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Research Article

Fibonomial and Lucanomial sums through well-poised q-series

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Abstract

By making use of known identities of terminating well-poised q-series, we shall demonstrate several remarkable summation formulae involving products of two Fibonomial/Lucanomial coefficients or quotients of two such coefficients over a third one.

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1. Introduction and outline

For an indeterminate x, Horadam and Mahon [11, 15] introduced two Fibonacci–like polynomial sequences $\{U_n\}$ and $\{V_n\}$ by the linear recurrence relations

$$U_n = 2xU_{n-1} + U_{n-2} \quad \text{and} \quad V_n = 2xV_{n-1} + V_{n-2}$$

with different initial conditions

$$U_0 = 0,\ U_1 = 1 \quad \text{and} \quad V_0 = 2,\ V_1 = 2x.$$

The Binet forms for these polynomials read as

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n$$

where

$$\alpha := \alpha(x) = x + \sqrt{x^2+1} \quad \text{and} \quad \beta := \beta(x) = x - \sqrt{x^2+1}$$

with

$$\alpha + \beta = 2x \quad \text{and} \quad \alpha\beta = -1.$$

When $\alpha, \beta = \frac{1\pm\sqrt{5}}{2}$ (or equivalently $x = \frac{1}{2}$), these sequences $\{U_n\}$ and $\{V_n\}$ will reduce, respectively, to the Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$. Instead, for $\alpha, \beta = 1 \pm \sqrt{2}$ (or equivalently $x = 1$), these sequences $\{U_n\}$ and $\{V_n\}$ will become the Pell sequence $\{P_n\}$ and the Pell–Lucas sequence $\{Q_n\}$.

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For $n, k \in \mathbb{N}$ with $n \geq k \geq 1$, define the generalized Fibonomial and Lucanomial coefficients by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{U_m} = \prod_{j=1}^{k} \frac{U_{(n-j+1)m}}{U_{jm}} = \frac{U_{nm}U_{(n-1)m}\cdots U_{(n-k+1)m}}{U_m U_{2m}\cdots U_{km}},$$

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{V_m} = \prod_{j=1}^{k} \frac{V_{(n-j+1)m}}{U_{jm}} = \frac{V_{nm}V_{(n-1)m}\cdots V_{(n-k+1)m}}{V_m V_{2m}\cdots V_{km}};$$

with the boundary conditions

$$\left\{ \begin{matrix} n \\ 0 \end{matrix} \right\}_{U_m} = \left\{ \begin{matrix} n \\ 0 \end{matrix} \right\}_{V_m} = \left\{ \begin{matrix} n \\ n \end{matrix} \right\}_{U_m} = \left\{ \begin{matrix} n \\ n \end{matrix} \right\}_{V_m} = 1.$$

In particular for $m = 1$, the reduced coefficients $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{U_1}$ and $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{V_1}$ will be denoted briefly by $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{U}$ and $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{V}$, respectively. When $U_n = F_n$ and $V_n = L_n$, the generalized Fibonomial and Lucanomial coefficients become the usual Fibonomial and Lucanomial coefficients (cf. [13, 14, 16, 17]), explicitly given by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{F} = \frac{F_n F_{n-1}\dots F_{n-k+1}}{F_1 F_2 \dots F_k} \quad \text{and} \quad \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{L} = \frac{L_n L_{n-1}\dots L_{n-k+1}}{L_1 L_2 \dots L_k}.$$

For an indeterminate y, recall the shifted factorial in base q

$$(y;q)_0 \equiv 1 \text{ and } (y;q)_n = (1-y)(1-qy)\cdots(1-q^{n-1}y) \text{ for } n \in \mathbb{N}.$$

Then the generalized Gaussian binomial coefficient (cf. [7, 8]) is defined by

$$\begin{bmatrix} m \\ k \end{bmatrix}_{p,q} = \frac{(p;q)_m}{(p;q)_k (p;q)_{m-k}} = (-p/q)^k q^{km-\binom{k}{2}} \frac{(q^{1-m}/p;q)_k}{(p;q)_k}, \tag{1.1}$$

where $m, k \in \mathbb{N}_0$ with $k \leq m$. When $p = q$, this reduces to the usual q-binomial coefficient

$$\begin{bmatrix} m \\ k \end{bmatrix}_{q} = \begin{bmatrix} m \\ k \end{bmatrix}_{q,q} = (-1)^k q^{km-\binom{k}{2}} \frac{(q^{-m};q)_k}{(q;q)_k}. \tag{1.2}$$

By introducing the following function of α and β

$$\rho := \rho(x) = \frac{\beta}{\alpha} = -\alpha^{-2} = -\beta^2 = \frac{x - \sqrt{x^2+1}}{x + \sqrt{x^2+1}}, \tag{1.3}$$

we can express

$$U_n = \alpha^{n-1}\frac{1-\rho^n}{1-\rho} \quad \text{and} \quad V_n = \alpha^n(1+\rho^n). \tag{1.4}$$

Consequently, we are led to the following two fundamental relations between the generalized Fibonomial/Lucanomial coefficients and the generalized Gaussian binomial coefficients:

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{U} = \alpha^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{\rho} \quad \text{and} \quad \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{V} = \alpha^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{-\rho,\rho}. \tag{1.5}$$

Analogous to binomial coefficients, there has been growing interest in studying Fibonomial coefficients and their properties (see for example [4, 12, 13, 17]). By converting the q-binomial coefficients into Fibonomial coefficients, the authors [7, 8] recently examined quadratic and cubic sums of Fibonomial and Lucanomial coefficients.

