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Some New Results for the J-Iterative Scheme in Kohlenbach Hyperbolic Space

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Abstract

In the present paper, we study the J-iterative scheme of Bhutia and Tiwary (J. Linear Topol. Algebra, **8**(4), (2019), 237-250) in Kohlenbach hyperbolic space. We prove the weak w^2-stability and data dependence theorems of this iterative scheme for contraction mappings. We also give some $\triangle$-convergence and strong convergence theorems for generalized α-nonexpansive mappings and finite families of total asymptotically nonexpansive mappings using J-iterative scheme. The results presented here can be viewed as a generalization of several well-known results in CAT(0) space and uniformly convex Banach space.

Keywords: *Data dependence; fixed point; hyperbolic space; J-iterative scheme; strong convergence; weak* w^2*-stability;* $\triangle$*-convergence.*
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1. Introduction

Kohlenbach [8] introduced the concept of hyperbolic space, defined below, which plays a significant role in many branches of mathematics. A *hyperbolic space* is a triple (X,d,W) where (X,d) is a metric space and $W : X\times X\times[0,1]\to X$ is a mapping such that
(H1) $d(z,W(x,y,\alpha))\leq(1-\alpha)d(z,x)+\alpha d(z,y)$,
(H2) $d(W(x,y,\alpha),W(x,y,\beta))=|\alpha-\beta|\,d(x,y)$,
(H3) $W(x,y,\alpha)=W(y,x,1-\alpha)$,
(H4) $d(W(x,z,\alpha),W(y,w,\alpha))\leq(1-\alpha)d(x,y)+\alpha d(z,w)$
for all $x,y,z,w\in X$ and $\alpha,\beta\in[0,1]$.
A mapping $\eta:(0,\infty)\times(0,2]\to(0,1]$ which provides $\delta=\eta(r,\varepsilon)$ for given $r>0$ and $\varepsilon\in(0,2]$ is called a *modulus of uniform convexity of* X. The function η is *monotone* if it decreases with r for a fixed ε.
In [10], it is noticed that any normed space is a hyperbolic space with the mapping $W(x,y,\gamma)=(1-\gamma)x+\gamma y$ and it is proved that CAT(0) space is uniformly convex hyperbolic space with the quadratic modulus of uniform convexity $\eta(r,\varepsilon)=\frac{\varepsilon^2}{8}$. Thus, the class of uniformly convex hyperbolic space is a natural generalization of both uniformly convex Banach space and CAT(0) space.
Remember that a sequence $\{x_n\}_{n=1}^{\infty}$ in X is said to be $\triangle$*-convergent* to $x\in X$ if x is the unique asymptotic center which is denoted by $A(X,\{u_{n_k}\})=\{x\}$ (see [11, 17]) of $\{u_{n_k}\}_{k=1}^{\infty}$ for every subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$. In this case, we write $\triangle$-$\lim_{n\to\infty}x_n=x$ and call x as $\triangle$-limit of $\{x_n\}_{n=1}^{\infty}$.
In 2019, Bhutia and Tiwary [3] introduced a new iterative scheme in Banach space which is called J-iterative scheme, as follows:

$$\begin{cases} x_1\in C,\\ z_n=T[(1-\beta_n)x_n+\beta_nTx_n],\\ y_n=T[(1-\alpha_n)z_n+\alpha_nTz_n],\\ x_{n+1}=Ty_n,\quad \forall n\geq 1. \end{cases}$$

They proved that this iterative scheme is faster than the recent schemes such as K-iterative [4], K*-iterative [19, 24], M*-iterative [23] and M-iterative [7, 17, 25] for contraction mappings. Also, they obtained a result for Suzuki generalized nonexpansive mappings under J-iterative scheme. In 2021, Izhar-ud-din et al. [5] modified the J-iterative scheme and proved some $\triangle$-convergence and strong convergence theorems of the modified J-iterative scheme in CAT(0) space using total asymptotically nonexpansive mappings defined in [1].

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Motivated by these papers, we study the weak w^2-stability, data dependence and convergence theorems of the J-iterative scheme in Kohlenbach hyperbolic space. This paper contains four sections. In Section 2, we establish the weak w^2-stability and data dependence results of the J-iterative scheme for contraction mappings. In Section 3, we prove some $\triangle$-convergence and strong convergence theorems of the J-iterative scheme for the class of generalized α-nonexpansive mappings which contains the class of Suzuki generalized nonexpansive mappings. In Section 4, we also prove some $\triangle$-convergence and strong convergence theorems for a finite family of total asymptotically nonexpansive mappings using the J-iterative scheme. Our results generalize the corresponding theorems of Bhutia and Tiwary [3] for uniformly convex Banach space and the theorems of Izhar-ud-din et al. [5] for CAT(0) space and many others in this direction.

2. The weak w^2-stability and data dependence results

We first extend the J-iterative scheme into the hyperbolic space as follows:

$$\begin{cases} x_1 \in C, \\ z_n = T(W(x_n, Tx_n, \beta_n)), \\ y_n = T(W(z_n, Tz_n, \alpha_n)), \\ x_{n+1} = Ty_n, \quad \forall n \geq 1. \end{cases} \tag{2.1}$$

Throughout the paper, we presume that C is a nonempty, closed, convex subset of a hyperbolic space X and $T : C \to C$ is a contraction mapping such that the fixed point set $F(T)$ is nonempty. In this case, it is known that the fixed point of T is unique.
The following theorem is a generalization of Theorem 2.1 in [3] to hyperbolic space.

