aligned}
||S_{\alpha,\beta} - S_{\bar{\alpha},\bar{\beta}}||^2 &= ||\sum_{i=\bar{\alpha}+1}^{\alpha}\sum_{j=\bar{\beta}+1}^{\beta} e_{ij}\phi_{ij}(x)||^2 \\
&=< \sum_{i=\bar{\alpha}+1}^{\alpha}\sum_{j=\bar{\beta}+1}^{\beta} e_{ij}\phi_{ij}(x), \sum_{r=\bar{\alpha}+1}^{\alpha}\sum_{s=\bar{\beta}+1}^{\beta} e_{rs}\phi_{rs}(x) > \\
&= \sum_{i,r=\bar{\alpha}+1}^{\alpha}\sum_{j,s=\bar{\beta}+1}^{\beta} e_{ij}e_{rs} < \phi_{ij}(x), \phi_{rs}(x) > \\
&= \sum_{i=\bar{\alpha}+1}^{\alpha}\sum_{j=\bar{\beta}+1}^{\beta} |e_{ij}|^2.
\end{aligned}
$$

Bessel's inequality [4] states that $\sum_{i=\bar{\alpha}+1}^{\alpha}\sum_{j=\bar{\beta}+1}^{\beta} |e_{ij}|^2$ converges as $\alpha, \beta \to \infty$ which implies the convergence of $||S_{\alpha,\beta} - S_{\bar{\alpha},\bar{\beta}}||^2$. This shows that $S_{\alpha,\beta}$ is a Cauchy sequence in Hilbert space and hence it converges to a sum $\bar{y}(x)$.

Let us show that $\bar{y}(x) = y(x)$:

$$
\begin{aligned}
< \bar{y}(x) - y(x), \phi_{ij}(x) > &=< \bar{y}(x), \phi_{ij}(x) > - < y(x), \phi_{ij}(x) > \\
&= \lim_{\alpha,\beta\to\infty} < S_{\alpha,\beta}, \phi_{ij}(x) > - < e_{ij}\phi_{ij}(x), \phi_{ij}(x) > \\
&= e_{ij} - e_{ij} \\
&= 0
\end{aligned}
$$

which gives that the series converges to $y(x)$ as $\alpha, \beta \to \infty$.

□

Theorem 2.2. *The error bound for the approximation of the series given in Eq. (2.2) to a function $y(x)$ on $[0,1]$ is [42],*

$$||y(x) - \bar{y}(x)|| \le \frac{1}{q!2^{q(k-1)}} \sup_{x\in[0,1]} |y^{(q)}(x)|$$

Proof. Let $\bar{y}(x)$ be the Taylor wavelet approximation of order q to $y(x)$ which is a q-times continuously differentiable function on $[0,1]$. Consider the approximation of $y(x)$ on the subintervals $[\frac{n-1}{\alpha}, \frac{n}{\alpha}]$ as follows

$$
\begin{aligned}
||y(x)-\bar{y}(x)||^2 &= \sum_{n=1}^{\alpha} \int_{\frac{n-1}{\alpha}}^{\frac{n}{\alpha}} [y(x)-\bar{y}(x)]^2 dx \\
&\leq \sum_{n=1}^{\alpha} \int_{\frac{n-1}{\alpha}}^{\frac{n}{\alpha}} [y(x)-\tilde{y}(x)]^2 dx \\
&\leq \sum_{n=1}^{\alpha} \int_{\frac{n-1}{\alpha}}^{\frac{n}{\alpha}} \Big[\frac{1}{q!2^{q(k-1)}} \sup_{x\in[0,1]} |y^{(q)}(x)|\Big]^2 dx \\
&\leq \int_0^1 \Big[\frac{1}{q!2^{q(k-1)}} \sup_{x\in[0,1]} |y^{(q)}(x)|\Big]^2 dx = [\frac{1}{q!2^{q(k-1)}} \sup_{x\in[0,1]} |y^{(q)}(x)|]^2
\end{aligned}
$$

where $\alpha = 2^{k-1}$ and $\tilde{y}(x)$ is the q^{th} order interpolation of $y(x)$.