By employing known identities from the q-series theory (cf. [10]), we shall further investigate Fibonomial/Lucanomial sums in this paper. In Section 2, we shall prove three main theorems that evaluate, in closed form, twelve sums of generalized Gaussian coefficients in different bases. Then these summation theorems will be utilized in Section 3 to deduce 18 further formulae involving products of two Fibonomial/Lucanomial coefficients or quotients of two such coefficients over a third one. To our knowledge, these formulae about Fibonomial/Lucanomial coefficients don't seem to have appeared previously in the literature.

2. q-Binomial sums with different bases

Following Bailey [1, Chapter 8] and Gasper–Rahman [10], the basic hypergeometric series (shortly called q-series) reads as

$$ {}_{1+\lambda}\phi_\lambda\left[\begin{matrix}a_0, a_1, \cdots, a_\lambda\\ b_1, \cdots, b_\lambda\end{matrix}\Big| q; z\right] = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} \frac{(a_0;q)_n (a_1;q)_n \cdots (a_\lambda;q)_n}{(b_1;q)_n;q)_n \cdots (b_\lambda;q)_n}. $$

Throughout this section, we shall utilize known summation formulae for the following terminating well–poised q-series

$$ {}_3\phi_2\left[\begin{matrix}q^{-2n}, & b, & d\\ q^{1-2n}/b, & q^{1-2n}/d\end{matrix}\Big| q; \frac{q^{\varepsilon-n}}{bd}\right], $$

where $n \in \mathbb{N}_0$, $\varepsilon \in \mathbb{Z}$ and $b, d, \in \mathbb{C}$. Some partial results appeared in [2, 3, 5, 9]. A full coverage was made by the first author [6, §1.2]. Three main theorems will be proved that express q-binomial sums in closed formulae.

Theorem 2.1 ($n \in \mathbb{N}_0$)**.**

$$ \text{(a)} \quad \sum_{k=0}^{2n} (-1)^k \begin{bmatrix}2n\\k\end{bmatrix}_q^2 \begin{bmatrix}2n\\k\end{bmatrix}_{-q,q}^{-1} q^{\binom{k}{2}-kn-k} = \frac{2(-1)^n \Lambda_n(q)}{q^{2n+1}(1+q^n)(1+q^{n-1})} \begin{bmatrix}2n\\n\end{bmatrix}_q \begin{bmatrix}2n\\n\end{bmatrix}_{-q,q}^{-1}, $$

$$ \text{(b)} \quad \sum_{k=0}^{2n} (-1)^k \begin{bmatrix}2n\\k\end{bmatrix}_q^2 \begin{bmatrix}2n\\k\end{bmatrix}_{-q,q}^{-1} q^{\binom{k}{2}-kn} = \frac{2(-1)^n}{1+q^n} \begin{bmatrix}2n\\n\end{bmatrix}_q \begin{bmatrix}2n\\n\end{bmatrix}_{-q,q}^{-1}, $$

$$ \text{(c)} \quad \sum_{k=0}^{2n} (-1)^k \begin{bmatrix}2n\\k\end{bmatrix}_q^2 \begin{bmatrix}2n\\k\end{bmatrix}_{-q,q}^{-1} q^{\binom{k}{2}-kn+k} = \frac{2(-q)^n}{1+q^n} \begin{bmatrix}2n\\n\end{bmatrix}_q \begin{bmatrix}2n\\n\end{bmatrix}_{-q,q}^{-1}, $$

$$ \text{(d)} \quad \sum_{k=0}^{2n} (-1)^k \begin{bmatrix}2n\\k\end{bmatrix}_q^2 \begin{bmatrix}2n\\k\end{bmatrix}_{-q,q}^{-1} q^{\binom{k}{2}-kn+2k} = \frac{2(-q)^n \Lambda_n(q)}{q(1+q^n)(1+q^{n-1})} \begin{bmatrix}2n\\n\end{bmatrix}_q \begin{bmatrix}2n\\n\end{bmatrix}_{-q,q}^{-1}; $$

where $\Lambda_n(q)$ stands for

$$ \Lambda_n(q) = \Big\{2q^n + 2q^{2n+1} - 1 - q^{3n+1}\Big\}. $$

Proof. According to (1.1) and (1.2), define the finite sum $A_n(\lambda)$ by

$$ \begin{aligned} A_n(\lambda) &= \sum_{k=0}^{2n} (-1)^k \begin{bmatrix}2n\\k\end{bmatrix}_q^2 \begin{bmatrix}2n\\k\end{bmatrix}_{-q,q}^{-1} q^{\binom{k}{2}-k(n+1)+k\lambda} \\ &= \sum_{k=0}^{2n} (-1)^k \frac{(-q;q)_k (q^{-2n};q)_k^2}{(q;q)_k^2 (-q^{-2n};q)_k} q^{k(\lambda-1+n)}. \end{aligned} $$

The last sum can be written in terms of well–poised q-series and then evaluated in closed form (cf. [6, §1.2]) as

$$
{}_3\phi_2\left[\begin{matrix} q^{-2n}, -q, \; q^{-2n} \\ q, -q^{-2n} \end{matrix}\Big| q; -q^{\lambda-1+n}\right] = \frac{(-1;q)_n(q^{-2n};q)_n}{(q;q)_n(-q^{-2n};q)_n} \times \begin{cases} \dfrac{1-2q^n-2q^{2n+1}+q^{3n+1}}{-q^{2n+1}(1+q^{n-1})}, & \lambda=0; \\ 1, & \lambda=1; \\ q^n, & \lambda=2; \\ \dfrac{1-2q^n-2q^{2n+1}+q^{3n+1}}{-q^{1-n}(1+q^{n-1})}, & \lambda=3. \end{cases}
$$

