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Araştırma Makalesi / Research Article

On Approximation Properties of Gamma Type Operator Based on (p,q)-Integer

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Abstract

In the literature; extensive work on the q and (p,q)-calculus has contributed greatly to describing the different generalizations of many operators involving the q and (p,q)-integers. In this study, we will present to you, that we define Gamma type operator based on (p,q)-integer. We get some direct output including asymptotic formula and error estimation in terms of modulus continuity. In addition, as a result of the research, we estimate the convergence rate of these operators in a weighted space.

Gama Tipi Operatörün (p,q)-tamsayı İkililerine Dayalı Yaklaşım Özellikleri

Anahtar kelimeler
Gama tipi operatör; Ağırlıklı süreklilik modülü; Lokal yaklaşım; (p;q)-Tamsayı ikilileri

Öz

Literatürde; q ve (p,q)-hesabı üzerindeki kapsamlı çalışma q ve (p,q)-tamsayı ikililerini içeren birçok operatörün farklı genellemelerinin tanımlanmasına büyük ölçüde katkıda bulunmuştur. Size sunacağımız bu çalışmada (p,q)-tamsayı ikililerine göre gama tipi operatörü tanımlayarak süreklilik modül açısından asimtotik formül ve hata tahmini içeren bazı doğrudan sonuçlar elde edeceğiz. Ayrıca, bu operatörlerin ağırlıklı bir uzayda yakınsaklığını araştırarak ve yakınsaklık oranını tahmin ediyoruz.

1. Introduction

Recently Karsli (2007,2010) defined and studied some approximation properties following gamma type operatör;

$$L_n(f;x)$$

$$=\int_0^\infty\int_0^\infty g_{n+2}(x,u).\,g_n(x,t).\,f(t)$$

$$=\frac{(2n+3)!\,x^{n+3}}{n!\,(n+2)!}\int_0^\infty \frac{t^n}{(1+t)^{2n+4}}f(t)dt$$

Later Mao (2007) introduced following generalized Gamma type operatör;

$$M_{n,k}(f\,;x)$$

$$=\int_0^\infty\int_0^\infty g_{(n-k)}(x\,,u)g_n(x\,,t)f(t).\,dt$$

$$=\frac{(2n-k+1)!\,x^{(n+1)}}{n!\,(n-k)!}\int_0^\infty \frac{t^{(n-k)}}{(1+t)^{(2n-k+2)}}f(t)dt$$

Karsli *et al.* (2015) introduced stancu type modification of the q-type of general Gamma operators as follows:

$$M_{n,k,q}^{\alpha,\beta}\quad(f\,;x)$$

$$=\frac{[2n-k+1]_q!\left(q^{\frac{2n-k+1}{2}}.x\right)^{(n+1)}}{[n]_q!\,[n-k]_q!}q^{\frac{(n-k)(n-k+1)}{2}}$$

$$\int_0^{\frac{\infty}{A}} \frac{t^{n-k}}{\left(q^{\frac{2n-k+1}{2}}x+t\right)^{2n-k+2}} . f\left(\frac{[n]_q . t+\alpha}{[n]_q+\beta}\right) d_q t.$$

They also present Voronovskaja type theorem and established some direst results.

These days, in the field of approximation theory one and two variable quantum calculus and its applications seem to have created a new field of study with different results.

Major work on the q and (p,q) −calculus has played an important role in arriving at different generalizations of many operators involving q and (p,q) −integers. As a result, some researchers have examined many positive linear operators based on q and (p,q)-integers and have brought them to the literature by giving their definitions (Karaisa 2015, 2016), (Gupta 2006, 2018), (Acar 2016, 2018), (Mursaleen 2015), (Khursheed and Karaisa 2017), (Sadiang 2018).

In this article motivated by these studies in the literature, we introduce the approximation properties of Gamma type operator based on (p,q) −integer.

Since the (p,q) −calculus has been the most intrusting area of research, this motivated us to introduce (p,q) −analogue of the generalized Gamma operators. First, let's start by examining some definitions and notations of the (p,q) −calculus concept.

