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# NOTES ON SOME PROPERTIES OF THE NATURAL RIEMANN EXTENSION

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Abstract. Let $(M,\nabla)$ be an $n$-dimensional differentiable manifold with a torsion-free linear connection and $T^*M$ its cotangent bundle. In this context we study some properties of the natural Riemann extension (M. Sekizawa (1987), O. Kowalski and M. Sekizawa (2011)) on the cotangent bundle $T^*M$. First, we give an alternative definition of the natural Riemann extension with respect to horizontal and vertical lifts. Secondly, we investigate metric connection for the natural Riemann extension. Finally, we present geodesics on the cotangent bundle $T^*M$ endowed with the natural Riemann extension.

## 1. Introduction

Let $(M,\nabla)$ be an $n$-dimensional $C^\infty$-manifold with a torsion-free linear connection and $\pi : T^*M \to M$ be the natural projection from its cotangent bundle $T^*M$ to $M$. For any local chart $\left(U, x^j\right),\ j=1,...,n$ around $x \in M$ induces a local chart $\left(\pi^{-1}(U), x^j, x^{\bar{j}} = p_j\right),\ \bar{j} = n+1,...,2n$ around $(x,p) \in T^*M$, where $x^{\bar{j}} = p_j$ are the components of the covector $p$ in each cotangent spaces $T^*_xM,\ x \in U$ endowed with the natural coframe $\left\{dx^j\right\},\ j=1,...,n$. By $\Im^r_s(M)\,(\Im^r_s(T^*M))$ we take the module over $F(M)\,(F(T^*M))$ of $C^\infty$ tensor fields of type $(r,s)$ on $M(T^*M)$.

In [18] Patterson and Walker defined a semi-Riemannian metric of signature $(n,n)$ on the cotangent bundle $T^*M$ of $(M,\nabla)$, called the Riemann extension. The Riemann extension described by

$$^R\nabla\left(^CV, {^CZ}\right) = -\gamma\left(\nabla_V Z + \nabla_Z V\right),$$

where $^CV$ and $^CZ$ denote the complete lifts of the vector fields $V$ and $Z$ on $M$ to $T^*M$ and $\gamma\left(\nabla_V Z + \nabla_Z V\right) = p_h\left(V^j\nabla_j Z^h + Z^j\nabla_j V^h\right)$.

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Since the tensor field ${}^{R}\nabla \in \Im_2^0(T^*M)$ is completely determined by its action upon the vector fields of type ${}^{H}\mathrm{V}$ and ${}^{V}\vartheta$, Aslanci *et al.*[3] give the following alternative definition for ${}^{R}\nabla$ by

$$\begin{aligned} {}^{R}\nabla\left({}^{H}V, {}^{H}Y\right) &= {}^{R}\nabla\left({}^{V}\vartheta, {}^{V}\omega\right) = 0, \\ {}^{R}\nabla\left({}^{V}\vartheta, {}^{H}Y\right) &= {}^{V}\left(\vartheta\left(Y\right)\right) = \vartheta\left(Y\right) \circ \pi \end{aligned}$$

for any $V, Y \in \Im_0^1(M)$ and $\vartheta, \omega \in \Im_1^0(M)$. The geometry of the Riemann extension and its generalization were intensively studied in many papers (see for example [2, 4, 8-11, 14, 15-17, 19, 21]).

The natural Riemann extension $\bar{g}$ as a generalization of the Riemann extension is given by Sekizawa in [20] (see also Kowalski and Sekizawa [12]) and defined by the three identities:

$$\begin{aligned} \bar{g}\left({}^{C}V, {}^{C}Z\right) &= -a^{V}\left(\nabla_V Z + \nabla_Z V\right) + b^{V}V^{V}Z, \\ \bar{g}\left({}^{C}V, {}^{V}\omega\right) &= a^{V}\left(\omega(V)\right), \\ \bar{g}\left({}^{V}\vartheta, {}^{V}\omega\right) &= 0 \end{aligned} \tag{1}$$

for any $V, Z \in \Im_0^1(M)$ and $\vartheta, \omega \in \Im_1^0(M)$, where ${}^{V}V = {}^{V}V_{(x,p)} = p(V_x) = \sum_{k=1}^{n} p_k V^k$ is a function and $a, b$ are arbitrary constants. We may assume $a > 0$ without loss of generality. When $b \neq 0$ (resp. $b = 0$), $\bar{g}$ is called a proper (resp. a non-proper) natural Riemannian extension. As a particular situation, when $a = 1$ and $b = 0$, we get the Riemannian extension. For further references relation to the natural Riemann extension, see [5-7,13].

In this paper, we give an alternative definition of the natural Riemann extension with respect to horizontal lifts of vector fields and vertical lifts of covector fields. Also, we present the Levi-Civita connection and Christoffel symbols with respect to the adapted frame. In Sect. 4, we show that the horizontal lift ${}^{H}\nabla$ of the torsion-free connection $\nabla$ to the cotangent bundle $T^*M$ is a metric connection with respect to the natural Riemann extension. In Theorem 3 , we find that the metric connection${}^{H}\nabla$ has a vanishing scalar curvature with respect to the natural Riemann extension. In Sect. 5, we investigate the geodesics on the cotangent bundle $T^*M$ with respect to the natural Riemann extension.

## 2. Preliminaries

Let $\vartheta = \vartheta_k dx^k$ and $V = V^k \frac{\partial}{\partial x^k}$ be the local statements in $U \subset M$ of a covector field (1-form) $\vartheta \in \Im_1^0(M)$ and a vector field $V \in \Im_0^1(M)$, respectively. The vertical lift ${}^{V}\vartheta$ of $\vartheta$, the horizontal and complete lift ${}^{H}V$, ${}^{C}V$ of $V$ are defined, respectively, by

$$\begin{aligned} {}^{V}\vartheta &= \sum_k \vartheta_k \partial_{\bar{k}}, \\ {}^{H}V &= V^k \partial_k + \sum_k p_h \Gamma_{kj}^h V^j \partial_{\bar{k}}, \end{aligned} \tag{2}$$

$$^{C}V = V^{k}\frac{\partial}{\partial x^{k}} - \sum_{k} p_{h}\partial_{k}V^{h}\frac{\partial}{\partial x^{\bar{k}}},$$

where $\frac{\partial}{\partial x^{k}} = \partial_{k}$, $\frac{\partial}{\partial x^{\bar{k}}} = \partial_{\bar{k}}$ and $\Gamma^{h}_{kj}$ are the components of $\nabla$ on $M$ [21].

