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Some Results Related to Strongly s-η Convex Functions of Hermite-Hadamard-Fejer Type Inequalities

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Abstract

The inequalities of Hermite-Hadamard-Fejer and Hemite-Hadamard are one of the most fundamental inequalities of the concept of convex functions. New inequalities and results are obtained from applying these inequalities to many convex functions. This paper deals with Hermite-Hadamard-Fejer type inequality for the strongly s-η-convex function consisting of strong s-convex and strong η-convex functions. Some new results related to the left-hand side of Hermite-Hadamard-type inequalities are obtained such that the modules of the second derivatives of the functions are strongly s-η-convex.

Hermite-Hadamard-Fejer Tipi Eşitsizliklerin Güçlü s-η Konveks Fonksiyonlarıyla İlgili Bazı Sonuçlar

Anahtar kelimeler

Güçlü s-η-Konveks; Güçlü η-Konveks; Güçlü s-Konveks; Güçlü Konveks; Hermite-Hadamard Tipli Eşitsizlik; Hermite-Hadamard-Fejer Tipli Eşitsizlik

Öz

Hermite-Hadamard ve Hemite-Hadamard-Fejer eşitsizlikleri konveks fonksiyonlar kavramının en temel eşitsizliklerinden birileridir. Bu eşitsizlikler birçok konveks fonksiyonlara uygulanarak yeni eşitsizlikler ve sonuçlar elde edilmiştir. Bu makale güçlü s-konveks ve güçlü η-konveks fonksiyonlardan oluşan güçlü s-η-konveks fonksiyonu için Hermite-Hadamard-Fejer tipli eşitsizliklerle ilgilenmektedir. Fonksiyonların ikinci türevlerinin mutlak değerleri güçlü s-η-konveks olacak şekilde Hermite-Hadamard tipli eşitsizliklerin sol tarafı ile ilgili bazı yeni sonuçlar elde edilmiştir.

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1. Introduction and Preliminaries

It is known that convex functions have resurfaced today as from the time of Archimedes and Euclid. The importance of convex functions in mathematics cannot be denied. To date, the convex functions have been developed by many researchers and have been used in many branches of science. Since the generalized convex functions are needed to have new developments regarding convex functions, many generalized convex functions and some forms of integral inequalities for these functions have been applied in recent years (Bayraktar and Gurbüz 2020, Awan *et al*. 2017, Song *et al*. 2018). In 1964, Polyak introduced the class of strongly convex functions and made a significant contribution to optimization development.

A function such that $\varphi: A \subset R \to R$ called strongly convex function with modulus, $\beta > 0$, if

$$\varphi(kz + (1-k)w) \leq k\varphi(z) + (1-k)\varphi(w)$$
$$-\beta k(1-k)(w-z)^2 \quad (1)$$

for all , $z, w \in A$ and $k \in [0,1]$.

Strongly convex functions attracted the attention of many researchers and studies of augmented generalized convex functions began to emerge.

Also, a function such that $\varphi: A \subset R \to R$ called strongly s-convex function with modulus, $\beta > 0$,if

$$\begin{aligned}\varphi(kz + (1-k)w) \\ \leq k^s\varphi(z) + (1-k)^s\varphi(w) \\ - \beta k(1-k)(w-z)^2 \qquad (2)\end{aligned}$$

for all, $z, w \in A$ and $k \in [0,1]$ (Erdem *et al*. 2017).

In 2018, M. Awan introduced the strong-convex functions class and demonstrated the Hermite-Hadamard type inequalities for these functions.

A function such that $\varphi: A \subset R \to R$ called strongly η-convex function with modulus, $\beta > 0$, if

$$\begin{aligned}\varphi(kz + (1-kl)w) \\ \leq \varphi(w) + k\eta\big(\varphi(z), \varphi(w)\big) \\ - \beta k(1-k)(w-z)^2 \qquad (3)\end{aligned}$$

for all, $z, w \in A$ and $k \in [0,1]$ (Erdem *et al*. 2017).

In 2020, M. Cortez introduced the class of $s-\eta$-convex functions, which is a combination of *s*-convex and η-convex functions.