Theorem 2.1. *Let $\{x_n\}_{n=1}^{\infty}$ be the iterative sequence given by (2.1) with the real sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ in $[0,1]$ satisfying $\sum_{n=1}^{\infty} \beta_n = \infty$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of T strongly.*

Proof. Let the unique fixed point be p. From (H1), (2.1) and the contractionness of T, we have

$$d(x_{n+1}, p) = d(Ty_n, p) \leq ad(y_n, p), \tag{2.2}$$

$$\begin{aligned} d(y_n, p) &= d(T(W(z_n, Tz_n, \alpha_n)), p) \\ &\leq ad(W(z_n, Tz_n, \alpha_n), p) \\ &\leq a[(1-\alpha_n)d(z_n, p) + \alpha_n d(Tz_n, p)] \\ &\leq a[(1-\alpha_n)d(z_n, p) + \alpha_n ad(z_n, p)] \\ &= a(1-\alpha_n(1-a))d(z_n, p) \leq ad(z_n, p) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} d(z_n, p) &= d(T(W(x_n, Tx_n, \beta_n)), p) \\ &\leq ad(W(x_n, Tx_n, \beta_n), p) \\ &\leq a[(1-\beta_n)d(x_n, p) + \beta_n d(Tx_n, p)] \\ &\leq a[(1-\beta_n)d(x_n, p) + \beta_n ad(x_n, p)] \\ &= a(1-\beta_n(1-a))d(x_n, p). \end{aligned} \tag{2.4}$$

Combining (2.2), (2.3) and (2.4), we obtain

$$\begin{aligned} d(x_{n+1}, p) &\leq a^3(1-\beta_n(1-a))d(x_n, p) \\ &\leq a^3(1-\beta_n(1-a))a^3(1-\beta_{n-1}(1-a))d(x_{n-1}, p) \\ &\leq \cdots \\ &\leq \left(a^3\right)^n \prod_{k=1}^{n}(1-\beta_k(1-a))d(x_1, p). \end{aligned} \tag{2.5}$$

It is well-known from the classical analysis that $1 - x \leq e^{-x}$ for all $x \in [0,1]$. Taking into account this fact together with (2.5), we have

$$d(x_{n+1}, p) \leq \left(a^3\right)^n e^{-(1-a)\sum_{k=1}^{n}\beta_k} d(x_1, p).$$

Since $\sum_{n=1}^{\infty} \beta_n = \infty$ and $a \in [0,1)$, then we get that $\lim_{n\to\infty} d(x_{n+1}, p) = 0$. Thus we obtain $x_n \to p \in F(T)$. □

Remark 2.2. *If the condition $\sum_{n=1}^{\infty} \beta_n = \infty$ replace with $\sum_{n=1}^{\infty} \alpha_n = \infty$ in Theorem 2.1, then we can rewrite (2.5) as*

$$d(x_{n+1}, p) \leq \left(a^3\right)^n \prod_{k=1}^{n}(1-\alpha_k(1-a))d(x_1, p).$$

Therefore, we get the same result.

Timiş [22] has defined the following concept of weak w^2-stability by adopting equivalent sequences instead of arbitrary sequences in the definition of T-stability in [2].

Definition 2.3. *(see [22, Definition 2.4]) Let (X,d) be a metric space, T be a self mapping on X and $\{x_n\}_{n=1}^{\infty} \subset X$ be an iterative sequence produced by a general relation of the form*

$$\begin{cases} x_1 \in X, \\ x_{n+1} = f(T,x_n), \quad \forall n \geq 1, \end{cases}$$

where $f(T,x_n)$ denotes all parameters in the given iterative scheme. Suppose that $\{x_n\}_{n=1}^{\infty}$ converges to $p \in F(T)$ strongly. If for any equivalent sequence $\{y_n\}_{n=1}^{\infty} \subset X$ of $\{x_n\}_{n=1}^{\infty}$,

$$\lim_{n\to\infty} d(y_{n+1}, f(T,y_n)) = 0 \Longrightarrow \lim_{n\to\infty} y_n = p,$$

then the iterative sequence $\{x_n\}_{n=1}^{\infty}$ is said to be weak w^2-stable with respect to T.

Next we show that the J-iteration process is weak w^2-stable with respect to T.

Theorem 2.4. *Suppose that the condition of Theorem 2.1 holds. Then the iteration process (2.1) is weak w^2-stable with respect to T.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be the iterative sequence given by (2.1) and $\{p_n\}_{n=1}^{\infty} \subset C$ be an equivalent sequence of $\{x_n\}_{n=1}^{\infty}$. Set

$$\varepsilon_n = d(p_{n+1}, Tq_n),$$

where $q_n = T(W(r_n, Tr_n, \alpha_n))$ with $r_n = T(W(p_n, Tp_n, \beta_n))$. Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$. It follows from (H4) and (2.1) that

$$\begin{aligned} d(p_{n+1},p) &\leq d(p_{n+1},x_{n+1}) + d(x_{n+1},p) \\ &\leq d(p_{n+1},Tq_n) + d(Tq_n,Ty_n) + d(x_{n+1},p) \\ &\leq \varepsilon_n + ad(y_n,q_n) + d(x_{n+1},p), \end{aligned}$$

$$\begin{aligned} d(y_n,q_n) &= d(T(W(z_n,Tz_n,\alpha_n)), T(W(r_n,Tr_n,\alpha_n))) \\ &\leq ad(W(z_n,Tz_n,\alpha_n), W(r_n,Tr_n,\alpha_n)) \\ &\leq a\left[(1-\alpha_n)d(z_n,r_n) + \alpha_n d(Tz_n,Tr_n)\right] \\ &\leq a\left[(1-\alpha_n)d(z_n,r_n) + \alpha_n ad(z_n,r_n)\right] \\ &= a(1-\alpha_n(1-a))d(z_n,r_n) \leq ad(z_n,r_n) \end{aligned}$$

and

$$\begin{aligned} d(z_n,r_n) &= d(T(W(x_n,Tx_n,\beta_n)), T(W(p_n,Tp_n,\beta_n))) \\ &\leq ad(W(x_n,Tx_n,\beta_n), W(p_n,Tp_n,\beta_n)) \\ &\leq a\left[(1-\beta_n)d(x_n,p_n) + \beta_n d(Tx_n,Tp_n)\right] \\ &\leq a\left[(1-\beta_n)d(x_n,p_n) + \beta_n ad(x_n,p_n)\right] \\ &= a(1-\beta_n(1-a))d(x_n,p_n). \end{aligned}$$