The (p,q) −integer of the number n given as follows:

$$[n]_{p,q} := \left(\frac{p^n-q^n}{p-q}\right), \qquad 0<q<p\leq 1, \quad n=1,2,..$$

The (p,q) −factorial $[n]_{p,q}!$ and the (p,q) −binomial coefficients are defined as:

$$[n]_{p,q}! := \begin{cases} [n]_{p,q} . [n-1]_{p,q} \dots\dots [1]_{p,q}, & n\in N \\ 1, & n=0 \end{cases}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}! . [n-k]_{p,q}!}, 0\leq k\leq n.$$

Further the (p,q) −binomial expansions are given as

$$(ax+by)^n_{p,q} = \sum_{k=0}^{n} p^{\binom{n-k}{2}} . q^{\binom{k}{2}} . a^{n-k} . b^k . x^{n-k} . y^k$$

and

$$(x-y)^n_{p,q} = (x-y).(px-qy).(p^2x-q^2y)\dots..(p^{n-1}x-q^{n-1}y).$$

Two different (p,q) −expansions named $E_{p,q}$ and $e_{p,q}$ of the exponential functions are given as follows;

$$e_{p,q}(x) = \sum_{k=0}^{\infty} \frac{p^{\frac{n.(n-1)}{2}}}{[n]_{p,q!}} x^n$$

and

$$E_{p,q}(x) = \sum_{k=0}^{\infty} \frac{q^{\frac{n.(n-1)}{2}}}{[n]_{p,q!}} x^n.$$

We can easily deduce $e_{p,q}(x).E_{p,q}(-x)=1$.

Aral and Gupta (2016) for any $n,m\in N$, proposed (p,q) −Beta, (p,q) −Gamma function of second kind follows:

$$B_{p,q}(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}_{p,q}} d_{p,q}x \qquad (1)$$

and

$$\Gamma_{p,q}(n) = \int_0^{\infty} p^{\frac{n.(n-1)}{2}} E_{p,q}(-qx) d_{p,q}x,$$

$$\Gamma_{p,q}(n+1) = [n]_{p,q}!.$$

Also they showed following relation of $B_{p,q}(m,n)$ and $\Gamma_{p,q}(n)$:

$$B_{p,q}(m,n) = q^{\frac{(2-m.(m-1))}{2}} . p^{\frac{(-m.(m-1))}{2}} . \frac{\Gamma_{p,q}(n).\Gamma_{p,q}(m)}{\Gamma_{p,q}(m+n)} \qquad (2)$$

Now for $0<q<p\leq 1$ we define (p,q)-analogue of generalized Gamma type operator as follows:

$$L_{n,k,p,q}(f;x) = \frac{[2n-k+1]_{p,q}! \, x^{n+1} q^{\frac{((n-k).(n-k+1)-2)}{2}} p^{\frac{((n-k).(n-k+1)-2)}{2}}}{[n]_{p,q}! \, [n-k]_{p,q}!}$$

$$.\int_0^\infty \frac{t^{n-k}}{(x+t)_{p,q}^{(2n-k+2)}}.f(p^{n-k+1}q^{n-k+1}).d_{p,q}t$$

Lemma 1.1.

$$L_n^{p,q}(t^m,x) = \frac{[n-k+m]_{p,q}!\,[n-m]_{p,q}!\quad p^{\frac{(-m(m-1))}{2}}q^{\frac{(-m(m-1))}{2}}}{[n]_{p,q}!\,[n-k]_{p,q}!}.x^m$$

Proof. By (1) and (2), we have

$$L_{n,k,p,q}(t^m;x) = \frac{[2n-k+1]_{p,q}!\,x^{n+1}q^{\frac{((n-k).(n-k+1)-2)}{2}}p^{\frac{(n-k).(n-k+1)}{2}}}{[n]_{p,q}!\,[n-k]_{p,q}!}$$

$$.p^{mn-km+m}.q^{mn-km+m}.\int_0^\infty \frac{t^{n-k+m}}{(x+t)_{p,q}^{2n-k+2}}d_{p,q}t$$