From (2), the complete lift $^{C}V$ of $V \in \Im^{1}_{0}(M)$ is expressed by

$$^{C}V = {}^{H}V - {}^{V}\left(p\left(\nabla V\right)\right), \tag{3}$$

where $p\left(\nabla V\right) = p_{j}\left(\nabla_{h}V^{j}\right)dx^{h}$.

In $U \subset M$ , we write

$$V_{(t)} = \frac{\partial}{\partial x^{t}},\ \vartheta^{(t)} = dx^{t},\ t = 1, 2, ..., n.$$

From (2) and the natural frame $\{\partial_{k}, \partial_{\bar{k}}\}$, we can see that these vector fields have, respectively, the local expressions

$$\begin{cases} {}^{V}\vartheta^{(t)} = \tilde{f}_{(\bar{t})} = \partial_{\bar{t}}, \\ {}^{H}V_{(t)} = \tilde{f}_{(t)} = \partial_{t} + \sum\limits_{h} p_{a}\Gamma^{a}_{ht}\partial_{\bar{h}}. \end{cases} \tag{4}$$

The set $\left\{{}^{H}V_{(t)}, {}^{V}\vartheta^{(t)}\right\} = \left\{\tilde{f}_{(t)}, \tilde{f}_{(\bar{t})}\right\} = \left\{\tilde{f}_{(\beta)}\right\}$ is called adapted frame to the connection $\nabla$ in $\pi^{-1}(U) \subset T^{*}M$.

We now consider local 1-forms $\tilde{\omega}^{\alpha}$ in $\pi^{-1}(U)$ defined by

$$\tilde{\omega}^{\alpha} = \bar{A}^{\alpha}{}_{B}dx^{B},$$

where

$$A^{-1} = \left(\bar{A}^{\alpha}{}_{B}\right) = \begin{pmatrix} \bar{A}^{i}{}_{j} & \bar{A}^{i}{}_{\bar{j}} \\ \bar{A}^{\bar{i}}{}_{j} & \bar{A}^{\bar{i}}{}_{\bar{j}} \end{pmatrix} = \begin{pmatrix} \delta^{i}_{j} & 0 \\ -p_{a}\Gamma^{a}_{ij} & \delta^{j}_{i} \end{pmatrix}. \tag{5}$$

The matrix (5) is the inverse of the matrix

$$A = \left(A_{\beta}{}^{A}\right) = \begin{pmatrix} A_{j}{}^{i} & A_{\bar{j}}{}^{i} \\ A_{j}{}^{\bar{i}} & A_{\bar{j}}{}^{\bar{i}} \end{pmatrix} = \begin{pmatrix} \delta^{i}_{j} & 0 \\ p_{a}\Gamma^{a}_{ij} & \delta^{j}_{i} \end{pmatrix} \tag{6}$$

of the transformation $\tilde{f}_{\beta} = A_{\beta}{}^{A}\partial_{A}$ (see [4]). In what follows, the set $\{\tilde{\omega}^{\alpha}\}$ is called the coframe dual of the adapted frame $\left\{\tilde{f}_{(\beta)}\right\}$, i.e. $\tilde{\omega}^{\alpha}\left(\tilde{f}_{\beta}\right) = \bar{A}^{\alpha}{}_{B}A_{\beta}{}^{B} = \delta^{\alpha}_{\beta}$.

The Lie bracket operations of the adapted frame $\left\{\tilde{f}_{(\beta)}\right\}$ on the cotangent bundle $T^{*}M$ are given by

$$\begin{aligned} \left[\tilde{f}_{(t)}, \tilde{f}_{(l)}\right] &= p_{a}R_{tlk}{}^{a}\tilde{f}_{(\bar{k})}, \\ \left[\tilde{f}_{(\bar{t})}, \tilde{f}_{(\bar{l})}\right] &= 0, \\ \left[\tilde{f}_{(t)}, \tilde{f}_{(\bar{l})}\right] &= -\Gamma^{l}_{tk}\tilde{f}_{(\bar{k})}, \end{aligned} \tag{7}$$

where ${R_{tlk}}^a$ being local components of the curvature tensor $R$ of $\nabla$ on $M$.

Hence we have the undermentioned components for vector fields ${}^V\vartheta$, ${}^HV$ and ${}^CV$ on $T^*M$

$$ {}^V\vartheta = \begin{pmatrix} 0 \\ \vartheta_j \end{pmatrix}, \quad {}^HV = \begin{pmatrix} V^j \\ 0 \end{pmatrix} \text{ and } {}^CV = \begin{pmatrix} V^j \\ -p_h \nabla_j V^h \end{pmatrix} \tag{8} $$

in the adapted frame $\left\{ \tilde{f}_{(\beta)} \right\}$.

## 3. The Natural Riemann Extension

Using (1) and (3), the natural Riemann extension $\bar{g}$ is determined by its action on ${}^V\vartheta$, ${}^HV$. Then we find

$$ \begin{aligned} &\bar{g}\left({}^HV, {}^HZ\right) = b^V V^V Z = bp(V)p(Z), \\ &\bar{g}\left({}^HV, {}^V\omega\right) = a^V\left(\omega(V)\right) = \left(\omega(V)\right) \circ \pi, \\ &\bar{g}\left({}^V\vartheta, {}^V\omega\right) = 0 \end{aligned} \tag{9} $$

for any $V, Z \in \Im_0^1(M)$ and $\vartheta, \omega \in \Im_1^0(M)$, where $a > 0,\ a, b$ are arbitrary constants and ${}^VV = {}^VV_{(x,p)} = p(V_x) = \sum_{k=1}^n p_k V^k = p(V)$ is a function. By virtue of (4) and (9), we obtain