These inequalities imply that

$$d(p_{n+1},p) \leq \varepsilon_n + a^3(1-\beta_n(1-a))d(x_n,p_n) + d(x_{n+1},p). \tag{2.6}$$

From Theorem 2.1, it follows that $\lim_{n\to\infty} d(x_{n+1},p) = 0$. Since $\{x_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ are equivalent sequences, we have $\lim_{n\to\infty} d(x_n,p_n) = 0$. Now taking the limit of both sides of (2.6) as $n \to \infty$ and then using the assumption $\lim_{n\to\infty} \varepsilon_n = 0$, we have $\lim_{n\to\infty} d(p_{n+1},p) = 0$. Thus $\{x_n\}_{n=1}^{\infty}$ is weak w^2-stable with respect to T. □

Next we prove the data dependence result for the J-iterative scheme.

Theorem 2.5. *Let $\overline{T} : C \to C$ be an approximate operator of T, that is $d(Tx,\overline{T}x) \leq \varepsilon$ for all $x \in C$ and for a fixed $\varepsilon > 0$. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{\overline{x}_n\}_{n=1}^{\infty}$ are two iterative sequences defined by (2.1) and*

$$\begin{cases} \overline{x}_1 \in C, \\ \overline{z}_n = \overline{T}(W(\overline{x}_n, \overline{T}\overline{x}_n, \beta_n)), \\ \overline{y}_n = \overline{T}(W(\overline{z}_n, \overline{T}\overline{z}_n, \alpha_n)), \\ \overline{x}_{n+1} = \overline{T}\overline{y}_n, \quad \forall n \geq 1, \end{cases} \tag{2.7}$$

respectively, where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are real sequences in $[0,1]$ satisfying $\sum_{n=1}^{\infty} \beta_n = \infty$. If $p = Tp$ and $\overline{p} = \overline{T}\overline{p}$ then we have

$$d(p,\overline{p}) \leq \frac{(a^3+2a^2+a+1)\varepsilon}{1-a^3},$$

where $a \in [0,1)$.

Proof. It follows from (2.1) and (2.7) that

$$\begin{aligned} d(x_{n+1},\bar{x}_{n+1}) &= d(Ty_n,\overline{T}\bar{y}_n) \\ &\le d(Ty_n,T\bar{y}_n)+d(T\bar{y}_n,\overline{T}\bar{y}_n) \\ &\le ad(y_n,\bar{y}_n)+\varepsilon, \end{aligned}$$

$$\begin{aligned} d(y_n,\bar{y}_n) &= d(T(W(z_n,Tz_n,\alpha_n)),\overline{T}(W(\bar{z}_n,\overline{T}\bar{z}_n,\alpha_n))) \\ &\le d(T(W(z_n,Tz_n,\alpha_n)),T(W(\bar{z}_n,\overline{T}\bar{z}_n,\alpha_n))) \\ &\quad +d(T(W(\bar{z}_n,\overline{T}\bar{z}_n,\alpha_n)),\overline{T}(W(\bar{z}_n,\overline{T}\bar{z}_n,\alpha_n))) \\ &\le ad(W(z_n,Tz_n,\alpha_n),W(\bar{z}_n,\overline{T}\bar{z}_n,\alpha_n))+\varepsilon \\ &\le a\left[(1-\alpha_n)d(z_n,\bar{z}_n)+\alpha_n d(Tz_n,\overline{T}\bar{z}_n)\right]+\varepsilon \\ &\le a(1-\alpha_n)d(z_n,\bar{z}_n)+a\alpha_n\left[d(Tz_n,T\bar{z}_n)+d(T\bar{z}_n,\overline{T}\bar{z}_n)\right]+\varepsilon \\ &\le a(1-\alpha_n)d(z_n,\bar{z}_n)+a\alpha_n[ad(z_n,\bar{z}_n)+\varepsilon]+\varepsilon \\ &= a(1-\alpha_n(1-a))d(z_n,\bar{z}_n)+a\alpha_n\varepsilon+\varepsilon \end{aligned}$$

and

$$\begin{aligned} d(z_n,\bar{z}_n) &= d(T(W(x_n,Tx_n,\beta_n)),\overline{T}(W(\bar{x}_n,\overline{T}\bar{x}_n,\beta_n))) \\ &\le d(T(W(x_n,Tx_n,\beta_n)),T(W(\bar{x}_n,\overline{T}\bar{x}_n,\beta_n))) \\ &\quad +d(T(W(\bar{x}_n,\overline{T}\bar{x}_n,\beta_n)),\overline{T}(W(\bar{x}_n,\overline{T}\bar{x}_n,\beta_n))) \\ &\le ad(W(x_n,Tx_n,\beta_n),W(\bar{x}_n,\overline{T}\bar{x}_n,\beta_n))+\varepsilon \\ &\le a\left[(1-\beta_n)d(x_n,\bar{x}_n)+\beta_n d(Tx_n,\overline{T}\bar{x}_n)\right]+\varepsilon \\ &\le a(1-\beta_n)d(x_n,\bar{x}_n)+a\beta_n\left[d(Tx_n,T\bar{x}_n)+d(T\bar{x}_n,\overline{T}\bar{x}_n)\right]+\varepsilon \\ &\le a(1-\beta_n)d(x_n,\bar{x}_n)+a\beta_n[ad(x_n,\bar{x}_n)+\varepsilon]+\varepsilon \\ &= a(1-\beta_n(1-a))d(x_n,\bar{x}_n)+a\beta_n\varepsilon+\varepsilon. \end{aligned}$$

Combining these inequalities, we get

$$\begin{aligned} d(x_{n+1},\bar{x}_{n+1}) &\le a^3(1-\alpha_n(1-a))(1-\beta_n(1-a))d(x_n,\bar{x}_n)+a^3(1-\alpha_n(1-a))\beta_n\varepsilon \\ &\quad +a^2(1-\alpha_n(1-a))\varepsilon+a^2\alpha_n\varepsilon+a\varepsilon+\varepsilon. \end{aligned} \tag{2.8}$$