$$= \frac{[2n-k+1]_{p,q}!\,x^{n+1}q^{\frac{(n-k).(n-k+1)-2}{2}}p^{\frac{(n-k).(n-k+1)}{2}}}{[n]_{p,q}!\,[n-k]_{p,q}!}$$

$$.p^{(mn-km+m)}q^{(mn-km+m)}.\int_0^\infty \frac{\left(\frac{t}{x}\right)^{(n-k+m)}}{\left(1+\frac{t}{x}\right)_{p,q}^{(2n-k+2)}}d_{p,q}\left(\frac{t}{x}\right)$$

$$= \frac{[2n-k+1]_{p,q}!\,x^{n+1}q^{\frac{((n-k).(n-k+1)-2)}{2}}p^{\frac{(n-k).(n-k+1)}{2}}}{[n]_{p,q}!\,[n-k]_{p,q}!}.$$

$$p^{(mn-km+m)}.q^{(mn-km+m)}.B_{p,q}((n-k+m+1),(n-m+1))$$

$$= \frac{[n-k+m]_{p,q}!\,[n-m]_{p,q}!\quad p^{\frac{-m(m-1)}{2}}q^{\frac{-m(m-1)}{2}}}{[n]_{p,q}!\,[n-k]_{p,q}!}x^m.$$

The completes the proof.

From Lemma 1.1. we obtain that.

Corollary 1.2.

$L_{n,k,p,q}(1\,;x) = 1$

$L_{n,k,p,q}(t\,;x) = \frac{[n-k+1]_{p,q}}{[n]_{p,q}}.x,$

$L_{n,k,p,q}(t^2\,;x) = \frac{[n-k+2]_{p,q}.[n-k+1]_{p,q}}{p.q.[n]_{p,q}.[n-1]_{p,q}}.x^2,$

$L_{n,k,p,q}(t^3\,;x) = \frac{[n-k+3]_{p,q}.[n-k+2]_{p,q}.[n-k+1]_{p,q}}{p^3.q^3.[n]_{p,q}.[n-1]_{p,q}.[n-2]_{p,q}}.x^3,$

$$L_{n,k,p,q}(t^4\,;x) = \frac{[n-k+4]_{p,q}.[n-k+3]_{p,q}.[n-k+2]_{p,q}.[n-k+1]_{p,q}}{p^6.q^6.[n]_{p,q}.[n-1]_{p,q}.[n-2]_{p,q}}x^4.$$

By linearity of $L_n^{p,q}$ we get.

Lemma 1.3.

$$L_{n,k,p,q}((t-x)\,;x) = \left(\frac{[n-k+1]_{p,q}}{[n]_{p,q}}-1\right).x,$$

$$L_{n,k,p,q}((t-x)^2\,;x) = \left(\frac{[n-k+2]_{p,q}.[n-k+1]_{p,q}}{p.q.[n]_{p,q}.[n-1]_{p,q}}-2\frac{[n-k+1]_{p,q}}{[n]_{p,q}}+1\right).x^2$$

2. Local approximation properties of $L_{n,p,q}(f\,;x)$

In this part of our work, we will examine the Korovkin's approximation property Altomare (1994) order of convergence under the usual modulus of continuity and Peetre's K-functional, and the rate of convergence if it is a member of the class $Lip_M(\alpha)$, etc.

Theorem 2.1. Let (p_n), (q_n) be sequences of real numbers such that $0 < q_n < p_n \le 1$ and $A > 0$. Then for each, $f \in C_m[0,\infty) := \{f \in C_m[0,\infty): |f(t)| \le M.(1+t)^m\ for\ some\ m > 0\ and\ M > 0\}$, then sequence of operators $L_{n,p_n,q_n}(f\,;x)$ converges to f uniformly on any finite closed subinterval $[0,A]$ if and only if $\lim_{n\to\infty} p_n = 1\ and\ \lim_{n\to\infty} q_n = 1$.

Proof. First, we assume that $\lim_{n\to\infty} p_n = 1$, $\lim_{n\to\infty} q_n = 1$. Now, we have to Show that $L_n(f,p_n,q_n\,;x)$ converges to f uniformly on $[0,A]$. From Corollary 1.2., we see that

$L_n(1,p_n,\ q_n\,;x) \to 1,$

$L_n(t,p_n,\ q_n\,;x) \to x,$

$L_n(t^2,p_n,\ q_n\,;x) \to x^2,$

uniformly on $[0,A]\ as\ n \to \infty$.