$$ \begin{aligned} &\bar{g}\left({}^HV_{(j)}, {}^HZ_{(k)}\right) = \bar{g}\left(\tilde{f}_{(j)}, \tilde{f}_{(k)}\right) = \bar{g}_{jk} = bp_jp_k, \\ &\bar{g}\left({}^HV_{(j)}, {}^V\vartheta^{(k)}\right) = \bar{g}\left(\tilde{f}_{(j)}, \tilde{f}_{(\bar{k})}\right) = \bar{g}_{j\bar{k}} = adx^k\left(\frac{\partial}{\partial x^j}\right) = a\delta_j^k, \\ &\bar{g}\left({}^V\vartheta^{(j)}, {}^HV_{(k)}\right) = \bar{g}\left(\tilde{f}_{(\bar{j})}, \tilde{f}_{(k)}\right) = \bar{g}_{\bar{j}k} = adx^j\left(\frac{\partial}{\partial x^k}\right) = a\delta_k^j, \\ &\bar{g}\left({}^V\vartheta^{(j)}, {}^V\omega^{(k)}\right) = \bar{g}\left(\tilde{f}_{(\bar{j})}, \tilde{f}_{(\bar{k})}\right) = \bar{g}_{\bar{j}\bar{k}} = 0. \end{aligned} $$

As corollary, the natural Riemann extension $\bar{g} = (\bar{g})_{JK}$ has the following components with respect to the adapted frame $\left\{ \tilde{f}_{(\beta)} \right\}$:

$$ \bar{g} = \bar{g}_{JK} = \begin{pmatrix} \bar{g}_{jk} & \bar{g}_{j\bar{k}} \\ \bar{g}_{\bar{j}k} & \bar{g}_{\bar{j}\bar{k}} \end{pmatrix} = \begin{pmatrix} bp_jp_k & a\delta_j^k \\ a\delta_k^j & 0 \end{pmatrix}. \tag{10} $$

Using $\bar{g}_{JK}\tilde{g}^{KI} = \delta_J^I$, we obtain the inverse $\tilde{g}^{JK}$ of the matrix $\bar{g}_{JK}$ as follows

$$ \tilde{g} = \tilde{g}^{JK} = \begin{pmatrix} 0 & \frac{1}{a}\delta_k^j \\ \frac{1}{a}\delta_j^k & -\frac{b}{a^2}p_jp_k \end{pmatrix}. \tag{11} $$

The Levi-Civita connection $\bar{\nabla}$ of the natural Riemann extension $\bar{g}$ is given by the following formulas:

**Theorem 1.** *In adapted frame $\left\{\tilde{f}_{(\beta)}\right\}$, the Levi-Civita connection $\bar{\nabla}$ of the natural Riemann extension $\bar{g}$ on $T^*M$ is given by the following equations:*

$$
\begin{aligned}
&i)\ \bar{\nabla}_{\tilde{f}_i}\tilde{f}_j=\left(\Gamma^l_{ij}-\frac{b}{2a}\left(\delta^l_ip_j+\delta^l_jp_i\right)\right)\tilde{f}_l+\left(\frac{b}{a}p_kp_l\Gamma^k_{ji}-p_kR_{jli}{}^k\right)\tilde{f}_{\bar{l}},\\
&ii)\ \bar{\nabla}_{\tilde{f}_i}\tilde{f}_{\bar{j}}=\left(\frac{b}{2a}\left(\delta^j_lp_i+\delta^j_ip_l\right)-\Gamma^j_{li}\right)\tilde{f}_{\bar{l}},\\
&iii)\ \bar{\nabla}_{\tilde{f}_{\bar{i}}}\tilde{f}_j=\frac{b}{2}\left(\delta^i_jp_l+\delta^i_lp_j\right)\tilde{f}_{\bar{l}},\\
&iv)\ \bar{\nabla}_{\tilde{f}_{\bar{i}}}\tilde{f}_{\bar{j}}=0
\end{aligned}
\tag{12}
$$

*where $R_{lji}{}^s$, $\Gamma^l_{ij}$ are respectively the components of the curvature tensor and coefficients of $\nabla$.*

*Proof.* The Koszul formula is given by

$$
\begin{aligned}
2\bar{g}\left(\bar{\nabla}_{\tilde{V}}\tilde{W},\tilde{Z}\right) &= \tilde{V}\left(\bar{g}\left(\tilde{W},\tilde{Z}\right)\right)+\tilde{W}\left(\bar{g}\left(\tilde{Z},\tilde{V}\right)\right)-\tilde{Z}\left(\bar{g}\left(\tilde{V},\tilde{W}\right)\right)-\bar{g}\left(\tilde{V},\left[\tilde{W},\tilde{Z}\right]\right)\\
&+ \bar{g}\left(\tilde{W},\left[\tilde{Z},\tilde{V}\right]\right)+\bar{g}\left(\tilde{Z},\left[\tilde{V},\tilde{W}\right]\right)
\end{aligned}
$$

for any $\tilde{V},\tilde{W},\tilde{Z}\in\Im^1_0(T^*M)$. In Koszul formula, we put $\tilde{V}=\tilde{f}_i,\tilde{f}_{\bar{i}},\tilde{W}=\tilde{f}_j,\tilde{f}_{\bar{j}},\ \tilde{Z}=\tilde{f}_k,\ \tilde{f}_{\bar{k}}$.
*i)* By using (4), (7) and (10), we have