If $a^3\in(0,1)$ then we can find a real number $k\in(0,1)$ such that $a^3=1-k$. Hence, by the facts of α_n, $\beta_n\le 1$, $1-\alpha_n(1-a)\le 1$ and $1-\beta_n(1-a)\le 1$ for all $n\ge 1$, we can rewrite (2.8) as

$$d(x_{n+1},\bar{x}_{n+1})\le(1-k)d(x_n,\bar{x}_n)+k\frac{a^3\varepsilon+2a^2\varepsilon+a\varepsilon+\varepsilon}{k}.$$

By Lemma 2.2 in [20], we have

$$d(p,\bar{p})\le\frac{(a^3+2a^2+a+1)\varepsilon}{1-a^3}.$$

If $a^3=0$, from (2.8), we get $d(p,\bar{p})\le\varepsilon$. This completes the proof. □

Remark 2.6. *In the proof of Theorem 2.5, we can also rewrite (2.8) as*

$$d(x_{n+1},\bar{x}_{n+1})\le(1-k)d(x_n,\bar{x}_n)+k\frac{a^3\beta_n\varepsilon+a^2\varepsilon+a^2\alpha_n\varepsilon+a\varepsilon+\varepsilon}{1-a^3}.$$

If the condition $\lim_{n\to\infty}\alpha_n=\lim_{n\to\infty}\beta_n=0$ is added for the sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ in the hypotheses of Theorem 2.5 then we obtain that

$$d(p,\bar{p})\le\frac{\varepsilon}{1-a}.$$

3. Some convergence results for a generalized α-nonexpansive mapping

The following theorem is a generalization of the results in Section 3 of [3].

Theorem 3.1. *Let C be a nonempty, closed, convex subset of a complete, uniformly convex hyperbolic space X with the monotone modulus of uniform convexity η and $T:C\to C$ be a generalized α-nonexpansive mapping. Let $\{x_n\}_{n=1}^{\infty}$ be the iterative sequence (2.1) with real sequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ in $[a,b]$ for some $a,b\in(0,1)$.*
(a) If $F(T)\neq\emptyset$, then $\lim_{n\to\infty}d(x_n,p)$ exists for each $p\in F(T)$.
(b) Then, $F(T)\neq\emptyset$ if and only if $\{x_n\}_{n=1}^{\infty}$ is bounded and $\lim_{n\to\infty}d(x_n,Tx_n)=0$.

Proof. (a) Let $p\in F(T)$. By Proposition 3.5 in [13], we have

$$d(x_{n+1},p)=d(Ty_n,p)\leq d(y_n,p), \tag{3.1}$$

$$\begin{aligned} d(y_n,p) &= d(T(W(z_n,Tz_n,\alpha_n)),p) \\ &\leq d(W(z_n,Tz_n,\alpha_n),p) \\ &\leq (1-\alpha_n)d(z_n,p)+\alpha_n d(Tz_n,p) \\ &\leq (1-\alpha_n)d(z_n,p)+\alpha_n d(z_n,p)=d(z_n,p) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} d(z_n,p) &= d(T(W(x_n,Tx_n,\beta_n)),p) \\ &\leq d(W(x_n,Tx_n,\beta_n),p) \\ &\leq (1-\beta_n)d(x_n,p)+\beta_n d(Tx_n,p) \\ &\leq (1-\beta_n)d(x_n,p)+\beta_n d(x_n,p)=d(x_n,p). \end{aligned} \tag{3.3}$$

By (3.1), (3.2) and (3.3), we obtain

$$d(x_{n+1},p)\leq d(x_n,p). \tag{3.4}$$

Hence the sequence $\{d(x_n,p)\}_{n=1}^{\infty}$ is non-increasing and bounded below, which implies that

$$\lim_{n\to\infty} d(x_n,p) \text{ exists for all } p\in F(T). \tag{3.5}$$

(b) Suppose $F(T)\neq\emptyset$ and choose $p\in F(T)$. Then, by (3.5), $\lim_{n\to\infty} d(x_n,p)$ exists and $\{x_n\}_{n=1}^{\infty}$ is bounded. Let

$$\lim_{n\to\infty} d(x_n,p)=c \quad \text{for some } c\geq 0. \tag{3.6}$$

Noting $d(Tx_n,p)\leq d(x_n,p)$, by (3.6) we have

$$\limsup_{n\to\infty} d(Tx_n,p)\leq c. \tag{3.7}$$

Taking the lim sup on both sides of (3.3), we obtain

$$\limsup_{n\to\infty} d(z_n,p)\leq c. \tag{3.8}$$

By (3.1) and (3.2), we get

$$d(x_{n+1},p)\leq d(z_n,p),$$

which yields that

$$c\leq \liminf_{n\to\infty} d(z_n,p). \tag{3.9}$$

From the estimates of (3.8) and (3.9), we have that

$$\lim_{n\to\infty} d(z_n,p)=c. \tag{3.10}$$

Thus, from (3.3), (3.6) and (3.10), we obtain

$$\lim_{n\to\infty} d(W(x_n,Tx_n,\beta_n),p)=c. \tag{3.11}$$

With the help of (3.6), (3.7), (3.11) and Lemma 2.5 in [9], we get

$$\lim_{n\to\infty} d(x_n,Tx_n)=0. \tag{3.12}$$

Conversely, we assume that $\{x_n\}_{n=1}^{\infty}$ is bounded and $\lim_{n\to\infty} d(x_n,Tx_n)=0$. Let $p\in A(C,\{x_n\})$. By Lemma 5.2 in [13], we have

$$\begin{aligned} r(Tp,\{x_n\}) &= \limsup_{n\to\infty} d(x_n,Tp) \\ &\leq \left(\frac{3+\alpha}{1-\alpha}\right)\limsup_{n\to\infty} d(x_n,Tx_n)+\limsup_{n\to\infty} d(x_n,p) \\ &= \limsup_{n\to\infty} d(x_n,p)=r(p,\{x_n\}). \end{aligned}$$

Hence, we conclude that $Tp\in A(C,\{x_n\})$. Since the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded, by Proposition 3.3 in [11], $A(C,\{x_n\})$ consists of a unique element. Hence, we have $Tp=p$. Thus, $F(T)\neq\emptyset$. □

We now prove the $\triangle$-convergence theorem of the iterative sequence $\{x_n\}_{n=1}^{\infty}$ defined by (2.1) for a generalized α-nonexpansive mapping in a hyperbolic space.