Observing further that

$$
\frac{(-1;q)_n(q^{-2n};q)_n}{(q;q)_n(-q^{-2n};q)_n} = (-1)^n \frac{(-1;q)_n(-q;q)_n(q;q)_{2n}}{(q;q)_n^2(-q;q)_{2n}} = \frac{2(-1)^n}{1+q^n} \begin{bmatrix} 2n \\ n \end{bmatrix}_q \begin{bmatrix} 2n \\ n \end{bmatrix}_{-q,q}^{-1},
$$

we deduce the four formulae in Theorem 2.1 from $A_n(\lambda)$ with $\lambda = 0, 1, 2, 3$. □

Theorem 2.2 ($n \in \mathbb{N}_0$)**.**

$$
\text{(a)} \quad \sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 4n \\ 2k \end{bmatrix}_q \begin{bmatrix} 2n \\ k \end{bmatrix}_{q^2} q^{3k(k-2n)-3k} = \frac{(-1)^n \Delta_n(q)}{q^{3n^2+n+1}(1-q^{6n-1})} \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2} \begin{bmatrix} 3n \\ n \end{bmatrix}_{q,q^2},
$$

$$
\text{(b)} \quad \sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 4n \\ 2k \end{bmatrix}_q \begin{bmatrix} 2n \\ k \end{bmatrix}_{q^2} q^{3k(k-2n)-k} = \frac{(-1)^n}{q^{3n^2+n}} \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2} \begin{bmatrix} 3n \\ n \end{bmatrix}_{q,q^2},
$$

$$
\text{(c)} \quad \sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 4n \\ 2k \end{bmatrix}_q \begin{bmatrix} 2n \\ k \end{bmatrix}_{q^2} q^{3k(k-2n)+k} = \frac{(-1)^n}{q^{3n^2-n}} \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2} \begin{bmatrix} 3n \\ n \end{bmatrix}_{q,q^2},
$$

$$
\text{(d)} \quad \sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 4n \\ 2k \end{bmatrix}_q \begin{bmatrix} 2n \\ k \end{bmatrix}_{q^2} q^{3k(k-2n)+3k} = \frac{(-1)^n \Delta_n(q)}{q^{3n^2-5n+1}(1-q^{6n-1})} \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2} \begin{bmatrix} 3n \\ n \end{bmatrix}_{q,q^2};
$$

where we denote $\Delta_n(q)$ for brevity by

$$
\Delta_n(q) = \left\{1 + 2q - 2q^{2n} - q^{2n+1}\right\}.
$$

Proof. According to (1.1) and (1.2), define the finite sum $B_n(\mu)$ by

$$
\begin{aligned}
B_n(\mu) &= \sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 4n \\ 2k \end{bmatrix}_q \begin{bmatrix} 2n \\ k \end{bmatrix}_{q^2} q^{3k(k-1-2n)+2k\mu} \\
&= \sum_{k=0}^{2n} \frac{(q^{-4n};q)_{2k}(q^{-4n};q^2)_k}{(q;q)_{2k}(q^2;q^2)_k} q^{k(2\mu-1+6n)}.
\end{aligned}
$$

Expressing the last sum in terms of well–poised q-series, we can evaluate it in closed form (cf. [6, §1.2]) as

$$ {}_3\phi_2\left[\begin{matrix} q^{-4n}, q^{-4n}, q^{1-4n} \\ q,\ q^2 \end{matrix}\Big| q^2; q^{2\mu-1+6n}\right] = \frac{(q^{-4n};q^2)_n(q^{1+4n};q^2)_n}{(q;q^2)_n(q^2;q^2)_n} \times \begin{cases} \dfrac{1+2q-2q^{2n}-q^{2n+1}}{q(1-q^{6n-1})}, & \mu=0; \\ 1, & \mu=1; \\ q^{2n}, & \mu=2; \\ \dfrac{q^{6n-1}(1+2q-2q^{2n}-q^{2n+1})}{(1-q^{6n-1})}, & \mu=3. \end{cases} $$

Taking into account that

$$ \frac{(q^{-4n};q^2)_n(q^{1+4n};q^2)_n}{(q;q^2)_n(q^2;q^2)_n} = (-1)^n\frac{(-q;q)_{2n}(q^{1+4n};q^2)_n}{q^{3n^2+n}(q^2;q^2)_n} = \frac{(-1)^n}{q^{3n^2+n}}\begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2}\begin{bmatrix} 3n \\ n \end{bmatrix}_{q,q^2}, $$

we find the four identities in Theorem 2.2 from $B_n(\mu)$ with $\mu = 0, 1, 2, 3$. □

Theorem 2.3 ($n \in \mathbb{N}_0$).

(a) $$\sum_{k=0}^{2n}(-1)^k\begin{bmatrix} 2n \\ k \end{bmatrix}_q^2\begin{bmatrix} 2n \\ k \end{bmatrix}_{q^3}^{-1} q^{kn-\binom{k}{2}-2k} = \frac{(q;q)_n(q;q)_{2n}}{q^{3n}(q^3;q^3)_n}\begin{bmatrix} 2n+1 \\ n \end{bmatrix}_q\begin{bmatrix} 2n+2 \\ n \end{bmatrix}_q\begin{bmatrix} 2n \\ n \end{bmatrix}_{q^3}^{-1},$$

(b) $$\sum_{k=0}^{2n}(-1)^k\begin{bmatrix} 2n \\ k \end{bmatrix}_q^2\begin{bmatrix} 2n \\ k \end{bmatrix}_{q^3}^{-1} q^{kn-\binom{k}{2}-k} = \frac{q^{-n}(q;q)_{2n}^2}{(q;q)_n(q^3;q^3)_n}\begin{bmatrix} 2n+1 \\ n \end{bmatrix}_q\begin{bmatrix} 2n \\ n \end{bmatrix}_{q^3}^{-1},$$