□

3. Implementation of the technique

In this section, we show the implementation of the technique to Troesch and Duffing problems which are both nonlinear and second order differential equations. Recall that the unknown function $y(x)$ is approximated by the Taylor wavelets in Eq. (2.2). We can approximate $y'(x)$ and $y''(x)$ in a similar way as follows:

$$y'(x) = \mathbf{E}\mathbf{\Phi}'(x) \quad \text{and} \quad y''(x) = \mathbf{E}\mathbf{\Phi}''(x) \tag{3.1}$$

where $\mathbf{\Phi}'(x)$ and $\mathbf{\Phi}''(x)$ are the first two derivatives of $\mathbf{\Phi}(x)$ defined before.

3.1. Application to Troesch's equation

Troesch's equation is defined by [48]

$$y''(x) = \lambda \sinh(\lambda y(x)) \quad , \quad x \in [0,1] \tag{3.2}$$

$$y(0) = 0 \quad , \quad y(1) = 1. \tag{3.3}$$

where $\lambda > 0$ is the Troesch's parameter.

The closed form of the solution of Eq. (3.2) is given in [39] as follows

$$y(x) = \frac{2}{\lambda}\sinh^{-1}\Big[\sqrt{1-r}\, sc(\lambda x|r)\Big]$$

where $sc(\lambda|r)$ is the Jacobi elliptic function and r is the solution of $sc(\lambda|r)\sqrt{1-r} = \sinh\left(\frac{\lambda}{2}\right)$.

Recall that the unknown function is expanded by Taylor wavelets in the form

$$y(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} e_{nm}\phi_{nm}(x) = \mathbf{E}\mathbf{\Phi}(x)$$

which includes $2^{k-1}M$ unknowns. Hence, we need $2^{k-1}M$ equations. Two of them are produced by rewriting the boundary conditions in Eq. (3.3) in terms of the above approximation as follows

$$
\begin{aligned}
y(0) &= \mathbf{E}\mathbf{\Phi}(0) \\
y(1) &= \mathbf{E}\mathbf{\Phi}(1)
\end{aligned}
\tag{3.4}
$$

Since we need $2^{k-1}(M-2)$ more equations, we insert the approximations in Eq. (2.2) and (3.1) into Eq. (3.2) to get

$$\mathbf{E\Phi}''(x) - \lambda \sinh\left[\lambda \mathbf{E\Phi}(x)\right] = 0$$

afterwards we insert the roots x_i, $i = 1, 2, \ldots,\ 2^{k-1}(M-2)$, of shifted Chebyshev polynomials as collocation points in the above equation

$$\mathbf{E\Phi}''(x_i) - \lambda \sinh\left[\lambda \mathbf{E\Phi}(x_i)\right] = 0. \tag{3.5}$$

Finally this system of nonlinear equations obtained from Eqs. (3.4) and (3.5) is solved for e_{nm} by using MATLAB tools and we find the approximate solution.

3.2. Application to Duffing equation

As the second case, let us consider the Duffing problem, [8]

$$y''(x) + \alpha y'(x) + \alpha_1 y(x) + \alpha_2 y^3(x) = f(x), \tag{3.6}$$

$$y(0) = \beta_1,\ y'(0) = \beta_2. \tag{3.7}$$

where α, α_1, α_2, β_1 and β_2 are real constants and $f(x)$ is a given force function. When $\alpha = \alpha_2 = 0$, the equation describes a simple harmonic motion. The case $\alpha = \alpha_2 \neq 0$ describes the motion of a damped oscillator with a more complex potential.

Once we substitute the approximations in Eq. (2.2) and (3.1) into the initial conditions in Eq. (3.7) and the differential equation (3.6), we get a system of nonlinear equations

$$\begin{aligned}
&\mathbf{E\Phi}(0) = \beta_1\\
&\mathbf{E\Phi}'(0) = \beta_2,\\
&\mathbf{E\Phi}''(x_i) + \alpha \mathbf{E\Phi}'(x_i) + \alpha_1 \mathbf{E\Phi}(x_i) + \alpha_2 \left[\mathbf{E\Phi}(x_i)\right]^3 = f(x_i)
\end{aligned}$$

where x_i, $i = 1, 2, \ldots, 2^{k-1}(M-2)$, are the first $2^{k-1}(M-2)$ roots of the shifted Chebyshev polynomial. Finally, we solve this system by MATLAB tools and obtain the approximation polynomial.