Theorem 3.2. *Let X,C,T and $\{x_n\}_{n=1}^{\infty}$ be the same as in Theorem 3.1 and $F(T)\neq\emptyset$. Then the sequence $\{x_n\}_{n=1}^{\infty}$ is $\triangle$-convergent to a fixed point of T.*

Proof. By Proposition 3.3 in [11], the sequence $\{x_n\}_{n=1}^{\infty}$ has a unique asymptotic center $A(C,\{x_n\})=\{x\}$. Let $\{u_{n_k}\}_{k=1}^{\infty}$ be any subsequence of $\{x_n\}_{n=1}^{\infty}$ such that $A(C,\{u_{n_k}\})=\{u\}$. Then, by Theorem 3.1, we have that $\lim_{k\to\infty} d(u_{n_k},Tu_{n_k})=0$. It follows similarly from the proof of Theorem 3.1 that u is a fixed point of T. Next, we claim that the fixed point u is the unique asymptotic center for each subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$. Assume on the contrary that $x\neq u$. Since $\lim_{n\to\infty} d(x_n,u)$ exists, by the uniqueness of asymptotic center, therefore we have

$$\begin{aligned}\limsup_{k\to\infty} d(u_{n_k},u) &< \limsup_{k\to\infty} d(u_{n_k},x)\\ &\leq \limsup_{n\to\infty} d(x_n,x)\\ &< \limsup_{n\to\infty} d(x_n,u)\\ &= \limsup_{k\to\infty} d(u_{n_k},u).\end{aligned}$$

This is a contradiction. Hence $x=u$. Since $\{u_{n_k}\}_{k=1}^{\infty}$ is an arbitrary subsequence of $\{x_n\}_{n=1}^{\infty}$, therefore $A(C,\{u_{n_k}\})=\{u\}$ for all subsequences $\{u_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$. It is proved that the sequence $\{x_n\}_{n=1}^{\infty}$ is $\triangle$-convergent to a fixed point of T. □

Next, we prove the strong convergence theorem.

Theorem 3.3. *Suppose that all conditions of Theorem 3.2 hold. Then the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of T strongly if and only if $\liminf_{n\to\infty} d(x_n,F(T))=0$ or $\limsup_{n\to\infty} d(x_n,F(T))=0$, where $d(x,F(T))=\inf\{d(x,p):p\in F(T)\}$.*

Proof. If the sequence $\{x_n\}_{n=1}^{\infty}$ converges to $p\in F(T)$ strongly then $\lim_{n\to\infty} d(x_n,p)=0$. Since $0\leq d(x_n,F(T))\leq d(x_n,p)$, we have $\liminf_{n\to\infty} d(x_n,F(T))=\limsup_{n\to\infty} d(x_n,F(T))=0$.
Conversely, suppose that $\liminf_{n\to\infty} d(x_n,F(T))=0$ or $\limsup_{n\to\infty} d(x_n,F(T))=0$. It follows from (3.5) that $\lim_{n\to\infty} d(x_n,F(T))$ exists and hence $\lim_{n\to\infty} d(x_n,F(T))=0$. Therefore, there exist a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ and $\{p_k\}_{k=1}^{\infty}$ in $F(T)$ such that $d(x_{n_k},p_k)<\frac{1}{2^k}$ for all $k\geq 1$. By (3.4), we have

$$d(x_{n_{k+1}},p_{k+1})\leq d(x_{n_k},p_k)<\frac{1}{2^k},$$

which implies that

$$d(p_{k+1},p_k)\leq d(p_{k+1},x_{n_{k+1}})+d(x_{n_{k+1}},p_k)<\frac{1}{2^{k+1}}+\frac{1}{2^k}<\frac{1}{2^{k-1}}\to 0 \quad \text{as } k\to\infty.$$

Hence, we conclude that $\{p_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $F(T)$ and so it converges to some p strongly. By Lemma 3.6 in [13], $F(T)$ is closed and so $p\in F(T)$. By (3.5), $\lim_{n\to\infty} d(x_n,p)$ exists and hence p is the strong limit of $\{x_n\}_{n=1}^{\infty}$. □

Now we prove the following strong convergence theorem using the concepts of condition (I) which is defined in [14] and compact set.

Theorem 3.4. *Under the assumptions of Theorem 3.2, if T satisfies the condition (I) or C is a compact subset of X, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of T strongly.*

Proof. If T satisfies the condition (I), then by (3.12), we have

$$\lim_{n\to\infty} f(d(x_n,F(T)))\leq \lim_{n\to\infty} d(x_n,Tx_n) \ =0.$$

Therefore, we get that $\lim_{n\to\infty} f(d(x_n,F(T)))=0$. Since f is a non-decreasing function satisfying $f(0)=0$ and $f(r)>0$ for all $r\in(0,\infty)$, we have $\lim_{n\to\infty} d(x_n,F(T))=0$. The rest of the proof follows in lines of Theorem 3.3.
If C is compact subset of X, then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\{x_{n_k}\}_{k=1}^{\infty}$ converges strongly to p for some $p\in C$. By Lemma 5.2 in [13] and (3.12), we have

$$\lim_{k\to\infty} d(x_{n_k},Tp)\leq\left(\frac{3+\alpha}{1-\alpha}\right)\lim_{k\to\infty} d(x_{n_k},Tx_{n_k})+\lim_{k\to\infty} d(x_{n_k},p)=0.$$