$$
\begin{aligned}
2\bar{g}\left(\bar{\nabla}_{\tilde{f}_i}\tilde{f}_j,\tilde{f}_t\right) &= \tilde{f}_i\left(\bar{g}\left(\tilde{f}_j,\tilde{f}_t\right)\right)+\tilde{f}_j\left(\bar{g}\left(\tilde{f}_t,\tilde{f}_i\right)\right)-\tilde{f}_t\left(\bar{g}\left(\tilde{f}_i,\tilde{f}_j\right)\right)-\bar{g}\left(\tilde{f}_i,\left[\tilde{f}_j,\tilde{f}_t\right]\right)\\
&+ \bar{g}\left(\tilde{f}_j,\left[\tilde{f}_t,\tilde{f}_i\right]\right)+\bar{g}\left(\tilde{f}_t,\left[\tilde{f}_i,\tilde{f}_j\right]\right)\\
&= \left(\partial_i+p_k\Gamma^k_{hi}\partial_{\bar{h}}\right)bp_jp_t+\left(\partial_j+p_k\Gamma^k_{hj}\partial_{\bar{h}}\right)bp_tp_i-\left(\partial_t+p_k\Gamma^k_{ht}\partial_{\bar{h}}\right)bp_ip_j\\
&- ap_kR_{jtl}{}^k\delta^l_i+ap_kR_{til}{}^k\delta^l_j+ap_kR_{ijl}{}^k\delta^l_t\\
&= bp_k\Gamma^k_{hi}\left(p_t\delta^h_j+p_j\delta^h_t\right)+bp_k\Gamma^k_{hj}\left(p_i\delta^h_t+p_t\delta^h_i\right)-bp_k\Gamma^k_{ht}\left(p_j\delta^h_i+p_i\delta^h_j\right)\\
&- ap_kR_{jti}{}^k+ap_kR_{tij}{}^k+ap_kR_{ijt}{}^k\\
&= 2bp_kp_t\Gamma^k_{ji}-2ap_kR_{jti}{}^k\\
&= \left(2\frac{b}{a}p_kp_l\Gamma^k_{ji}-2p_kR_{jli}{}^k\right)a\delta^l_t\\
&= 2\bar{g}\left(\left(\frac{b}{a}p_kp_l\Gamma^k_{ji}-p_kR_{jli}{}^k\right)\tilde{f}_{\bar{l}},\tilde{f}_t\right)
\end{aligned}
$$

and

$$
\begin{aligned}
2\bar{g}\left(\bar{\nabla}_{\tilde{f}_i}\tilde{f}_j,\tilde{f}_{\bar{t}}\right) &= \tilde{f}_i\left(\bar{g}\left(\tilde{f}_j,\tilde{f}_{\bar{t}}\right)\right)+\tilde{f}_j\left(\bar{g}\left(\tilde{f}_{\bar{t}},\tilde{f}_i\right)\right)-\tilde{f}_{\bar{t}}\left(\bar{g}\left(\tilde{f}_i,\tilde{f}_j\right)\right)-\bar{g}\left(\tilde{f}_i,\left[\tilde{f}_j,\tilde{f}_{\bar{t}}\right]\right)\\
&\quad +\bar{g}\left(\tilde{f}_j,\left[\tilde{f}_{\bar{t}},\tilde{f}_i\right]\right)+\bar{g}\left(\tilde{f}_{\bar{t}},\left[\tilde{f}_i,\tilde{f}_j\right]\right)\\
&= -\partial_{\bar{t}}\left(bp_ip_j\right)+a\Gamma^t_{jk}\delta^k_i+a\Gamma^t_{ik}\delta^k_j
\end{aligned}
$$

$$= \quad 2a\Gamma^l_{ij}\delta^t_l - b\left(\delta^l_i p_j + \delta^l_j p_i\right)\delta^t_l$$
$$= \quad 2\bar{g}\left(\left(\Gamma^l_{ij} - \frac{b}{2a}\left(\delta^l_i p_j + \delta^l_j p_i\right)\right)\tilde{f}_l, \tilde{f}_{\bar{t}}\right).$$

For *ii), iii)* and *iv)* we get calculations similar to those above. □

Then we write $\bar{\nabla}_{\tilde{f}_\alpha}\tilde{f}_\beta = \bar{\Gamma}^\delta_{\alpha\beta}\tilde{f}_\delta$ in the adapted frame $\left\{\tilde{f}_{(\alpha)}\right\}$ of $T^*M$, where $\bar{\Gamma}^\delta_{\alpha\beta}$ is the coeffients of $\bar{\nabla}$. Using Theorem 1, we obtain

**Corollary 1.** *In adapted frame $\left\{\tilde{f}_{(\beta)}\right\}$, the components of the Christoffel symbols $\bar{\Gamma}^\delta_{\alpha\beta}$ of $\bar{\nabla}$ on $(T^*M, \bar{g})$ are found as follows*

$$\bar{\Gamma}^l_{ij} = \Gamma^l_{ij} - \frac{b}{2a}\left(\delta^l_i p_j + \delta^l_j p_i\right), \qquad \bar{\Gamma}^{\bar{l}}_{ij} = \frac{b}{a}p_k p_l \Gamma^k_{ji} - p_k R_{jli}{}^k,$$
$$\bar{\Gamma}^{\bar{l}}_{i\,\bar{j}} = \frac{b}{2a}\left(\delta^j_l p_i + \delta^j_i p_l\right) - \Gamma^j_{li}, \qquad \bar{\Gamma}^{\bar{l}}_{\bar{i}\,j} = \frac{b}{2}\left(\delta^i_j p_l + \delta^i_l p_j\right),$$
$$\bar{\Gamma}^{\bar{l}}_{\bar{i}\,\bar{j}} = \bar{\Gamma}^l_{\bar{i}\,\bar{j}} = \bar{\Gamma}^l_{\bar{i}\,j} = \bar{\Gamma}^l_{i\,\bar{j}} = 0. \tag{13}$$

Let $\tilde{V} = \tilde{V}^\alpha\tilde{f}_{(\alpha)} = \tilde{V}^i\tilde{f}_{(i)} + \tilde{V}^{\bar{i}}\tilde{f}_{(\bar{i})}$ be a vector field on $T^*M$. The covariant derivative of $\tilde{V}$ with respect to the Levi-Civita connection $\bar{\nabla}$ of the natural Riemann extension $\bar{g}$ is given by

$$\bar{\nabla}_\beta\tilde{V}^\alpha = \tilde{f}_{(\beta)}\tilde{V}^\alpha + \Gamma^\alpha_{\beta\gamma}\tilde{V}^\gamma.$$