4. Numerical solutions and discussion

This section describes the solution of Troesch and Duffing equations for different problem variables. The efficiency of the method is shown by comparing the absolute errors obtained from the current technique with other numerical techniques, and presented in terms of tables and figures.

4.1. Troesch's equation

First, we consider the Troesch's equation for $\lambda = 0.5$ and $\lambda = 1$. This problem has also been studied with many numerical techniques such as the perturbation method with Pade approximation (PM-Pade) [16], the Homotopy perturbation method (HPM) [34], the variational iteration method (VIM) [35], the modified nonlinear Shooting method (MNLSM) [2] and the decomposition method (DM) [11]. The exact solutions of the problem for both values of λ are calculated by Mathematica. In this problem, we take $k = 1$ and $M = 7$.

Table 1 presents the exact solution, the Taylor wavelet method (TWM) and other numerical method solutions for $\lambda = 0.5$. We can interpret that the TWM results match at up to six decimal places with the exact solution, whereas other numerical methods match at fewer decimal places. We can also see that the VIM does not give accurate results.

Table 1. Exact and numerical results for $\lambda = 0.5$

x_i	Exact Solution	TWM	HPM [34]	PM-Pade [16]	MNLSM [2]	VIM [35]	DM [11]
0.1	0.095944	0.095944	0.095948	0.095941	0.095972	0.100042	0.095938
0.2	0.192128	0.192128	0.192135	0.192123	0.192185	0.200334	0.192118
0.3	0.288794	0.288794	0.288804	0.288786	0.288879	0.301128	0.288780
0.4	0.386184	0.386185	0.386196	0.386174	0.386298	0.402677	0.386168
0.5	0.484547	0.484547	0.484559	0.484534	0.484416	0.505241	0.484530
0.6	0.584133	0.584133	0.584145	0.584117	0.584281	0.609082	0.584116
0.7	0.685201	0.685201	0.685212	0.685182	0.685256	0.714470	0.685186
0.8	0.788017	0.788017	0.788025	0.787994	0.788079	0.821682	0.788005
0.9	0.892854	0.892854	0.892859	0.892829	0.892926	0.931008	0.892848

Figure 1 illustrates the absolute errors obtained from the TWM, HPM, PM-Pade, MNLSM and DM. We observe that the HPM and DM errors are close to each other, PM-Pade errors are similar to HPM and DM up to 0.6, MNLSM has the highest errors at each point and TWM has the least errors. In the TWM, 6^{th} order approximation polynomial is used whereas in the HPM and PM-Pade, 9^{th} order polynomials are used. In the MNLSM, step size is taken as $h = 0.1$. We can conclude from this figure that the TWM is more accurate than these methods.