A function such that $\varphi: A \subset R \to R$ called strongly s-η-convex function with modulus, β> 0, if

$$\begin{aligned}\varphi(kz + (1-k)w) \\ \leq \varphi(w) + k^s\eta\big(\varphi(z), \varphi(w)\big) \qquad (4)\end{aligned}$$

for all, $z, w \in A$ and $k \in [0,1]$

(Vivas-Cortez *et al*. 2020).

Inspired by these studies, we present the following definition.

Definition 1. The function such that $\varphi: A \subset R \to R$ called strongly s-η-convex function with respect to $\eta: R \times R \to R$ and modulus $\beta > 0$, where $0 < s < 1$, if

$$\varphi(kz + (1-k)w) \leq \varphi(w) + k^s\eta\big(\varphi(z), \varphi(w)\big)$$
$$-\beta k(1-k)(w-z)^2 \qquad (5)$$

for all, $z, w \in A$ and $k \in [0,1]$

(Öğünmez and Toigombaeva 2021).

Remark 1. If $z = w$ in (5), then we have

$$\varphi(z) \leq \varphi(z) + k^s\eta\big(\varphi(z), \varphi(z)\big)$$
$$- \beta k(1-k)(z-z)^2$$
$$k^s\eta\big(\varphi(z), \varphi(z)\big) \geq 0.$$

If $k = 1$ in (5), we get

$$\varphi(z) \leq \varphi(w) + \eta(\varphi(z), \varphi(w))$$
$$\varphi(z) - \varphi(w) \leq \eta(\varphi(z), \varphi(w)).$$

If $\eta\big(\varphi(z), \varphi(w)\big) = \varphi(z) - \varphi(w)$ in (5) then we also obtain

$$\begin{aligned}&\varphi(kz + (1-k)w) \\ &\quad \leq \varphi(w) + k^s(\varphi(z) - \varphi(w)) \\ &\quad - \beta k(1-k)(w-z)^2 \\ &= k^s\varphi(z) + (1-k^s)\varphi(w) \\ &\quad - \beta k(1-k)(w-z)^2 \\ &\leq \varphi(w) + k\eta(\varphi(z), \varphi(w)) - \beta k(1-k)(w-z)^2\end{aligned}$$

In (2013), M. E. Özdemir *et al*. obtained some new results for the left-hand side of Hermite-Hadamard type inequalities for the *s*-convex function in the second sense.

Lemma 1. Let $\varphi: A \subset R \to R$ is a smooth map on A, where $e, f \in A$ and $e < f$. If $\varphi'' \in L([e,f])$, then we state the following result

$$\begin{aligned}&\frac{1}{f-e}\int_e^f \varphi(z)dz - \varphi\left(\frac{e+f}{2}\right) \\ &= \frac{(f-e)^2}{16}\left[\int_0^1 k^2\varphi''\left(k\frac{e+f}{2} + (1-k)e\right)dk \right. \\ &\left. + \int_0^1 (1-k)^2\varphi''\left(kf + (1-k)\frac{e+f}{2}\right)dk\right]\end{aligned}$$

(Ozdemir *et al*. 2013).

Theorem 1. Suppose that $\varphi: A \subset [0,\infty) \to R$ be a twice smooth map on A^0 such that $\varphi'' \in L([e,f])$, where $e,f \in A$ and $e < f$. If $|\varphi''|$ is strongly s-η-convex on $[e,f]$, for some $s \in (0,1]$ with modulus $\beta > 0$, then we have the following equality

$$\left|\varphi\left(\frac{e+f}{2}\right) - \frac{1}{f-e}\int_e^f \varphi(z)dz\right|$$
$$\leq \frac{(f-e)^2}{8(s+1)(s+2)(s+3)} \times \left\{|\varphi''(e)| + (s+1)(s+2)\left|\varphi''\left(\frac{e+f}{2}\right)\right| + |\varphi''(f)|\right\} - \frac{\beta}{160}(f-e)^2 \quad (7)$$
$$\leq \frac{(f-e)^2}{8(s+1)(s+2)(s+3)} \times \left\{[1+(s+2)2^{1-s}][|\varphi''(e)| + |\varphi''(f)|] - \frac{[1+(s+1)(s+2)2^{1-s}]}{12}\beta(f-e)^2\right\} - \frac{\beta}{160}(f-e)^2$$

(Erdem *et al*. 2017).