(c) $$\sum_{k=0}^{2n}(-1)^k\begin{bmatrix} 2n \\ k \end{bmatrix}_q^2\begin{bmatrix} 2n \\ k \end{bmatrix}_{q^3}^{-1} q^{kn-\binom{k}{2}} = \frac{(q;q)_{2n}^2}{(q;q)_n(q^3;q^3)_n}\begin{bmatrix} 2n+1 \\ n \end{bmatrix}_q\begin{bmatrix} 2n \\ n \end{bmatrix}_{q^3}^{-1},$$

(d) $$\sum_{k=0}^{2n}(-1)^k\begin{bmatrix} 2n \\ k \end{bmatrix}_q^2\begin{bmatrix} 2n \\ k \end{bmatrix}_{q^3}^{-1} q^{kn-\binom{k}{2}+k} = \frac{(q;q)_n(q;q)_{2n}}{(q^3;q^3)_n}\begin{bmatrix} 2n+1 \\ n \end{bmatrix}_q\begin{bmatrix} 2n+2 \\ n \end{bmatrix}_q\begin{bmatrix} 2n \\ n \end{bmatrix}_{q^3}^{-1}.$$

Proof. According to (1.1) and (1.2), define the finite sum $C_n(\nu)$ by

$$ \begin{aligned} C_n(\nu) &= \sum_{k=0}^{2n}(-1)^k\begin{bmatrix} 2n \\ k \end{bmatrix}_q^2\begin{bmatrix} 2n \\ k \end{bmatrix}_{q^3}^{-1} q^{k(n-2)-\binom{k}{2}+k\nu} \\ &= \sum_{k=0}^{2n}\frac{(q^{-2n};q)_k^2(q^3;q^3)_k}{(q;q)_k^2(q^{-6n};q^3)_k} q^{k(\nu-2-n)}. \end{aligned} $$

Writing the last sum in terms of well–poised q-series and denoting by ω a cubic root of the unity, we can evaluate it in closed form (cf. [6, §1.2]) as

$$ {}_3\phi_2\left[\begin{matrix} q^{-2n},\ q\omega,\ q\omega^2 \\ q^{-2n}\omega, q^{-2n}\omega^2 \end{matrix}\Big| q; q^{\nu-2-n}\right] = \frac{(q^{-2n};q)_n(q^{-1-2n};q)_n}{(q^{-2n}\omega;q)_n(q^{-2n}\omega^2;q)_n} \times \begin{cases} \dfrac{(1+q^{n+1})(1-q^{2n+1})}{q^{2n}(1-q^{n+2})}, & \nu=0; \\ 1, & \nu=1; \\ q^n, & \nu=2; \\ \dfrac{q^n(1+q^{n+1})(1-q^{2n+1})}{(1-q^{n+2})}, & \nu=3. \end{cases} $$

In view of the equality

$$\frac{(q^{-2n};q)_n(q^{-1-2n};q)_n}{(q^{-2n}\omega;q)_n(q^{-2n}\omega^2;q)_n} = q^{-n}\frac{(1-q^{2n+1})(q^{n+1};q)_n^3}{(1-q^{n+1})(q^{3n+3};q^3)_n} = \frac{q^{-n}(q;q)_{2n}^2}{(q;q)_n(q^3;q^3)_n}\begin{bmatrix}2n+1\\ n\end{bmatrix}_q\begin{bmatrix}2n\\ n\end{bmatrix}_{q^3}^{-1},$$

the four identities displayed in Theorem 2.3 are consequently obtained from $C_n(\nu)$ for $\nu = 0, 1, 2, 3$. □

3. Applications to Fibonomial sums

For the summation formulae established in the last section, by specifying $q = \rho$ and then combining the resulting equations, we shall derive further identities for Fibonomial and Lucanomial coefficients.

Firstly, by applying Theorem 2.1, we deduce six identities below in Proposition 3.1 and Corollary 3.2. Because proofs for them are almost identical, we shall only give a detailed proof for the first one (a) in Proposition 3.1.

Proposition 3.1 ($n \in \mathbb{N}_0$)**.**

(a) $$\sum_{k=0}^{2n} U_k\begin{Bmatrix}2n\\ k\end{Bmatrix}_U^2\begin{Bmatrix}2n\\ k\end{Bmatrix}_V^{-1}(-1)^{\binom{k}{2}+k(n+1)} = 2(-1)^n\frac{U_n}{V_n}\begin{Bmatrix}2n\\ n\end{Bmatrix}_U\begin{Bmatrix}2n\\ n\end{Bmatrix}_V^{-1},$$

(b) $$\sum_{k=0}^{2n} U_k^3\begin{Bmatrix}2n\\ k\end{Bmatrix}_U^2\begin{Bmatrix}2n\\ k\end{Bmatrix}_V^{-1}(-1)^{\binom{k}{2}+kn} = 2(-1)^n\frac{U_n^3V_{2n+1}}{V_{n-1}}\begin{Bmatrix}2n\\ n\end{Bmatrix}_U\begin{Bmatrix}2n\\ n\end{Bmatrix}_V^{-1},$$

(c) $$\sum_{k=0}^{2n} U_{3k}\begin{Bmatrix}2n\\ k\end{Bmatrix}_U^2\begin{Bmatrix}2n\\ k\end{Bmatrix}_V^{-1}(-1)^{\binom{k}{2}+kn} = 2(-1)^n\begin{Bmatrix}2n\\ n\end{Bmatrix}_U\begin{Bmatrix}2n\\ n\end{Bmatrix}_V^{-1} \times\frac{U_{3n}V_{n+1}}{V_nV_{n-1}}\Big\{V_{3n+1}-2(-1)^nV_{n+1}\Big\}.$$