Then, we obtain $Tp=p$, that is, $p\in F(T)$. It follows from (3.5) that $\lim_{n\to\infty} d(x_n,p)$ exists for every $p\in F(T)$ and hence $\{x_n\}_{n=1}^{\infty}$ converges to p strongly. □

4. Some convergence results for a finite family of total asymptotically nonexpansive mappings

First, we modify the J-iterative scheme for a finite family of mappings into hyperbolic space:

$$\begin{cases} x_1\in C,\\ z_n=T_i^n(W(x_n,T_i^n x_n,\beta_n)),\\ y_n=T_i^n(W(z_n,T_i^n z_n,\alpha_n)),\\ x_{n+1}=T_i^n y_n, \quad \forall n\geq 1,\end{cases} \tag{4.1}$$

where $T_i=T_{i(\text{mod}N)}$(here the function mod N takes values in $\{1,2,...,N\}$) and for each $i=1,2,...,N, T_i:C\to C$ is an uniformly L_i-Lipschitzian and $(\{v_n^{(i)}\},\{\mu_n^{(i)}\},\zeta^{(i)})$-total asymptotically nonexpansive mapping.

Remark 4.1. *In fact, letting*

$$\begin{aligned} L &= \max\{L_i; i=1,2,...,N\}, v_n = \max\{v_n^{(i)}; i=1,2,...,N\}, \\ \mu_n &= \max\{\mu_n^{(i)}; i=1,2,...,N\}, \zeta = \max\{\zeta^{(i)}; i=1,2,...,N\}, \end{aligned}$$

then $\{T_i\}_{i=1}^N$ is a finite family of uniformly L-Lipschitzian and $(\{v_n\},\{\mu_n\},\zeta)$-total asymptotically nonexpansive mappings.

From now on for a finite family $\{T_i\}_{i=1}^N$, we denote $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$.

We prove some convergence theorems of the iterative sequence $\{x_n\}_{n=1}^\infty$ defined by (4.1) for a finite family of total asymptotically nonexpansive mappings in a hyperbolic space.

Theorem 4.2. *Let C be a nonempty, closed, convex subset of a complete, uniformly convex hyperbolic space X with the monotone modulus of uniform convexity η. Let $\{T_i\}_{i=1}^N$ be a finite family of uniformly L-Lipschitzian and $(\{v_n\},\{\mu_n\},\zeta)$-total asymptotically nonexpansive self mappings on C. If the following conditions are satisfied:*
(i) $\sum_{n=1}^\infty v_n < \infty$ and $\sum_{n=1}^\infty \mu_n < \infty$;
(ii) there exist constants $a, b \in (0,1)$ such that $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty \subset [a,b]$;
(iii) there exists a constant $M > 0$ such that $\zeta(r) \leq Mr, \forall r \geq 0$;
then
(a) the sequence $\{x_n\}_{n=1}^\infty$ defined by (4.1) is $\triangle$-convergent to a point in F.
(b) the sequence $\{x_n\}_{n=1}^\infty$ converges to some $p \in F$ strongly if and only if $\liminf_{n\to\infty} d(x_n,F) = 0$ or $\limsup_{n\to\infty} d(x_n,F) = 0$.

Proof. (a) Let $p \in F$. Since $\{T_i\}_{i=1}^N$ is a finite family of total asymptotically nonexpansive mappings, by the condition (iii), we get

$$\begin{aligned} d(z_n,p) &= d(T_i^n(W(x_n,T_i^nx_n,\beta_n)),p) \\ &\leq d(W(x_n,T_i^nx_n,\beta_n),p) + v_n\zeta(d(W(x_n,T_i^nx_n,\beta_n),p)) + \mu_n \\ &\leq (1+v_nM)d(W(x_n,T_i^nx_n,\beta_n),p) + \mu_n \\ &\leq (1+v_nM)[(1-\beta_n)d(x_n,p) + \beta_n d(T_i^nx_n,p)] + \mu_n \\ &\leq (1+v_nM)[(1-\beta_n)d(x_n,p) + \beta_n\{d(x_n,p) + v_n\zeta(d(x_n,p)) + \mu_n\}] + \mu_n \\ &\leq (1+v_nM)[(1+\beta_nv_nM)d(x_n,p) + \beta_n\mu_n] + \mu_n \\ &\leq (1+v_nM)^2 d(x_n,p) + (2+v_nM)\mu_n. \end{aligned} \tag{4.2}$$

Similarly, we obtain

$$\begin{aligned} d(y_n,p) &= d(T_i^n(W(z_n,T_i^nz_n,\alpha_n)),p) \\ &\leq (1+v_nM)d(W(z_n,T_i^nz_n,\alpha_n),p) + \mu_n \\ &\leq (1+v_nM)^2 d(z_n,p) + (2+v_nM)\mu_n. \end{aligned} \tag{4.3}$$

Substituting (4.2) into (4.3), we have

$$d(y_n,p) \leq (1+v_nM)^4 d(x_n,p) + (2+v_nM)(1+(1+v_nM)^2)\mu_n. \tag{4.4}$$

Also, we obtain

$$\begin{aligned} d(x_{n+1},p) &= d(T_i^ny_n,p) \leq d(y_n,p) + v_n\zeta(d(y_n,p)) + \mu_n \\ &\leq (1+v_nM)d(y_n,p) + \mu_n. \end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5), we have

$$d(x_{n+1},p) \leq (1+\sigma_n)d(x_n,p) + \xi_n, \ \forall n \geq 1 \text{ and } p \in F(T),$$

where $\sigma_n = 5(v_nM) + 10(v_nM)^2 + 10(v_nM)^3 + 5(v_nM)^4 + (v_nM)^5$ and $\xi_n = 1 + (1+v_nM)(2+v_nM)(1+(1+v_nM)^2)$. By virtue of the condition (i), we get

$$\sum_{n=1}^\infty \sigma_n < \infty \text{ and } \sum_{n=1}^\infty \xi_n < \infty.$$