2. Main Results

Theorem 2. Let $\varphi: A \to R$ be a strongly s-η-convex function with modulus $\beta > 0$. Let the function $\psi: A \to R^+$ be an integrable and suppose that it is a symmetric function with respect to $\frac{e+f}{2}$. Then we have

$$\varphi\left(\frac{e+f}{2}\right)\int_e^f \psi(z)dz - \frac{M_\eta}{2^s}\int_e^f \psi(z)dz + \frac{\beta}{4}\int_e^f (2z-e-f)^2\psi(z)dz$$
$$\leq \int_e^f \varphi(z)\psi(z)\,dz \quad (8)$$
$$\leq \frac{\varphi(e)+\varphi(f)}{2}\int_e^f \psi(z)dz + \frac{M_\eta}{2(f-e)^s}\int_e^f (z-e)^s\psi(z)dz - \beta\int_e^f (z-e)(f-z)\psi(z)dz.$$

Here M_η is the constant which is an upper bound of the η-convex function.

Proof. Since φ is a strongly s-η-convex function, then we get

$$\varphi\left(\frac{e+f}{2}\right) = \varphi\left(\frac{ek+f(1-k)+fk+e(1-k)}{2}\right)$$
$$\leq \varphi(fk+e(1-k)) + \frac{1}{2^s}\eta\big(\varphi(ek+f(1-k)),\varphi(fk+e(1-k))\big) - \frac{\beta}{4}(fk+e(1-k)-ek-f(1-k))^2$$
$$= \varphi\big(fk+e(1-k)\big) + \frac{1}{2^s}\eta\big(\varphi(ek+f(1-k)),\varphi(fk+e(1-k))\big) - \frac{\beta}{4}(2k-1)^2(f-e)^2 \quad (9)$$
$$\leq \varphi\big(fk+e(1-k)\big) + \frac{1}{2^s}M_\eta - \frac{\beta}{4}(2k-1)^2(f-e)^2.$$

If we multiply both sides of (9) by $\psi\big(fk+e(1-k)\big)$ and integrate with respect to k on [0,1], then we derive

$$\varphi\left(\frac{e+f}{2}\right)\int_0^1 \psi\big(fk+e(1-k)\big)dk$$
$$\leq \int_0^1 \varphi\big(fk+e(1-k)\big)\psi\big(fk+e(1-k)\big)\,dk + \int_0^1 \frac{1}{2^s}M_\eta\,\psi\big(fk+e(1-k)\big)dk \quad (10)$$

$$-\int_0^1 \frac{\beta}{4}(2k-1)^2(f-e)^2\psi\big(fk+e(1-k)\big)dk.$$

Next, if we use the changing variables

$z=fk+e(1-k)$ and $fk=\frac{1}{f-e}dz$, then it follows that

$$\frac{1}{f-e}\varphi\left(\frac{e+f}{2}\right)\int_e^f \varphi(z)dz$$

$$\leq \frac{1}{f-e}\int_e^f \varphi(z)\psi(z)\,dz+\frac{M_\eta}{2^s(f-e)}\int_e^f \psi(z)dz$$

$$-\frac{\beta}{4(f-e)}\int_e^f \left(2\frac{z-e}{f-e}-1\right)^2(f-e)^2\,\psi(z)dz$$

$$=\frac{1}{f-e}\int_e^f \varphi(z)\psi(z)\,dz+\frac{M_\eta}{2^s(f-e)}\int_e^f \psi(z)dz$$

$$-\frac{\beta}{4(f-e)}\int_e^f (2z-f-e)^2\,\psi(z)dz. \quad (11)$$

Multiplying both sides of (11) by $(f-e)$, we have

$$\varphi\left(\frac{e+f}{2}\right)\int_e^f \varphi(z)dz$$

$$-\frac{M_\eta}{2^s}\int_e^f \psi(z)dz-\frac{\beta}{4}\int_e^f (2z-e-f)^2\,\psi(z)dz$$

$$\leq \int_e^f \varphi(z)\psi(z)\,dz. \quad (12)$$

Now, we obtain the left hand side of the desired inequality. By using the fact that φ is strongly s-η-convex function and $\psi(z)$ is symmetric, then we have