Proof. First recalling (1.4), we have the equalities

$$U_k = \frac{1-\rho^k}{1-\rho}(\rho^{-\frac12}\mathbf{i})^{k-1}, \tag{3.1}$$

$$U_k^3 = \frac{(1-\rho^k)^3}{(1-\rho)^3}(\rho^{-\frac12}\mathbf{i})^{3k-3}, \tag{3.2}$$

$$U_{3k} = \frac{1-\rho^{3k}}{1-\rho}(\rho^{-\frac12}\mathbf{i})^{3k-1}. \tag{3.3}$$

Then by means of (1.5), we can rewrite

$$\begin{Bmatrix}2n\\ k\end{Bmatrix}_U^2 = \begin{bmatrix}2n\\ k\end{bmatrix}_\rho^2(\rho^{-\frac12}\mathbf{i})^{2k(2n-k)},$$

$$\begin{Bmatrix}2n\\ k\end{Bmatrix}_V^{-1} = \begin{bmatrix}2n\\ k\end{bmatrix}_{-\rho,\rho}^{-1}(\rho^{-\frac12}\mathbf{i})^{k(k-2n)}.$$

Now, the sum on the left hand side in Equation (a) can be expressed as

$$\Phi = \frac{-\rho^{\frac12}\mathbf{i}}{1-\rho}\sum_{k=0}^{2n}(-1)^k\begin{bmatrix}2n\\ k\end{bmatrix}_\rho^2\begin{bmatrix}2n\\ k\end{bmatrix}_{-\rho,\rho}^{-1}\rho^{\binom{k}{2}-kn}\{1-\rho^k\}.$$

Splitting further the last sum into two according to the terms inside braces "$\{\cdots\}$" and then evaluating them by formulae (b) and (c) in Theorem 2.1, we get the following expression

$$\begin{aligned}\Phi &= \frac{-\rho^{\frac{1}{2}}\mathbf{i}}{1-\rho}\begin{bmatrix}2n\\ n\end{bmatrix}_{\rho}\begin{bmatrix}2n\\ n\end{bmatrix}^{-1}_{-\rho,\rho}\left\{\frac{2(-1)^n}{1+\rho^n}-\frac{2(-\rho)^n}{1+\rho^n}\right\}\\ &= \frac{2(-1)^{n+1}\rho^{\frac{1}{2}}\mathbf{i}(1-\rho^n)}{(1+\rho^n)(1-\rho)}\begin{bmatrix}2n\\ n\end{bmatrix}_{\rho}\begin{bmatrix}2n\\ n\end{bmatrix}^{-1}_{-\rho,\rho}\\ &= 2(-1)^n\frac{U_2}{V_n}\begin{Bmatrix}2n\\ n\end{Bmatrix}_U\begin{Bmatrix}2n\\ n\end{Bmatrix}^{-1}_V,\end{aligned}$$

which confirmed the first identity (a). The two other identities (b) and (c) can be done in exactly the same manner except that we have to make use use of (3.2) and (3.3), instead of (3.1). ☐

Again from (1.4), we can express

$$V_k = (1+\rho^k)(\rho^{-\frac{1}{2}}\mathbf{i})^k, \tag{3.4}$$

$$V_k^3 = (1+\rho^k)^3(\rho^{-\frac{1}{2}}\mathbf{i})^{3k}, \tag{3.5}$$

$$V_{3k} = (1+\rho^{3k})(\rho^{-\frac{1}{2}}\mathbf{i})^{3k}. \tag{3.6}$$

By following the same procedure as the proof for Proposition 3.1, we can establish three identities as in the corollary below.

Corollary 3.2 ($n \in \mathbb{N}_0$).

$$\text{(A)}\quad \sum_{k=0}^{2n} V_k\begin{Bmatrix}2n\\ k\end{Bmatrix}^2_U\begin{Bmatrix}2n\\ k\end{Bmatrix}^{-1}_V(-1)^{\binom{k}{2}+k(n+1)} = 2(-1)^n\begin{Bmatrix}2n\\ n\end{Bmatrix}_U\begin{Bmatrix}2n\\ n\end{Bmatrix}^{-1}_V,$$

$$\text{(B)}\quad \sum_{k=0}^{2n} V_k^3\begin{Bmatrix}2n\\ k\end{Bmatrix}^2_U\begin{Bmatrix}2n\\ k\end{Bmatrix}^{-1}_V(-1)^{\binom{k}{2}+kn} = 2(-1)^n\begin{Bmatrix}2n\\ n\end{Bmatrix}_U\begin{Bmatrix}2n\\ n\end{Bmatrix}^{-1}_V \times\left\{5+3\frac{V_{n+1}}{V_{n-1}}-3(-1)^n\frac{V_{3n+1}}{V_{n-1}}+\frac{V_{5n+1}}{V_{n-1}}\right\},$$

$$\text{(C)}\quad \sum_{k=0}^{2n} V_{3k}\begin{Bmatrix}2n\\ k\end{Bmatrix}^2_U\begin{Bmatrix}2n\\ k\end{Bmatrix}^{-1}_V(-1)^{\binom{k}{2}+kn} = 2(-1)^n\begin{Bmatrix}2n\\ n\end{Bmatrix}_U\begin{Bmatrix}2n\\ n\end{Bmatrix}^{-1}_V \times\frac{1-(-1)^nV_{2n}}{V_{n-1}}\left\{2V_{n+1}-(-1)^nV_{3n+1}\right\}.$$

Proposition 3.3 ($n \in \mathbb{N}_0$).