By Lemma 2 in [12],

$$\lim_{n\to\infty} d(x_n,p) \text{ exists for each } p \in F. \tag{4.6}$$

We may assume that

$$\lim_{n\to\infty} d(x_n,p) = c \geq 0. \tag{4.7}$$

Taking lim sup on both sides of the inequality (4.2), we have

$$\limsup_{n\to\infty} d(z_n,p) \leq c. \tag{4.8}$$

Since

$$\begin{aligned} d(T_i^n z_n, p) &\leq d(z_n, p) + v_n \zeta(d(z_n, p)) + \mu_n \\ &\leq (1 + v_n M) d(z_n, p) + \mu_n, \ \forall n \geq 1, \end{aligned}$$

we have

$$\limsup_{n\to\infty} d(T_i^n z_n, p) \leq c. \tag{4.9}$$

Similarly, we get

$$\limsup_{n\to\infty} d(T_i^n x_n, p) \leq c. \tag{4.10}$$

Now, we can write

$$\begin{aligned} d(x_{n+1}, p) &\leq (1 + v_n M) d(y_n, p) + \mu_n \\ &\leq (1 + v_n M)\left[(1 + v_n M)^2 d(z_n, p) + (2 + v_n M)\mu_n\right] + \mu_n \\ &= (1 + v_n M)^3 d(z_n, p) + [1 + (1 + v_n M)(2 + v_n M)]\,\mu_n. \end{aligned}$$

Taking lim inf on both sides of the above inequality, we have that $\liminf_{n\to\infty} d(z_n, p) \geq c$. Combining with (4.8), it yields that

$$\lim_{n\to\infty} d(z_n, p) = c. \tag{4.11}$$

On the other hand, since

$$\begin{aligned} \lim_{n\to\infty} d(z_n, p) &\leq \lim_{n\to\infty} d(T_i^n(W(x_n, T_i^n x_n, \beta_n)), p) \\ &\leq \lim_{n\to\infty} [(1 + v_n M) d(W(x_n, T_i^n x_n, \beta_n), p) + \mu_n] \\ &= \lim_{n\to\infty} d(W(x_n, T_i^n x_n, \beta_n), p) \\ &\leq \lim_{n\to\infty} [(1 - \beta_n) d(x_n, p) + \beta_n d(T_i^n x_n, p)] \\ &\leq \lim_{n\to\infty} [(1 + \beta_n v_n M) d(x_n, p) + \beta_n \mu_n] \\ &= \lim_{n\to\infty} d(x_n, p), \end{aligned}$$

we have

$$\lim_{n\to\infty} d(W(x_n, T_i^n x_n, \beta_n), p) = c. \tag{4.12}$$

By Lemma 2.5 in [9] and (4.7), (4.10), (4.12), we get

$$\lim_{n\to\infty} d(x_n, T_i^n x_n) = 0. \tag{4.13}$$

From (4.4) and (4.5), we conclude that

$$\limsup_{n\to\infty} d(y_n, p) \leq c \ \text{ and } \liminf_{n\to\infty} d(y_n, p) \geq c,$$

respectively. Hence, $\lim_{n\to\infty} d(y_n, p) = c$. Likewise, since

$$\begin{aligned} \lim_{n\to\infty} d(y_n, p) &\leq \lim_{n\to\infty} d(T_i^n(W(z_n, T_i^n z_n, \alpha_n)), p) \\ &\leq \lim_{n\to\infty} [(1 + v_n M) d(W(z_n, T_i^n z_n, \alpha_n), p) + \mu_n] \\ &= \lim_{n\to\infty} d(W(z_n, T_i^n z_n, \alpha_n), p) \\ &\leq \lim_{n\to\infty} [(1 - \alpha_n) d(z_n, p) + \alpha_n d(T_i^n z_n, p)] \\ &\leq \lim_{n\to\infty} [(1 + \alpha_n v_n M) d(z_n, p) + \alpha_n \mu_n] \\ &= \lim_{n\to\infty} d(z_n, p), \end{aligned}$$