$$\varphi(ke+(1-k)f)+\varphi(kf+(1-k)e)$$

$$\leq \varphi(f)+k^s\eta\big(\varphi(e),\varphi(f)\big)-\beta k(1-k)(f-e)^2$$

$$+\varphi(e)+k^s\eta\big(\varphi(f),\varphi(e)\big)-\beta k(1-k)(f-e)^2$$

$$=[\varphi(e)+\varphi(f)] \quad (13)$$

$$+k^s\Big(\eta\big(\varphi(e),\varphi(f)\big)+\eta\big(\varphi(f),\varphi(e)\big)\Big)$$

$$-2\beta k(1-k)(f-e)^2$$

$$\leq [\varphi(e)+\varphi(f)]+k^sM_\eta$$

$$-2\beta k(1-k)(f-e)^2.$$

If we multiply both sides of the inequality (13) by $\psi(kf+(1-k)e)$ and integrate with respect to k on [0,1], we deduce

$$\int_0^1 \varphi(ke+(1-k)f)\psi(kf+(1-k)e)dk$$

$$+\int_0^1 \varphi(ke+(1-k)f)\psi(kf+(1-k)e)dk$$

$$\leq [\varphi(e)+\varphi(f)]\int_0^1 \psi(kf+(1-k)e)dk$$

$$+M_\eta\int_0^1 k^s\psi(kf+(1-k)e)dk$$

$$-2\beta(f-e)^2\int_0^1 k(1-k)\varphi(kf+(1-k)e)dk$$

By changing variables $z=ke+(1-k)f$ and $dz=(e-f)dk$, we also get

$$\frac{2}{f-e}\int_e^f \varphi(z)\psi(z)dz$$

$$\leq \frac{\varphi(e)+\varphi(f)}{f-e}\int_e^f \psi(z)dz$$

$$+\frac{M_\eta}{f-e}\int_e^f \left(\frac{z-e}{f-e}\right)^s\psi(z)dz$$

$$-\frac{2\beta(f-e)^2}{f-e}\int_e^f \left(\frac{z-e}{f-e}\right)\left(1-\frac{z-e}{f-e}\right)\psi(z)dz$$

$$=\frac{\varphi(e)+\varphi(f)}{f-e}\int_e^f \psi(z)dz \quad (14)$$

$$+\frac{M_\eta}{(f-e)^{s+1}}\int_e^f (z-e)^s\psi(z)dz$$

$$-\frac{2\beta}{f-e}\int_e^f (z-e)(f-z)\psi(z)dz.$$

Then multiplying both sides of (14) by $(f-e)/2$, we have

$$\int_e^f \varphi(z)\psi(z)dz \leq \frac{\varphi(e)+\varphi(f)}{2}\int_e^f \psi(z)dz$$

$$+\frac{M_\eta}{2(f-e)^s}\int_e^f (z-e)^s\psi(z)dz \qquad (15)$$

$$-\beta\int_e^f (z-e)(f-z)\psi(z)dz.$$

This completes the proof.

Theorem 3. Suppose that $\varphi: A \subset [0,\infty) \to R$ is a twice smooth map on A, where $e, f \in A$ and e$< f$. If $|\varphi''|$ is strongly s-η-convex on $[e,f]$ with modulus, $\beta > 0$, then the following satisfies

$$\left|\frac{1}{f-e}\int_e^f \varphi(z)dz - \varphi\left(\frac{e+f}{2}\right)\right|$$

$$\leq \frac{(f-e)^2}{16}\left[\frac{1}{3}\left(|\varphi''(e)| + \left|\varphi''\left(\frac{e+f}{2}\right)\right|\right)\right.$$

$$+\frac{1}{(s+1)(s+2)(s+3)} \qquad (16)$$

$$\times\left((s+2)(s+3)\eta\left(\left|\varphi''\left(\frac{e+f}{2}\right)\right|, |\varphi''(e)|\right)\right.$$

$$\left.\left.+2\,\eta\left(|\varphi''(f)|, \left|\varphi''\left(\tfrac{e+f}{2}\right)\right|\right)\right)\right] - \frac{\beta}{160}(f-e)^2.$$