Property of the Korovkin theorem implies that $L_n(f,p_n,q_n;x)$ converges to f uniformly on $[0,A]$ provided $f \in C_m[0,\infty)$.

Let's show the opposite with a contradiction. Suppose that $p_n$and $q_n \nrightarrow 1$. Then they must contain subsequences $p_{n_k} \in (0,1)$, $q_{n_k} \in (0,1)$, $p_{n_k} \to a \in [0,1)\ and\ q_{n_k} \to b \in [0,1)$ as $k \to \infty$ respectively. Thus,

$$\frac{1}{[n_k]_{p_{n_k},q_{n_k}}} = \frac{p_{n_k}-q_{n_k}}{(p_{n_k})^{n_k}-(q_{n_k})^{n_k}} \to 0 \quad \text{as } k \to \infty$$

and we get;

$$L_n\left(t^2, p_{n_k}, q_{n_k}; x\right) - x^2 = \frac{[n-k+2]_{p_{n_k},q_{n_k}}.[n-k+1]_{p_{n_k},q_{n_k}}}{p_{n_k}.q_{n_k}.[n]_{p_{n_k},q_{n_k}}.[n-1]_{p_{n_k},q_{n_k}}}.x^2 - x^2$$

$= \frac{x^2}{ab} - x^2 \neq 0$ as $k \to \infty$.

This appears as a contradiction and like this $p_n \to 1$ and $q_n \to 1$ as $n \to \infty$.

Theorem 2.2. Let $f \in C_B[0,\infty)$, $q = q_n \in (0,1)$ and $p = p_n \in (q,1]$ such that $p_n \to 1$ and $q_n \to 1$ as $n \to \infty$ and $\omega_{a+1}(f,\delta)$ be modulus of continuity on the finite interval $[0,(a+1)] \subset [0,\infty)$, where $a > 0$. Then $|L_n(f,p,q;x) - f(x)| \leq 4M_f(1+a^2)\delta_n^2(x) + 2\omega_{a+1}(f,\delta_n(x))$ where $\delta_n(x) = \sqrt{L_n((t-x)^2,p,q;x)}$ calculated in Lemma 1.3.

Proof. For $x \in [0,a]$ and $t > (a+1)$ since $(t-x) > 1$,

$$\begin{aligned}|f(t) - f(x)| &\leq M_f(2 + x^2 + t^2) \\ &\leq M_f(2 + 3x^2 + 2(t-x)^2) \\ &\leq M_f((4+3x^2).(t-x)^2) \\ &\leq 4M_f(1+a^2).(t-x)^2 \qquad (3)\end{aligned}$$

For $x \in [0,a]$ and $t \leq (a+1)$

$$|f(t) - f(x)| \leq \omega_{a+1}(f,|t-x|) \leq \left(1 + \frac{|t-x|}{\delta}\right).\omega_{a+1}(f,\delta) \text{ with } \delta > 0 \qquad (4)$$

From (3) and (4) we may write

$$|f(t) - f(x)| \leq 4M_f(1+a^2).(t-x)^2 + \left(1 + \frac{|t-x|}{\delta}\right).\omega_{a+1}(f,\delta)$$

for $x \in [0,a]$ and $t > 0$. (5)

Thus by applying the Cauchy-Schwarz's inequality we get,

$$\begin{aligned}&|L_n(f,p,q;x) - f(x)| \\ &\leq L_n(|f(t) - f(x)|,p,q;x) \\ &\leq 4M_f(1+a^2).L_n((t-x)^2,p,q;x) \\ &+ \left(1 + \frac{1}{\delta}.\sqrt{L_n((t-x)^2,p,q;x)}\right).\omega_{a+1}(f,\delta) \\ &\leq 4M_f(1+a^2).\delta_n^2(x) + 2\omega_{a+1}(f,\delta_n(x))\end{aligned}$$

on choosing $\delta := \delta_n(x)$. This completes the proof.