$$\text{(a)}\quad \sum_{k=0}^{2n}(-1)^k\begin{Bmatrix}4n\\ 2k\end{Bmatrix}_U\begin{Bmatrix}2n\\ k\end{Bmatrix}_{U_2}U_{2k} = (-1)^nU_{2n}\begin{Bmatrix}6n\\ 2n\end{Bmatrix}_U\begin{Bmatrix}2n\\ n\end{Bmatrix}_{U_2}\begin{Bmatrix}3n\\ n\end{Bmatrix}^{-1}_{U_2},$$

$$\text{(b)}\quad \sum_{k=0}^{2n}(-1)^k\begin{Bmatrix}4n\\ 2k\end{Bmatrix}_U\begin{Bmatrix}2n\\ k\end{Bmatrix}_{U_2}U_{2k}^3 = (-1)^{n-1}\frac{U_{4n}U_{2n}^2}{U_{6n-1}}\begin{Bmatrix}6n\\ 2n\end{Bmatrix}_U\begin{Bmatrix}2n\\ n\end{Bmatrix}_{U_2}\begin{Bmatrix}3n\\ n\end{Bmatrix}^{-1}_{U_2},$$

$$\text{(c)}\quad \sum_{k=0}^{2n}(-1)^k\begin{Bmatrix}4n\\ 2k\end{Bmatrix}_U\begin{Bmatrix}2n\\ k\end{Bmatrix}_{U_2}U_{6k} = (-1)^n\frac{U_{6n}}{U_{6n-1}}\begin{Bmatrix}6n\\ 2n\end{Bmatrix}_U\begin{Bmatrix}2n\\ n\end{Bmatrix}_{U_2}\begin{Bmatrix}3n\\ n\end{Bmatrix}^{-1}_{U_2} \times\left\{3U_{2n-1}-V_{2n}\right\}.$$

Proof. According to (1.4), we have the equalities

$$\begin{aligned} U_{2k} &= \frac{1-\rho^{2k}}{1-\rho}(\rho^{-\frac{1}{2}}\mathbf{i})^{2k-1}, && (3.7)\\ U_{2k}^3 &= \frac{(1-\rho^{2k})^3}{(1-\rho)^3}(\rho^{-\frac{1}{2}}\mathbf{i})^{6k-3}, && (3.8)\\ U_{6k} &= \frac{1-\rho^{6k}}{1-\rho}(\rho^{-\frac{1}{2}}\mathbf{i})^{6k-1}. && (3.9) \end{aligned}$$

In view of (1.5), we can express

$$\begin{aligned} U_{2k}^3 &= \frac{(1-\rho^{2k})^3}{(1-\rho)^3}\rho^{\frac{3}{2}-3k}\mathbf{i}^{2k+3},\\ \left\{\begin{matrix}4n\\2k\end{matrix}\right\}_U &= \begin{bmatrix}4n\\2k\end{bmatrix}_\rho \rho^{2k^2-4nk},\\ \left\{\begin{matrix}2n\\k\end{matrix}\right\}_{U_2} &= \begin{bmatrix}2n\\k\end{bmatrix}_{\rho^2} \rho^{k^2-2nk}(-1)^k. \end{aligned}$$

They are utilized to convert the sum on the left hand side in Equation (b) to the following one

$$\Psi = \frac{-\rho^{\frac{3}{2}}\mathbf{i}}{(1-\rho)^3}\sum_{k=0}^{2n}(-1)^k\begin{bmatrix}4n\\2k\end{bmatrix}_\rho\begin{bmatrix}2n\\k\end{bmatrix}_{\rho^2}\rho^{3k(k-2n)-3k}\Big\{1-\rho^{2k}\Big\}^3.$$

Now making expansion into four terms

$$\Big\{1-\rho^{2k}\Big\}^3 = 1-3\rho^{2k}+3\rho^{4k}-\rho^{6k}$$

and then evaluating the corresponding sums by Theorem 2.2, the resulting expression can be written as

$$\begin{aligned} \Psi =& \frac{-\rho^{\frac{3}{2}}\mathbf{i}}{(1-\rho)^3}\begin{bmatrix}2n\\n\end{bmatrix}_{\rho^2}\begin{bmatrix}3n\\n\end{bmatrix}_{\rho,\rho^2}\\ &\times\left\{\frac{(-1)^n\Delta_n(\rho)}{\rho^{3n^2+n+1}(1-\rho^{6n-1})}-\frac{3(-1)^n}{\rho^{3n^2+n}}+\frac{3(-1)^n}{\rho^{3n^2-n}}-\frac{(-1)^n\Delta_n(\rho)}{\rho^{3n^2-5n+1}(1-\rho^{6n-1})}\right\}\\ =& \frac{-\rho^{\frac{3}{2}}\mathbf{i}}{(1-\rho)^3}\begin{bmatrix}2n\\n\end{bmatrix}_{\rho^2}\begin{bmatrix}3n\\n\end{bmatrix}_{\rho,\rho^2}\left\{\frac{(-1)^n\Delta_n(\rho)(1-\rho^{6n})}{\rho^{3n^2+n+1}(1-\rho^{6n-1})}-\frac{3(-1)^n}{\rho^{3n^2+n}}(1-\rho^{2n})\right\}. \end{aligned}$$

Keeping in mind that

$$\Delta_n(\rho) = \Big\{1+2\rho-2\rho^{2n}-\rho^{2n+1}\Big\},$$

we can factorize the above expression inside the braces "$\{\cdots\}$" into

$$(-1)^n\frac{(1-\rho)(1-\rho^{4n})(1-\rho^{2n})^2}{\rho^{3n^2+n+1}(1-\rho^{6n-1})}.$$