Proof: Taking modulus on both sides of Lemma 1. Using the fact that $|\varphi''|$ is strongly $s-\eta$-convex, we have

$$\left|\frac{1}{f-e}\int_e^f \varphi(z)dz - \varphi\left(\frac{e+f}{2}\right)\right|$$

$$\leq \frac{(f-e)^2}{16}\left[\int_0^1 k^2\left|\varphi''\left(k\frac{e+f}{2}+(1-k)e\right)\right|dk\right.$$

$$\left.+\int_0^1 (1-k)^2\left|\varphi''\left(kf+(1-k)\frac{e+f}{2}\right)\right|dk\right]$$

$$\leq \frac{(f-e)^2}{16}\left[\int_0^1 k^2\left(|\varphi''(e)|\right.\right.$$

$$+k^s\eta\left(\left|\varphi''\left(\frac{e+f}{2}\right)\right|, |\varphi''(e)|\right)$$

$$\left.-\beta k(1-k)(f-e)^2\right)dk$$

$$+\int_0^1 (1-k)^2\left(\left|\varphi''\left(\frac{e+f}{2}\right)\right|\right.$$

$$+k^s\eta\left(|\varphi''(f)|, \left|\varphi''\left(\frac{e+f}{2}\right)\right|\right)$$

$$\left.\left.-\beta k(1-k)(f-e)^2\right)dk\right] \qquad (17)$$

$$=\frac{(f-e)^2}{16}\left[\int_0^1\left(k^2|\varphi''(e)|\right.\right.$$

$$+k^{s+2}\eta\left(\left|\varphi''\left(\frac{e+f}{2}\right)\right|, |\varphi''(e)|\right)$$

$$\left.-\beta k^3(1-k)(f-e)^2\right)dk$$

$$+\int_0^1\left((1-k)^2\left|\varphi''\left(\frac{e+f}{2}\right)\right|\right)dk$$

$$+k^s(1-k)^2\eta\left(|\varphi''(f)|, \left|\varphi''\left(\frac{e+f}{2}\right)\right|\right)$$

$$-\beta k(1-k)^3(f-e)^2].$$

By calculating the required integral in (17), we get

$$\left|\frac{1}{f-e}\int_e^f \varphi(z)dz - \varphi\left(\frac{e+f}{2}\right)\right|$$

$$\leq \frac{(f-e)^2}{16}\left[\frac{1}{3}|\varphi''(e)|\right.$$

$$+\frac{1}{s+3}\eta\left(\left|\varphi''\left(\frac{e+f}{2}\right)\right|, |\varphi''(e)|\right) - \frac{\beta}{20}(f-e)^2$$

$$+\frac{1}{3}\left|\varphi''\left(\frac{e+f}{2}\right)\right|$$

$$+\frac{2}{(s+1)(s+2)(s+3)}\eta\left(|\varphi''(f)|, \left|\varphi''\left(\frac{e+f}{2}\right)\right|\right)$$

$$\left.-\frac{\beta}{20}(f-e)^2\right]$$

$$=\frac{(f-e)^2}{16}\left[\frac{1}{3}\left(|\varphi''(e)| + \left|\varphi''\left(\frac{e+f}{2}\right)\right|\right)\right.$$

$$\left.+\frac{1}{(s+1)(s+2)(s+3)}\right]$$

$$\times\left((s+2)(s+3)\eta\left(\left|\varphi''\left(\frac{e+f}{2}\right)\right|,|\varphi''(e)|\right)\right.$$
$$\left.+2\,\eta\left(|\varphi''(f)|,\left|\varphi''\left(\frac{e+f}{2}\right)\right|\right)\right)$$
$$-\frac{\beta}{160}(f-e)^2.$$

This completes the proof.

3. Conclusion and Discussion

To obtain new results related to Hermite-Hadamard inequalities, the generalized convex functions and the generalized strongly convex functions are needed. In recent years, some new convex functions have been introduced by the combinations of generalized convex functions such as *s*-*h*-convex, *s*-*m*-convex, *s*-*η*-convex (Bayraktar and Gurbüz 2020, Öğünmez and Toigombaeva 2021, Obeidat 2019) etc.

In our previous work, we introduced the concept of strongly *s*-*η*-convex functions. This paper concerns about Hermite Hadamard-Fejer type inequalities for strongly s-η-convex functions. In future works, researchers are able to derive new integral inequalities using the strongly *s*-*η*-convex functions.

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