Now we will present to you the local approximation theorem for the Kantorovich type of (p,q) −Szász operators. Let $C_B[0,\infty)$ be the space of all real-valued continuous bounded functions f on $[0,\infty)$, endowed with the norm

$$\|f\| = sup_{x\in[0,\infty)}|f(x)|.$$

The Peetre's K-functional is defined by

$$K_2(f,\delta) = inf_{\lambda\in C^2[0,\infty)}\{\|f-\lambda\| + \delta\|\lambda''\|\}$$

where $C_B^2[0,\infty) := \{\lambda \in C_B[0,\infty): \lambda', \lambda'' \in C_B[0,\infty)\}$.

By Devore (1993, $p.$ 177, $Theorem$ 2.4) there exists an absolute constant $M > 0$.

$K_2(f,\delta) \leq M\omega_2(f,\sqrt{\delta})$, where $\delta > 0$ and

$$\omega_2(f,\delta) = sup_{0<h\leq\delta}sup_{x\in[0,\infty)}|f(x+2h) - 2f(x+h) + f(x)|,$$

where $f \in C_B[0,\infty)$ and $\delta > 0$.

Theorem 2.3.
Let $f \in C_B[0,\infty)$, $0 < q < p \leq 1$, $x \in [0,\infty)$. We have,

$$|K_n(f,p,q;x) - f(x)| \leq M\omega_2\left(f,\sqrt{\delta_n(x)}\right) + \omega\left(f,\left|\left(1 - \frac{[n-k+1]_{p,q}}{[n]_{p,q}}\right).x\right|\right),$$

where M is an absolute constant ω is the usual modulus of continuity and

$\delta_n(x) = L_n((t-x)^2, p, q; x) + (1 - \frac{[n-k+1]_{p,q}}{[n]_{p,q}})^2 x^2.$

Proof. We consider the auxiliary operators L_n^* is given as follows:

$$L_n^*(f, p, q\,; x) = L_n(f, p, q\,; x) - f\left(\frac{1}{[2]_{p,q}[n]_{p,q}} + \frac{1}{q}.x\right) + f(x)$$

(6)

From Lemma 1.3. $L_n^*(f, p, q\,; x)$ are linear and reconstructs linear operators. Thus,

$$L_n^*((t-x), p, q\,; x) = L_n((t-x), p, q\,; x) - \left(\left(\frac{[n-k+1]_{p,q}}{[n]_{p,q}}.x\right) - x\right)$$
$$= L_n(t, p, q\,; x) - x.L_n(1, p, q\,; x) - \left(\frac{[n-k+1]_{p,q}}{[n]_{p,q}}.x\right) + x = 0.$$

Let $x \in [0,\infty)$ and $\lambda \in C_B^2[0,\infty)$. Using the Taylor's formula

$$\lambda(t) = \lambda(x) + \lambda'(x)(t-x) + \int_x^t (t-\theta)\lambda''(\theta)d\theta.$$

Applying L_n^* to either side of the given equation and using (6) we have,

$$L_n^*(\lambda, p, q\,; x) - \lambda(x) = L_n^*\big((t-x).\lambda'(x), p, q\,; x\big) + L_n^*\left(\int_x^t (t-\theta).\lambda''(\theta)d\theta, p, q\,; x\right)$$
$$= \lambda'(x).L_n^*\big((t-x), p, q\,; x\big) + L_n^{(p,q)}\left(\int_x^t (t-\theta).\lambda''(\theta)d\theta, p, q\,; x\right)$$
$$- \int_x^{\frac{[n-k+1]_{p,q}}{[n]_{p,q}}} \left(\frac{[n-k+1]_{p,q}}{[n]_{p,q}}.x - \theta\right).\lambda''(\theta)d\theta.$$

In addition to this since

$$\left|\int_x^t (t-\theta).\lambda''(\theta)d\theta\right| \le \int_x^t |t-\theta|.|\lambda''(\theta)|d\theta \le \|\lambda''\|.\int_x^t |t-\theta|d\theta \le (t-x)^2.\|\lambda''\|$$ and

$$\left|\int_x^{\frac{[n-k+1]_{p,q}}{[n]_{p,q}}} \left(\frac{[n-k+1]_{p,q}}{[n]_{p,q}}.x - \theta\right).\lambda''(\theta)d\theta\right| \le \left(\frac{[n-k+1]_{p,q}}{[n]_{p,q}}.x - \theta\right)^2.\|\lambda''\|.$$