Applying (4), (8) and (13), we find the following components for the covariant derivatives of the vector fields ${}^HV, {}^CV, {}^V\vartheta$ with respect to the Levi-Civita connection $\bar{\nabla}$ of the natural Riemann extension $\bar{g}$:

$$\bar{\nabla}_i{}^HV^j = \tilde{f}_{(i)}{}^HV^j + \bar{\Gamma}^j_{ik}{}^HV^k + \bar{\Gamma}^j_{i\bar{k}}{}^HV^{\bar{k}} = \nabla_i V^j - \frac{b}{2a}\left(p_i V^j + \delta^j_i p_k V^k\right),$$
$$\bar{\nabla}_{\bar{i}}{}^HV^j = \tilde{f}_{(\bar{i})}{}^HV^j + \bar{\Gamma}^j_{\bar{i}k}{}^HV^k + \bar{\Gamma}^j_{\bar{i}\bar{k}}{}^HV^{\bar{k}} = 0,$$
$$\bar{\nabla}_i{}^HV^{\bar{j}} = \tilde{f}_{(i)}{}^HV^{\bar{j}} + \bar{\Gamma}^{\bar{j}}_{ik}{}^HV^k + \bar{\Gamma}^{\bar{j}}_{i\bar{k}}{}^HV^{\bar{k}} = \frac{b}{a}p_t p_j \Gamma^t_{ki}V^k - p_t R_{kji}{}^t V^k,$$
$$\bar{\nabla}_{\bar{i}}{}^HV^{\bar{j}} = \tilde{f}_{(\bar{i})}{}^HV^{\bar{j}} + \bar{\Gamma}^{\bar{j}}_{\bar{i}k}{}^HV^k + \bar{\Gamma}^{\bar{j}}_{\bar{i}\bar{k}}{}^HV^{\bar{k}} = \frac{b}{2}\left(p_j V^i + \delta^i_j p_k V^k\right).$$
$$\bar{\nabla}_i{}^CV^j = \nabla_i V^j - \frac{b}{2a}\left(p_i V^j + \delta^j_i p_k V^k\right),$$
$$\bar{\nabla}_{\bar{i}}{}^CV^j = 0,$$
$$\bar{\nabla}_i{}^CV^{\bar{j}} = -p_t\nabla_i\nabla_j V^t + \frac{b}{a}p_t p_j \Gamma^t_{ki}V^k - \frac{b}{2a}p_t\left(p_i\nabla_j V^t + p_j\nabla_i V^t\right) - p_t R_{kji}{}^t V^k,$$
$$\bar{\nabla}_{\bar{i}}{}^CV^{\bar{j}} = -\nabla_j V^i + \frac{b}{2}\left(p_j V^i + \delta^i_j p_k V^k\right).$$
$$\bar{\nabla}_i{}^V\vartheta^j = 0,$$
$$\bar{\nabla}_{\bar{i}}{}^V\vartheta^j = 0,$$

$$\bar{\nabla}_i{}^V\vartheta^{\bar{j}} = \nabla_i\vartheta_j + \frac{b}{2a}\left(p_i\vartheta_j + p_j\vartheta_i\right),$$
$$\bar{\nabla}_{\bar{i}}{}^V\vartheta^{\bar{j}} = 0.$$

Then, we get the following theorem:

**Theorem 2.** *The horizontal and complete lifts ${}^HV, {}^CV \in \Im_0^1(T^*M)$ of $V \in \Im_0^1(M)$ and the vertical lift ${}^V\vartheta \in \Im_0^1(T^*M)$ of $\vartheta \in \Im_1^0(M)$ are not parallel with respect to the Levi-Civita connection $\bar{\nabla}$ of the natural Riemann extension $\bar{g}$.*

## 4. The Metric Connection with Respect to the Natural Riemann Extension $\bar{g}$

The Levi-Civita connection $\bar{\nabla}$ of the natural Riemann extension $\bar{g}$ on the cotangent bundle $T^*M$ is the unique connection which satisfies $\bar{\nabla}\bar{g} = 0$, and has no torsion. Further, there exists another connection which satisfies $\breve{\nabla}\bar{g} = 0$, and has non-trivial torsion tensor. This connection is called the metric connection of $\bar{g}$.

Now we consider the horizontal lift ${}^H\nabla$ of any connection $\nabla$ on the cotangent bundle $T^*M$ defined by

$$\begin{array}{ll} {}^H\nabla_{{}^V\vartheta}{}^V\omega = 0\ , & {}^H\nabla_{{}^V\vartheta}{}^HZ = 0, \\ {}^H\nabla_{{}^HV}{}^V\omega = {}^V\left(\nabla_V\omega\right)\ , & {}^H\nabla_{{}^HV}{}^HZ = {}^H\left(\nabla_VZ\right) \end{array} \tag{14}$$

for any $V, Z \in \Im_0^1(M)$ and $\vartheta, \omega \in \Im_1^0(M)$ [21].

Let ${}^H\Gamma_{\alpha\beta}^{\gamma}$ be coefficients of ${}^H\nabla$. Using the formula ${}^H\nabla_\alpha\tilde{f}_{(\beta)} = {}^H\Gamma_{\alpha\beta}^{\gamma}\tilde{f}_{(\gamma)}$, where ${}^H\nabla_\alpha = {}^H\nabla_{\tilde{f}_{(\alpha)}}$, we obtain

$$\begin{array}{ll} {}^H\Gamma_{ij}^k = {}^H\Gamma_{ij}^k\ , & {}^H\Gamma_{i\bar{j}}^{\bar{k}} = -{}^H\Gamma_{ik}^j, \\ {}^H\Gamma_{\bar{i}\bar{j}}^{\bar{k}} = {}^H\Gamma_{ij}^{\bar{k}} = {}^H\Gamma_{\bar{i}\bar{j}}^k = {}^H\Gamma_{\bar{i}j}^k = {}^H\Gamma_{\bar{i}j}^{\bar{k}} = {}^H\Gamma_{i\bar{j}}^k = 0. \end{array} \tag{15}$$

The torsion tensor $T$ of ${}^H\nabla$ is the skew–symmetric (1,2)-tensor field and satisfies the following:

$$T\left({}^V\vartheta, {}^V\omega\right) = 0, T\left({}^HV, {}^V\omega\right) = 0\ , T\left({}^HV, {}^HZ\right) = -\gamma R\left(V, Z\right),$$

where $R$ denotes the curvature tensor of $\nabla$ and $\gamma R(V, Z) = \sum_j p_h R_{klj}^h V^k Z^l \frac{\partial}{\partial x^{\bar{j}}}$ (see[21, p.287]).