we have

$$\lim_{n\to\infty} d(W(z_n, T_i^n z_n, \alpha_n), p) = c. \tag{4.14}$$

Again, by Lemma 2.5 in [9] and (4.9), (4.11), (4.14), we get

$$\lim_{n\to\infty} d(z_n, T_i^n z_n) = 0. \tag{4.15}$$

By (4.13) and (4.15), we have

$$\begin{aligned} d(T_i^n x_n, T_i^n z_n) &\leq d(x_n,z_n) + v_n\zeta(d(x_n,z_n)) + \mu_n \\ &\leq (1+v_nM)d(x_n, T_i^n(W(x_n,T_i^n x_n,\beta_n))) + \mu_n \\ &\leq (1+v_nM)\left[d(x_n,T_i^n x_n) + d(T_i^n x_n, T_i^n(W(x_n,T_i^n x_n,\beta_n)))\right] + \mu_n \\ &\leq (1+v_nM)d(x_n,T_i^n x_n) + (1+v_nM)[d(x_n,W(x_n,T_i^n x_n,\beta_n)) \\ &\quad + v_n\zeta(d(x_n,W(x_n,T_i^n x_n,\beta_n))) + \mu_n] + \mu_n \\ &\leq (1+v_nM)d(x_n,T_i^n x_n) + (1+v_nM)^2 d(x_n,W(x_n,T_i^n x_n,\beta_n)) + (2+v_nM)\mu_n \\ &\leq (1+v_nM)d(x_n,T_i^n x_n) + (1+v_nM)^2\beta_n d(x_n,T_i^n x_n) + (2+v_nM)\mu_n \\ &\to 0 \quad \text{as} \quad n\to\infty \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} d(T_i^n z_n, T_i^n y_n) &\leq d(z_n,y_n) + v_n\zeta(d(z_n,y_n)) + \mu_n \\ &\leq d(z_n,y_n) + v_n\zeta(d(z_n,y_n)) + \mu_n \\ &\leq (1+v_nM)d(z_n, T_i^n(W(z_n,T_i^n z_n,\alpha_n))) + \mu_n \\ &\leq (1+v_nM)\left[d(z_n,T_i^n z_n) + d(T_i^n z_n, T_i^n(W(z_n,T_i^n z_n,\alpha_n)))\right] + \mu_n \\ &\leq (1+v_nM)d(z_n,T_i^n z_n) + (1+v_nM)[d(z_n,W(z_n,T_i^n z_n,\alpha_n)) \\ &\quad + v_n\zeta(d(x_n,W(z_n,T_i^n z_n,\alpha_n))) + \mu_n] + \mu_n \\ &\leq (1+v_nM)d(z_n,T_i^n z_n) + (1+v_nM)^2 d(z_n,W(z_n,T_i^n z_n,\alpha_n)) + (2+v_nM)\mu_n \\ &\leq (1+v_nM)d(z_n,T_i^n z_n) + (1+v_nM)^2\alpha_n d(z_n,T_i^n z_n) + (2+v_nM)\mu_n \\ &\to 0 \quad \text{as} \quad n\to\infty, \end{aligned} \tag{4.17}$$

respectively. From (4.13), (4.16) and (4.17), we get

$$\begin{aligned} d(x_n,x_{n+1}) &= d(x_n,T_i^n y_n) \\ &\leq d(x_n,T_i^n x_n) + d(T_i^n x_n,T_i^n z_n) + d(T_i^n z_n,T_i^n y_n) \\ &\to 0 \text{ as } n\to\infty. \end{aligned} \tag{4.18}$$

Since $\{T_i\}_{i=1}^N$ is a finite family of uniformly L-Lipschitzian, we obtain

$$\begin{aligned} d(x_n,Tx_n) &\leq d(x_n,x_{n+1}) + d(x_{n+1},T_i^{n+1}x_{n+1}) + d(T_i^{n+1}x_{n+1},T_i^{n+1}x_n) + d(T_i^{n+1}x_n,T_ix_n) \\ &\leq (1+L)d(x_n,x_{n+1}) + d(x_{n+1},T_i^{n+1}x_{n+1}) + Ld(T_i^n x_n,x_n). \end{aligned}$$

Hence, (4.13) and (4.18) imply that

$$\lim_{n\to\infty} d(x_n,T_ix_n) = 0 \quad \text{for each } i = 1,2,...,N. \tag{4.19}$$

The rest of proof follows the pattern of Theorem 3.4 in [6].
(b) The necessity of the conditions is obvious. Thus, we only prove the sufficiency. It follows from (4.6) that $\lim_{n\to\infty} d(x_n,F)$ exists. Moreover, $\liminf_{n\to\infty} d(x_n,F) = 0$ or $\limsup_{n\to\infty} d(x_n,F) = 0$ implies that $\lim_{n\to\infty} d(x_n,F) = 0$. The rest of the proof is similar to Theorem 4 in [26] and therefore is omitted. □

By using the concept of semi-compactness which is defined in [18] and the condition (A) which is introduced by Khan et al. [9], we prove the following strong convergence theorem.

Theorem 4.3. *Under the assumptions of Theorem 4.2, if one of the mappings in the family $\{T_i\}_{i=1}^N$ is semi-compact or the family $\{T_i\}_{i=1}^N$ satisfies the condition (A), then the sequence $\{x_n\}_{n=1}^\infty$ converges to a point in F strongly.*

Proof. First, we assume that the mapping T_k in the family $\{T_i\}_{i=1}^N$ is semi-compact. By (4.19) and semi-compactness of T_k, there exists a subsequence $\{x_{n_k}\}_{k=1}^\infty \subset \{x_n\}_{n=1}^\infty$ such that $\{x_{n_k}\}_{k=1}^\infty$ converges to some point $p \in C$ strongly. Moreover, by the uniform continuity of $\{T_i\}_{i=1}^N$, we have

$$d(p,T_ip) = \lim_{k\to\infty} d(x_{n_k},T_ix_{n_k}) = 0 \text{ for each } i = 1,2,...,N.$$

This satisfies that $p \in F$. It follows from (4.6) that $\lim_{n\to\infty} d(x_n,p)$ exists and hence $\lim_{n\to\infty} d(x_n,p) = 0$. As a result, $\{x_n\}_{n=1}^\infty$ converges strongly to a point p in F.
Second, we can suppose that the family $\{T_i\}_{i=1}^N$ satisfies the condition (A). Then we have that

$$\max\{d(x,T_ix) : i = 1,2,...N\} \geq f(d(x,F)) \quad \text{for all} \quad x \in C \tag{4.20}$$

holds. Thus, from (4.19) and (4.20), we obtain $\lim_{n\to\infty} f(d(x_n,F)) = 0$. Since f is a non-decreasing mapping with $f(0) = 0$ and $f(r) > 0\ \forall r > 0$, we have $\lim_{n\to\infty} d(x_n,F) = 0$. The conclusion now can be seen from Theorem 4.2. □

Remark 4.4. *Theorems 4.2, 4.3 generalize the results of Izhar-ud-din et al. [5] in two ways: (i) from a total asymptotically nonexpansive mapping to a finite family of total asymptotically nonexpansive mappings, (ii) from a CAT(0) space to a uniformly convex hyperbolic space.*

5. Conclusion

In the above sections, we have modified the J-iterative scheme into the hyperbolic space and established the weak w^2-stability, data dependence results for contraction mappings and derived some convergence results for generalized α-nonexpansive mappings using this iterative scheme. Also, we have extended the J-iterative scheme for a finite family of total asymptotically nonexpansive mappings in hyperbolic space and have presented some convergence theorems of this iterative scheme.

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