By substitution, we have

$$\begin{aligned} \Psi &= \frac{-\rho^{\frac{3}{2}}\mathbf{i}}{(1-\rho)^3}\begin{bmatrix}2n\\n\end{bmatrix}_{\rho^2}\begin{bmatrix}3n\\n\end{bmatrix}_{\rho,\rho^2}\times(-1)^n\frac{(1-\rho)(1-\rho^{2n})^2(1-\rho^{4n})}{\rho^{3n^2+n+1}(1-\rho^{6n-1})}\\ &= -\rho^{-3n^3}\begin{bmatrix}2n\\n\end{bmatrix}_{\rho^2}\begin{bmatrix}3n\\n\end{bmatrix}_{\rho,\rho^2}\times\frac{\rho^{\frac{1}{2}-n}(-1)^n\mathbf{i}(1-\rho^{2n})^2(1-\rho^{4n})}{(1-\rho)^2(1-\rho^{6n-1})}. \end{aligned}$$

Observing that

$$\frac{\rho^{\frac{1}{2}-n}(-1)^n\mathbf{i}(1-\rho^{2n})^2(1-\rho^{4n})}{(1-\rho)^2(1-\rho^{6n-1})}=\frac{U_{4n}U_{2n}^2}{U_{6n-1}},$$

$$\begin{bmatrix}2n\\n\end{bmatrix}_{\rho^2}\begin{bmatrix}3n\\n\end{bmatrix}_{\rho,\rho^2}=\begin{bmatrix}6n\\2n\end{bmatrix}_{\rho}\begin{bmatrix}2n\\n\end{bmatrix}_{\rho^2}\begin{bmatrix}3n\\n\end{bmatrix}_{\rho^2}^{-1}$$

$$=(-1)^n\rho^{3n^2}\begin{Bmatrix}6n\\2n\end{Bmatrix}_U\begin{Bmatrix}2n\\n\end{Bmatrix}_{U_2}\begin{Bmatrix}3n\\n\end{Bmatrix}_{U_2}^{-1},$$

we arrive at

$$\Psi=(-1)^{n-1}\frac{U_{4n}U_{2n}^2}{U_{6n-1}}\begin{Bmatrix}6n\\2n\end{Bmatrix}_U\begin{Bmatrix}2n\\n\end{Bmatrix}_{U_2}\begin{Bmatrix}3n\\n\end{Bmatrix}_{U_2}^{-1},$$

which proves the identity (b). By applying (3.7) and (3.9), we can confirm the two other identities (a) and (c) in the same way. □

Instead of (3.7–3.9), we have, in accordance with (1.4), the equalities

$$V_{2k}=(1+\rho^{2k})(\rho^{-\frac{1}{2}}\mathbf{i})^{2k}, \tag{3.10}$$

$$V_{2k}^3=(1+\rho^{2k})^3(\rho^{-\frac{1}{2}}\mathbf{i})^{6k}, \tag{3.11}$$

$$V_{6k}=(1+\rho^{6k})(\rho^{-\frac{1}{2}}\mathbf{i})^{6k}. \tag{3.12}$$

By carrying out the same proof as for Proposition 3.3 and making use of the above three substitutions, we can derive the following three summation formulae.

Corollary 3.4 ($n\in\mathbb{N}_0$).

(A) $$\sum_{k=0}^{2n}(-1)^k\begin{Bmatrix}4n\\2k\end{Bmatrix}_U\begin{Bmatrix}2n\\k\end{Bmatrix}_{U_2}V_{2k}=(-1)^nV_{2n}\begin{Bmatrix}6n\\2n\end{Bmatrix}_U\begin{Bmatrix}2n\\n\end{Bmatrix}_{U_2}\begin{Bmatrix}3n\\n\end{Bmatrix}_{U_2}^{-1},$$

(B) $$\sum_{k=0}^{2n}(-1)^k\begin{Bmatrix}4n\\2k\end{Bmatrix}_U\begin{Bmatrix}2n\\k\end{Bmatrix}_{U_2}V_{2k}^3=(-1)^n\frac{U_{4n}}{U_{2n}}\begin{Bmatrix}6n\\2n\end{Bmatrix}_U\begin{Bmatrix}2n\\n\end{Bmatrix}_{U_2}\begin{Bmatrix}3n\\n\end{Bmatrix}_{U_2}^{-1}\times\left\{5+3\frac{U_{2n+1}}{U_{6n-1}}-3\frac{U_{2n-1}}{U_{6n-1}}-\frac{U_{6n+1}}{U_{6n-1}}\right\},$$

(C) $$\sum_{k=0}^{2n}(-1)^k\begin{Bmatrix}4n\\2k\end{Bmatrix}_U\begin{Bmatrix}2n\\k\end{Bmatrix}_{U_2}V_{6k}=\frac{(-1)^nV_{6n}}{U_{6n-1}}\begin{Bmatrix}6n\\2n\end{Bmatrix}_U\begin{Bmatrix}2n\\n\end{Bmatrix}_{U_2}\begin{Bmatrix}3n\\n\end{Bmatrix}_{U_2}^{-1}\times\left\{3U_{2n-1}-V_{2n}\right\}.$$

By utilizing the substituting equalities given in (3.1–3.3) and (3.4–3.6), we can prove from Theorem 2.3 further six formulae below in Proposition 3.5 and Corollary 3.6.

Proposition 3.5 ($n\in\mathbb{N}_0$).