From (9) and (14), we obtain

$$\begin{array}{rcl} \left({}^H\nabla_{{}^V\vartheta}\bar{g}\right)\left({}^V\omega, {}^V\varepsilon\right) & = & {}^H\nabla_{{}^V\vartheta}\bar{g}\left({}^V\omega, {}^V\varepsilon\right) - \bar{g}\left({}^H\nabla_{{}^V\vartheta}{}^V\omega, {}^V\varepsilon\right) - \bar{g}\left({}^V\omega, {}^H\nabla_{{}^V\vartheta}{}^V\varepsilon\right), \\ & = & 0 \\ \left({}^H\nabla_{{}^HV}\bar{g}\right)\left({}^V\vartheta, {}^V\omega\right) & = & {}^H\nabla_{{}^HV}\bar{g}\left({}^V\vartheta, {}^V\omega\right) - \bar{g}\left({}^H\nabla_{{}^HV}{}^V\vartheta, {}^V\omega\right) - \bar{g}\left({}^V\vartheta, {}^H\nabla_{{}^HV}{}^V\omega\right), \\ & = & 0 \\ \left({}^H\nabla_{{}^V\vartheta}\bar{g}\right)\left({}^V\omega, {}^HZ\right) & = & {}^H\nabla_{{}^V\vartheta}\bar{g}\left({}^V\omega, {}^HZ\right) - \bar{g}\left({}^H\nabla_{{}^V\vartheta}{}^V\omega, {}^HZ\right) - \bar{g}\left({}^V\vartheta, {}^H\nabla_{{}^V\omega}{}^HZ\right) \end{array}$$

$$= \; {}^{V}\vartheta\left(a\left({}^{V}\left(\omega\left(Z\right)\right)\right)\right) = 0,$$

$$\begin{aligned}
\left({}^{H}\nabla_{{}^{H}V}\bar{g}\right)\left({}^{V}\omega, {}^{H}Z\right) &= {}^{H}\nabla_{{}^{H}V}\bar{g}\left({}^{V}\omega, {}^{H}Z\right) - \bar{g}\left({}^{H}\nabla_{{}^{H}V}{}^{V}\omega, {}^{H}Z\right) - \bar{g}\left({}^{V}\omega, {}^{H}\nabla_{{}^{H}V}{}^{H}Z\right) \\
&= {}^{H}\nabla_{{}^{H}V}\left(a\left({}^{V}\left(\omega\left(Z\right)\right)\right)\right) - \bar{g}\left({}^{V}\left(\nabla_{V}\omega\right), {}^{H}Z\right) - \bar{g}\left({}^{V}\omega, {}^{H}\left(\nabla_{V}Z\right)\right) \\
&= \left({}^{H}\nabla_{{}^{H}V}a\right)\left({}^{V}\left(\omega\left(Z\right)\right)\right) + a\left({}^{V}\left(\nabla_{V}\left(\omega\left(Z\right)\right)\right)\right) - a\left({}^{V}\left(\left(\nabla_{V}\omega\right)\left(Z\right)\right)\right) \\
&+ a\left({}^{V}\left(\omega\left(\nabla_{V}Z\right)\right)\right) \\
&= a\left({}^{V}\left(\nabla_{V}\left(\omega\left(Z\right)\right)\right)\right) - a\left({}^{V}\left(\nabla_{V}\left(\omega\left(Z\right)\right)\right)\right) = 0,
\end{aligned}$$

$$\begin{aligned}
\left({}^{H}\nabla_{{}^{V}\vartheta}\bar{g}\right)\left({}^{H}Z, {}^{V}\varepsilon\right) &= {}^{H}\nabla_{{}^{V}\vartheta}\bar{g}\left({}^{H}Z, {}^{V}\varepsilon\right) - \bar{g}\left({}^{H}\nabla_{{}^{V}\vartheta}{}^{H}Z, {}^{V}\varepsilon\right) - \bar{g}\left({}^{H}Z, {}^{H}\nabla_{{}^{V}\vartheta}{}^{V}\varepsilon\right), \\
&= 0, \\
\left({}^{H}\nabla_{{}^{H}V}\bar{g}\right)\left({}^{H}Z, {}^{V}\varepsilon\right) &= {}^{H}\nabla_{{}^{H}V}\bar{g}\left({}^{H}Z, {}^{V}\varepsilon\right) - \bar{g}\left({}^{H}\nabla_{{}^{H}V}{}^{H}Z, {}^{V}\varepsilon\right) - \bar{g}\left({}^{H}Z, {}^{H}\nabla_{{}^{H}V}{}^{V}\varepsilon\right) \\
&= {}^{H}\nabla_{{}^{H}V}\left(a^{V}\left(\varepsilon\left(Z\right)\right)\right) - \bar{g}\left({}^{H}\left(\nabla_{V}Z\right), {}^{V}\varepsilon\right) - \bar{g}\left({}^{H}Z, {}^{V}\left(\nabla_{V}\varepsilon\right)\right) \\
&= {}^{V}\left(\nabla_{V}\left(a\varepsilon\left(Z\right)\right)\right) - a^{V}\left(\varepsilon\left(\nabla_{V}Z\right)\right) - a^{V}\left(\left(\nabla_{V}\varepsilon\right)\left(Z\right)\right) = 0, \\
\left({}^{H}\nabla_{{}^{V}\vartheta}\bar{g}\right)\left({}^{H}V, {}^{H}Z\right) &= {}^{H}\nabla_{{}^{V}\vartheta}\bar{g}\left({}^{H}V, {}^{H}Z\right) - \bar{g}\left({}^{H}\nabla_{{}^{V}\vartheta}{}^{H}V, {}^{H}Z\right) - \bar{g}\left({}^{H}V, {}^{H}\nabla_{{}^{V}\vartheta}{}^{H}Z\right), \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\left({}^{H}\nabla_{{}^{H}V}\bar{g}\right)\left({}^{H}Y, {}^{H}Z\right) &= {}^{H}\nabla_{{}^{H}V}\bar{g}\left({}^{H}Y, {}^{H}Z\right) - \bar{g}\left({}^{H}\nabla_{{}^{H}V}{}^{H}Y, {}^{H}Z\right) - \bar{g}\left({}^{H}Y, {}^{H}\nabla_{{}^{H}V}{}^{H}Z\right) \\
&= {}^{H}\nabla_{{}^{H}V}\left(bp\left(Y\right)p\left(X\right)\right) - {}^{V}\left(bp\left(\nabla_{V}Y\right)p\left(Z\right)\right) - {}^{V}\left(bp\left(Y\right)p\left(\nabla_{V}Z\right)\right) \\
&= {}^{V}\left(\nabla_{V}b\left(p\left(Y\right)\right)p\left(Z\right)\right) - {}^{V}\left(\nabla_{V}b\left(p\left(Y\right)\right)p\left(Z\right)\right) = 0
\end{aligned}$$

for any $V, Y, Z \in \Im_0^1(M)$ and $\vartheta, \omega, \varepsilon \in \Im_1^0(M)$, i.e. the horizontal lift ${}^{H}\nabla$ of $\nabla$ is a metric connection.