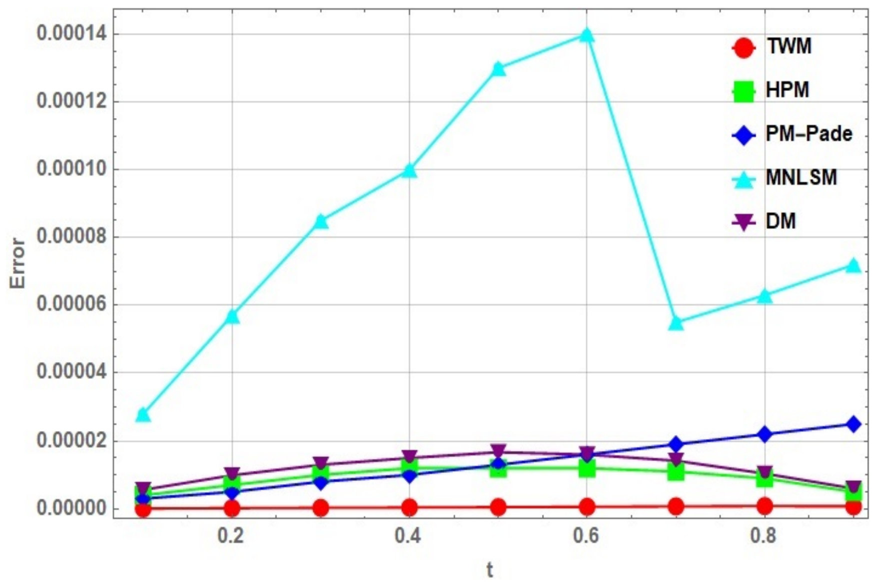

Figure 1. Absolute errors of the numerical techniques for $\lambda = 0.5$

In Table 2, we represent the analytical and numerical solutions of the problem for $\lambda = 1$. We use the same degree polynomial and observe that the technique still gives accurate results for increasing values of λ. We again observe the accuracy of the TWM results are higher than the other numerical results.

Table 2. Comparison of the results of several numerical methods for $\lambda = 1$

x_i	Exact Solution	TWM	HPM [34]	PM-Pade [16]	MNLSM [2]	VIM [35]	DM [11]
0.1	0.084661	0.084668	0.084934	0.084705	0.084730	0.100167	0.084248
0.2	0.170171	0.170186	0.170697	0.170260	0.170310	0.201339	0.169430
0.3	0.257393	0.257417	0.258133	0.257531	0.257603	0.304541	0.256414
0.4	0.347222	0.347254	0.348116	0.347413	0.347506	0.410841	0.346085
0.5	0.440599	0.440639	0.441572	0.440849	0.439937	0.521373	0.439401
0.6	0.538534	0.538582	0.539498	0.538848	0.538905	0.637362	0.537365
0.7	0.642128	0.642187	0.642987	0.642508	0.642093	0.760162	0.641083
0.8	0.752608	0.752676	0.753267	0.753043	0.752558	0.891287	0.751788
0.9	0.871362	0.871426	0.871733	0.871811	0.871310	1.032460	0.870908

To have a better understanding of the effectiveness of the current method, we present the absolute errors obtained from the TWM, HPM, PM-Pade, MNLSM and DM in Figure 2. We observe that the DM has the highest number of errors at each point where the TWM has the lowest number of errors. Thus, we may suggest that the TWM is more accurate than these methods for an increasing value of λ.

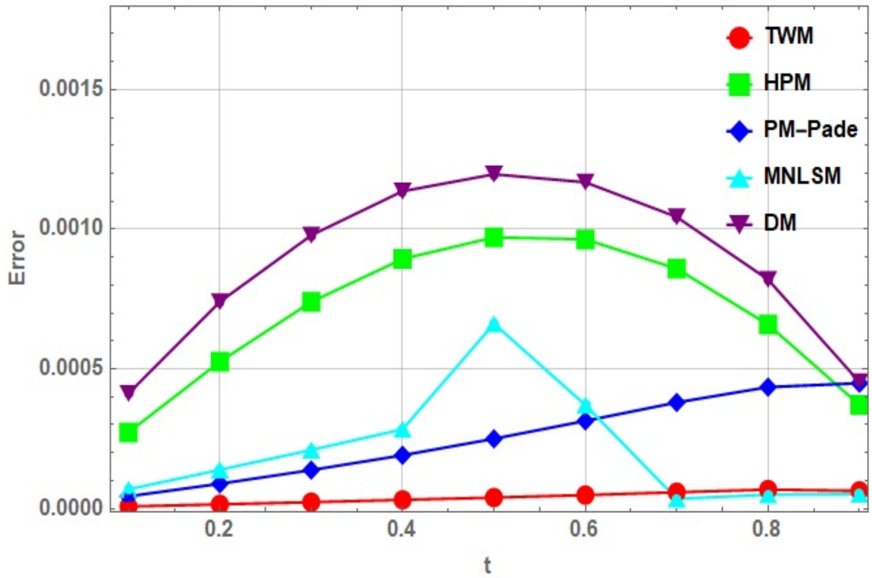

Figure 2. Absolute errors obtained from the numerical methods for $\lambda = 1$

4.2. Duffing equation

Let us examine the Duffing equation in two examples. We solve these problems by taking $k = 1$ for different values of M.