(a) $$\sum_{k=0}^{2n}U_k\begin{Bmatrix}2n\\k\end{Bmatrix}_U^2\begin{Bmatrix}2n\\k\end{Bmatrix}_{U_3}^{-1}(-1)^{\binom{k}{2}+n(k+1)}=(4+4x^2)^nU_n\prod_{k=n+1}^{2n}\frac{U_k^2U_{k+1}}{U_{3k}},$$

(b) $$\sum_{k=0}^{2n}U_k^3\begin{Bmatrix}2n\\k\end{Bmatrix}_U^2\begin{Bmatrix}2n\\k\end{Bmatrix}_{U_3}^{-1}(-1)^{\binom{k+1}{2}+n(k+1)}=(4+4x^2)^{n-1}\frac{U_n^2}{U_{n+2}}\prod_{k=n+1}^{2n}\frac{U_k^2U_{k+1}}{U_{3k}}\times\left\{V_{4n+2}-1-(-1)^nV_{2n}+2(-1)^nV_{2n+2}\right\},$$

(c) $$\sum_{k=0}^{2n}U_{3k}\begin{Bmatrix}2n\\k\end{Bmatrix}_U^2\begin{Bmatrix}2n\\k\end{Bmatrix}_{U_3}^{-1}(-1)^{\binom{k+1}{2}+n(k+1)}=(4+4x^2)^nU_{3n}\prod_{k=n+1}^{2n}\frac{U_kU_{k+1}U_{k+2}}{U_{3k}}.$$

Corollary 3.6 ($n \in \mathbb{N}_0$).

$$\text{(A)}\quad \sum_{k=0}^{2n} V_k \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 2n \\ k \end{Bmatrix}_{U_3}^{-1} (-1)^{\binom{k}{2}+n(k+1)} = (4+4x^2)^n V_n \prod_{k=n+1}^{2n} \frac{U_k^2 U_{k+1}}{U_{3k}},$$

$$\text{(B)}\quad \sum_{k=0}^{2n} V_k^3 \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 2n \\ k \end{Bmatrix}_{U_3}^{-1} (-1)^{\binom{k+1}{2}+n(k+1)} = (4+4x^2)^n \frac{V_n}{U_{n+2}} \prod_{k=n+1}^{2n} \frac{U_k^2 U_{k+1}}{U_{3k}}$$

$$\times \Big\{ U_n V_{4n+2} + 2V_{n+1} + 2U_{n+2} - (-1)^n U_{3n} \Big\},$$

$$\text{(C)}\quad \sum_{k=0}^{2n} V_{3k} \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 2n \\ k \end{Bmatrix}_{U_3}^{-1} (-1)^{\binom{k+1}{2}+n(k+1)} = (4+4x^2)^n V_{3n} \prod_{k=n+1}^{2n} \frac{U_k U_{k+1} U_{k+2}}{U_{3k}}.$$

Finally, we take the above identity (C) to exemplify the proofs. Keeping in mind (3.6) and then rewriting the two braced coefficients in the summand

$$\begin{Bmatrix} 2n \\ k \end{Bmatrix}_U^2 = \begin{bmatrix} 2n \\ k \end{bmatrix}_\rho^2 \rho^{k^2-2nk}(-1)^k,$$

$$\begin{Bmatrix} 2n \\ k \end{Bmatrix}_{U_3}^{-1} = \begin{bmatrix} 2n \\ k \end{bmatrix}_{\rho^3}^{-1} \rho^{3nk-\frac{3k^2}{2}}(-1)^{nk}\mathbf{i}^{k^2};$$

we can express the sum in the identity (C) as the ρ-binomial sum below

$$\Omega = (-1)^n \sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix}_\rho^2 \begin{bmatrix} 2n \\ k \end{bmatrix}_{\rho^3}^{-1} \rho^{kn-\binom{k}{2}-2k} \{1+\rho^{3k}\}.$$

The last sum can be evaluated by formulae (a) and (d) in Theorem 3.6 as follows:

$$\begin{aligned}
\Omega &= (-1)^n(1+\rho^{-3n}) \begin{bmatrix} 2n+1 \\ n \end{bmatrix}_\rho \begin{bmatrix} 2n+2 \\ n \end{bmatrix}_\rho \begin{bmatrix} 2n \\ n \end{bmatrix}_{\rho^3}^{-1} \frac{(\rho;\rho)_n(\rho;\rho)_{2n}}{(\rho^3;\rho^3)_n} \\
&= (-1)^n(1+\rho^{-3n}) \frac{(\rho;\rho)_{2n}(\rho;\rho)_{2n+1}(\rho;\rho)_{2n+2}(\rho^3;\rho^3)_n}{(\rho;\rho)_n(\rho;\rho)_{n+1}(\rho;\rho)_{n+2}(\rho^3;\rho^3)_{2n}} \\
&= (-1)^n(1+\rho^{-3n}) \frac{(\rho^{1+n};\rho)_n(\rho^{2+n};\rho)_n(\rho^{3+n};\rho)_n}{(\rho^{3+3n};\rho^3)_n} \\
&= (-1)^n(1+\rho^{-3n}) \prod_{k=n+1}^{2n} \frac{(1-\rho^k)(1-\rho^{k+1})(1-\rho^{k+2})}{(1-\rho^{3k})} \\
&= (-1)^n \frac{(1-\rho)^{2n}}{\rho^n} V_{3n} \prod_{k=n+1}^{2n} \frac{U_k U_{k+1} U_{k+2}}{U_{3k}}.
\end{aligned}$$

In view of (1.3), the identity (C) follows by making the simplification below

$$(-1)^n \frac{(1-\rho)^{2n}}{\rho^n} = (4+4x^2)^n. \qquad \square$$

4. Conclusive comments

The summation formulae presented in this paper show that there is a significantly deep connection between the basic hypergeometric series and the sums regarding Fibonomial and Lucanomial coefficients as well as their variants. The authors believe that through this connection, it is possible to employ the q-series theory to evaluate efficiently those seemingly difficult sums involving Fibonomial–like coefficients. The interested reader is encouraged to make further exploration.

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