In [21], the Ricci tensor field ${}^{H}R_{\gamma\beta}$ of ${}^{H}\nabla$ is given by:

$$\begin{aligned}
&{}^{H}R_{kj} = {}^{H}R_{\alpha kj}{}^{\alpha} = {}^{H}R_{ikj}{}^{i} + {}^{H}R_{\bar{i}kj}{}^{\bar{i}} = R_{ikj}{}^{i} = R_{kj}\;, \\
&{}^{H}R_{\bar{k}\bar{j}} = {}^{H}R_{\bar{k}j} = {}^{H}R_{k\bar{j}} = 0,
\end{aligned} \tag{16}$$

where $R_{kj}$ denotes the Ricci tensor field of $\nabla$ on $M$.

Now using (11) and (16) the natural Riemann extension $\bar{g}$, the scalar curvature of ${}^{H}\nabla$ is generated by

$${}^{H}r = \bar{g}^{\gamma\beta\,H}R_{\gamma\beta} = \bar{g}^{jk\,H}R_{jk} + \bar{g}^{\bar{j}k\,H}R_{\bar{j}k} + \bar{g}^{j\bar{k}\,H}R_{j\bar{k}} + \bar{g}^{\bar{j}\bar{k}\,H}R_{\bar{j}\bar{k}} = 0.$$

Thus we have

**Theorem 3.** *The cotangent bundle $T^{*}M$ with metric connection ${}^{H}\nabla$ has a vanishing scalar curvature with respect to the natural Riemann extension $\bar{g}$.*

## 5. Geodesics on the Cotangent Bundle with the Natural Riemann Extension

Let now we investigate the geodesics on the cotangent bundle with the natural Riemann extension. Let $C : x^h = x^h\,(t)$ be a curve in $M$ and $\omega_h\,(t)$ be a covector field along $C$. Also, we take that $\tilde{C}$ be a curve on $T^*M$ and locally given by

$$x^h = x^h\,(t)\,,\ x^{\bar{h}} \overset{def}{=} p_h = \omega_h\,(t)\,. \tag{17}$$

If the curve $C$ satisfies at all the points the relation

$$\frac{\delta\omega_h}{dt} = \frac{d\omega_h}{dt} - \Gamma^i_{jh}\frac{dx^j}{dt}\omega_i = 0,$$

then the curve $\tilde{C}$ is said to be a horizontal lift of the curve $C$ in $M$. Hence, the initial condition $\omega_h = \omega^0_h$ for $t = t_0$ is taken, there exists a unique horizontal lift given by (17).

If $t$ is the arc length of a curve $x^A = x^A\,(t)\,, A = (i, \bar{i})$ in $T^*M$, then the differential equations of the geodesic is given by

$$\frac{\delta^2 x^A}{dt^2} = \frac{d^2 x^A}{dt^2} + \bar{\Gamma}^A_{CB}\frac{dx^C}{dt}\frac{dx^B}{dt} = 0 \tag{18}$$

with respect to the induced coordinates $\left(x^i, x^{\bar{i}}\right) = \left(x^i, p_i\right)$ in $T^*M$, where $\bar{\Gamma}^A_{CB}$ are components of $\bar{\nabla}$ defined by (13).

Now, from (5), (6) and using the adapted frame $\left\{\tilde{f}_{(\beta)}\right\}$, we write the equation (18) as follow:

$$\theta^\alpha = \tilde{A}^\alpha{}_A dx^A,$$

i.e.

$$\theta^h = \tilde{A}^h{}_A dx^A = \delta^h_i dx^i = dx^h$$

for $\alpha = h$ and

$$\theta^{\bar{h}} = \tilde{A}^{\bar{h}}{}_A dx^A = -p_a\Gamma^a_{hj}dx^j + \delta^h_j dx^j = \delta p_h$$

for $\alpha = \bar{h}$. Also we put

$$\frac{\theta^h}{dt} = \tilde{A}^h{}_A\frac{dx^A}{dt} = \frac{dx^h}{dt},$$
$$\frac{\theta^{\bar{h}}}{dt} = \tilde{A}^{\bar{h}}{}_A\frac{dx^A}{dt} = \frac{\delta p_h}{dt}$$

along a curve $x^A = x^A\,(t)$ in $T^*M$. Hence,

$$\frac{d}{dt}\left(\frac{\theta^\alpha}{dt}\right) + \bar{\Gamma}^\alpha_{\gamma\beta}\frac{\theta^\gamma}{dt}\frac{\theta^\beta}{dt} = 0.$$

Using (18), we obtain

$$\begin{aligned}
&a)\ \frac{\delta^2 x^h}{dt^2}+\frac{b}{2a}\left(\delta_i^h p_j+\delta_j^h p_i\right)\frac{dx^i}{dt}\frac{dx^j}{dt}=0,\\
&b)\ \frac{\delta^2 p_h}{dt^2}+p_s\left(\frac{b}{a}p_h\Gamma_{ji}^s-R_{jhi}{}^s\right)\frac{dx^i}{dt}\frac{dx^j}{dt}+\frac{b}{2}\left(\delta_j^i p_h+\delta_h^i p_j\right)\frac{\delta p_i}{dt}\frac{dx^j}{dt}\\
&\qquad+\frac{b}{2a}\left(\delta_h^j p_i+\delta_i^j p_h\right)\frac{dx^i}{dt}\frac{\delta p_j}{dt}=0,
\end{aligned}\tag{19}$$

where $\frac{\delta^2 p_h}{dt^2}=\frac{d}{dt}\left(\frac{\delta p_h}{dt}\right)-\Gamma_{jh}^s\frac{\delta p_s}{dt}\frac{dx^j}{dt}$.