4.2.1. Example 1: First, consider the damping Duffing equation, [49]:

$$\begin{aligned} &y''(x) + 2y'(x) + y(x) + 8y^3(x) = e^{-3x}, \\ &y(0) = 0.5 \quad , \quad y'(0) = -0.5 \end{aligned}$$

with the analytical solution $y(x) = \dfrac{1}{2}e^{-x}$. This equation has also been solved by the standard Adomian decomposition method (ADM) and the restarted ADM (RADM) [49]. In that study, the force function, e^{-3x}, is approximated using a fourth order Maclaurin series. In the present method, this approximation is not necessary to obtain an accurate solution. Furthermore, the problem is solved using seven iterations in ADM, and $m = 3, n = 2$ in RADM which both yield to approximation polynomials whose orders are higher than 6. On the other hand, we solve the problem by using a 6^{th} order polynomial (M=7) and compare the absolute errors in Table 3. One can see from Table 3 that the ADM and RADM have similar errors at each collocation point which are higher than the present method. Thus, we may suggest that the TWM is the more accurate than the mentioned methods.

Table 3. Absolute errors obtained from the current method, the ADM and the RADM

x_i	TWM	ADM [49]	RADM [49]
0.1	1.61e-09	4.53e-09	4.53e-09
0.2	1.09e-09	5.47e-07	5.47e-07
0.3	3.28e-09	8.85e-06	8.81e-06
0.4	5.13e-09	6.29e-05	6.24e-05
0.5	1.04e-09	2.86e-04	2.81e-04
0.6	8.00e-10	9.86e-04	9.57e-04
0.7	1.36e-08	2.80e-03	2.67e-03
0.8	5.07e-11	6.93e-03	6.48e-03
0.9	2.93e-07	1.54e-02	1.40e-02
1	1.58e-06	3.18e-02	2.80e-02

4.2.2. Example 2: The last example is the Duffing equation given in [51]

$$\begin{aligned}&y''(x) + (0.4)y'(x) + (1.1)y(x) + y^3(x) = (2.1)\cos(1.8)x,\\&y(0) = 0.5 \quad , \quad y'(0) = -0.5\end{aligned}$$

This equation does not have an analytical solution. Thus, we solve the problem by using Mathematica for comparison purposes. The Mathematica solutions (MS), the TWM solutions with $M = 11$ and the Laplace decomposition method (LDM) solutions [51] are presented in Table 4.

Table 4. Results and absolute errors obtained from the Mathematica solution (MS), TWM and LDM [51]

x_i	MS	TWM	LDM [51]	Error in TWM	Error in LDM
0.1	0.083584545	0.083584532	0.083584535	1.32e-08	1.10e-08
0.2	-0.105091905	-0.105091922	-0.105091915	1.72e-08	9.41e-09
0.3	-0.266020292	-0.266020310	-0.266020303	1.76e-08	1.02e-08
0.4	-0.399977924	-0.399977960	-0.399977948	3.61e-08	2.36e-08
0.5	-0.508314781	-0.508314822	-0.508128865	4.08E-08	2.8e-08
0.6	-0.592890659	-0.592890702	-0.592890689	4.29e-08	2.97e-08
0.7	-0.656065476	-0.656065523	-0.656065502	4.62e-08	2.50e-08
0.8	-0.700676914	-0.700676946	-0.700676939	3.26e-08	2.48e-08
0.9	-0.729970824	-0.729970883	-0.732272085	5.89e-08	2.54e-08
1	-0.747476049	-0.747473382	-0.747476077	2.67e-06	2.75e-08

5. Conclusion

In this study, Taylor wavelets are introduced to find approximate solutions of Troesch and Duffing equations. The implementation of the technique transforms the indicated equations into a system of nonlinear equations. The efficiency of the technique is illustrated by comparing the numerical results with different numerical results and exact solutions. It is shown that accurate numerical results can be obtained by using low degree polynomials; the method is effectual, easy to implement and highly precise.

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