We conclude that,

$$L_n^*(\lambda, p, q\,; x) - \lambda(x) = \left|L_n\left(\int_x^t (t-\theta).\lambda''(\theta)d\theta, p, q\,; x\right) - \int_x^{\frac{[n-k+1]_{p,q}}{[n]_{p,q}}x} \left(\frac{[n-k+1]_{p,q}}{[n]_{p,q}}.x - \theta\right)\lambda''(\theta)d\theta\right|$$
$$\le \|\lambda''\|.L_n((t-x)^2, p, q\,; x) + \left(\frac{[n-k+1]_{p,q}}{[n]_{p,q}}.x - x\right)^2.\|\lambda''\|$$
$$= \delta_n(x).\|\lambda''\|. \qquad (7)$$

Keeping in mind the boundedness of L_n^* by (6) we have,

$$|L_n^*(f, p, q\,; x)| \le |L_n(f, p, q\,; x)| + 2\|f\| \le 3\|f\|. \qquad (8)$$

Using (7) and (8) in (6) we obtain

$$
\begin{aligned}
|L_n(f,p,q\,;x)-f(x)| &\leq |L_n^*(f,p,q\,;x)-f(x)| \\
&+\left|f(x)-f\left(\frac{[n-k+1]_{p,q}}{[n]_{p,q}}.x\right)\right| \\
&\leq \left|L_n^*\big((f-\lambda),p,q\,;x\big)\right. \\
&\left.-(f-\lambda)(x)\right| \\
&+\left|f(x)-f\left(\frac{[n-k+1]_{p,q}}{[n]_{p,q}}.x\right)\right| \\
&+|L_n^*(\lambda,p,q\,;x)-\lambda(x)| \\
&\leq \left|L_n^*\big((f-\lambda),p,q\,;x\big)\right. \\
&\left.-(f-\lambda)(x)\right| \\
&+\left|f(x)-f\left(\frac{[n-k+1]_{p,q}}{[n]_{p,q}}.x\right)\right| \\
&+|K_n^*(\lambda,p,q\,;x)-\lambda(x)| \\
&\leq 4\|f-\lambda\| \\
&+\omega\left(f,\left|\left(1-\frac{[n-k+1]_{p,q}}{[n]_{p,q}}\right).x\right|\right) \\
&+\delta_n(x).\|\lambda''\|.
\end{aligned}
$$

We have the following result;

$$
\begin{aligned}
&|L_n(f,p,q\,;x)-f(x)| \\
&\leq 4L_2\big(f,\delta_n(x)\big)+\omega\left(f,\left|\left(1\right.\right.\right. \\
&\qquad\left.\left.\left.-\frac{[n-k+1]_{p,q}}{[n]_{p,q}}.x\right)\right|\right).
\end{aligned}
$$

By the definition of $K-funcitional$ we get,

$$
\begin{aligned}
&|K_n(f,p,q\,;x)-f(x)| \\
&\leq M\omega_2\left(f\,,\sqrt{\delta_n(x)}\right)+\omega\left(f,\left|\left(1-\frac{[n-k+1]_{p,q}}{[n]_{p,q}}\right).x\right|\right).
\end{aligned}
$$

3. Concluding Remarks.

Karsli (2007,2010) investigated some approximation properties of gamma type operators for the first time in the literature and gave important definitions. Then Mao (2007) investigated the generalized case of this operator. Recently, one and two variable quantum calculus and its applications have been adopted by many researchers as a new field of study. Karsli et all. (2015) introduced the stancu-type modification of the q-type of general gamma operators with the help of gamma operators. These studies in the literature have motivated us to investigate the approach Properties of the (p,q)-integer-based gamma-type operator.

The results we obtained above; as a continuation of this study, it has contributed to our investigation of the generalized form and properties of this operator.

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