**Theorem 4.** *Let $\tilde{C}$ be a curve expressed locally by $x^h=x^h(t)$, $p_h=\omega_h(t)$ with respect to the induced coordinate system $\left(x^i,x^{\bar{i}}\right)=\left(x^i,p_i\right)$ on $T^*M$. If the curve $\tilde{C}$ satisfies the equation (19), then it is a geodesic of the natural Riemann extension $\bar{g}$.*

Let us assume that the curve (19) lies on a fibre, namely $x^h=const$. Then we obtain

$$\frac{\delta^2 p_h}{dt^2}=0.$$

Then we find $p_h=k_h t+n_h$, where $k_h$and $n_h$ are constant. With this selection, we have proved the following:

**Theorem 5.** *If geodesic $x^h=x^h(t)$, $p_h=p_h(t)$ lies on a fibre of $T^*M$ endowed with the natural Riemann extension $\bar{g}$, then: $x^h=c^h$, $p_h=k_h t+n_h$ where $c^h,k_h$ and $n_h$ are constant.*

Let now $\tilde{C}:x^h=x^h(t),x^{\bar{h}}=p_h(t)=\omega_h(t)$ be a horizontal lift ($\frac{\delta p_h}{dt}=\frac{\delta\omega_h}{dt}=0$) of the geodesic $C:x^h=x^h(t)$ ($\frac{\delta^2 x^h}{dt^2}=0$) in $M$ of $\nabla$. Then by virtue of (19), we obtain

**Theorem 6.** *Let $(M,\nabla)$ be an dimensional manifold with metric $g$ and $T^*M$ be its cotangent bundle with the natural Riemann extension $\bar{g}$. Then the horizontal lift of a geodesic on $M$ need not be a geodesic on $T^*M$ with respect to the connection $\bar{\nabla}$.*

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## References

[1] Abbasi, M. T. K., Amri, N., Bejan, C. L., Conformal vector fields and Ricci soliton structures on natural Riemannian extensions, *Mediterr. J. Math.*, 18(55) (2021), 1-16. https://doi.org/10.1007/s00009-020-01690-5

[2] Aslanci, S., Cakan, R., On a cotangent bundle with deformed Riemannian extension, *Mediterr. J. Math.*, 11(4) (2014), 1251–1260. DOI 10.1007/s00009-013-0337-2

[3] Aslanci, S., Kazimova, S., Salimov, A. A, Some notes concerning Riemannian extensions, *Ukrainian Math. J.*, 62(5) (2010), 661–675.

[4] Bejan, C. L., Eken, Ş., A characterization of the Riemann extension in terms of harmonicity, *Czech. Math. J.*, 67(1) (2017), 197–206. DOI: 10.21136/CMJ.2017.0459-15

[5] Bejan, C. L., Kowalski, O., On some differential operators on natural Riemann extensions, *Ann. Glob. Anal. Geom.,* 48 (2015), 171-–180. DOI 10.1007/s10455-015-9463-3

[6] Bejan, C. L., Nakova, G., Amost para-Hermitian and almost paracontact metric structures induced by natural Riemann extensions, *Resulth Math.*, 74(15) (2019). https://doi.org/10.1007/s00025-018-0939-x

[7] Bejan, C. L., Meriç, Ş. E., Kılıç, E., Einstein metrics induced by natural Riemann extensions, *Adv. Appl. Clifford Algebras,* 27(3) (2017), 2333–2343. DOI 10.1007/s00006-017-0774-2

[8] Bilen, L., Gezer, A., On metric connections with torsion on the cotangent bundle with modified Riemannian extension, *J. Geom.* 109(6) (2018), 1–17. https://doi.org/10.1007/s00022-018-0411-9

[9] Calvino-Louzao, E., García-Río, E., Gilkey, P., Vázquez-Lorenzo, A., The geometry of modified Riemannian extensions, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 465 (2107) (2009), 2023—2040. https://www.jstor.org/stable/30245448

[10] Dryuma, V., The Riemannian extension in theory of differential equations and their application, *Mat. Fiz. Anal. Geom.*, 10(3) (2003), 307-–325.

[11] Gezer, A., Bilen, L., Cakmak, A., Properties of modified Riemannian extensions, *Zh. Mat. Fiz. Anal. Geom.,* 11(2) (2015), 159-–173. https://doi.org/10.15407/mag11.02.159

[12] Kowalski, O., Sekizawa, M., On natural Riemann extensions, *Publ. Math. Debr.,* 78(3–4) (2011), 709—721. DOI: 10.5486/PMD.2011.4992

[13] Kowalski, O., Sekizawa, M., Almost Osserman structures on natural Riemann extensions, *Differ. Geom. Appl.,* 31 (2013), 140—149. https://doi.org/10.1016/j.difgeo.2012.10.007

[14] Ocak, F., Notes about a new metric on the cotangent bundle, *Int. Electron. J. Geom.,* 12(2) (2019), 241–249.

[15] Ocak, F., Some properties of the Riemannian extensions, *Konuralp J. of Math.*, 7(2) (2019), 359–362.

[16] Ocak, F., Some notes on Riemannian extensions, *Balkan J. Geom. Appl.*, 24(1) (2019), 45–50.

[17] Ocak, F., Kazimova, S., On a new metric in the cotangent bundle, *Transactions of NAS of Azerbaijan Series of Physical-Technical and Mathematical Sciences,* 38(1) (2018), 128—138.

[18] Patterson, E. M., Walker, A. G., Riemannian extensions, *Quant. Jour. Math.*, 3 (1952), 19–28.

[19] Salimov, A., Cakan, R., On deformed Riemannian extensions associated with twin Norden metrics, *Chinese Annals of Math. Ser. B.,* 36 (2015), 345–354. DOI: 10.1007/s11401-015-0914-8

[20] Sekizawa, M., Natural transformations of affine connections on manifolds to metrics on cotangent bundles, *Proc. 14th Winter School. Srní, Czech, 1986, Suppl. Rend. Circ. Mat. Palermo, Ser.,* 14(2) (1987), 129—142.

[21] Yano, K., Ishihara, S., Tangent and Cotangent Bundles, Mercel Dekker, Inc., New York, 1973.