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APPROXIMATION PROPERTIES OF THE FRACTIONAL *q*-INTEGRAL OF RIEMANN-LIOUVILLE INTEGRAL TYPE SZÁSZ-MIRAKYAN-KANTOROVICH OPERATORS

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Abstract. In the present paper, we introduce the fractional q-integral of Riemann-Liouville integral type Szász-Mirakyan-Kantorovich operators. Korovkin-type approximation theorem is given and the order of convergence of these operators are obtained by using Lipschitz-type maximal functions, second order modulus of smoothness and Peetre's K-functional. Weighted approximation properties of these operators in terms of modulus of continuity have been investigated. Then, for these operators, we give a Voronovskaya-type theorem. Moreover, bivariate fractional q- integral Riemann-Liouville fractional integral type Szász-Mirakyan-Kantorovich operators are constructed.The last section is devoted to detailed graphical representation and error estimation results for these operators.

1. Introduction

Approximation theory is a subject that serves as an important bridge between applied and pure mathematics. The approximation of functions by positive linear operators is an important research area in general mathematics. Especially, it plays an important role in mathematical analysis problems and in many fields of science. One of its most important advantages is that it provides powerful tools for application areas such as computer aided geometric design and numerical analysis. One of the best known of these operators is the Szász - Mirakyan operator (see [9] and [10]), which is generalizations of Bernstein polynomials to the infinite interval and defined as

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$$S_n(f;x)=\sum_{k=0}^{n}s_{n,k}(x)\ f\left(\frac{k}{n}\right),$$

where $n\in\mathbb{N}$, $x\in[0,\infty)$ and $s_{n,k}(x)=e^{-nx}\frac{(nx)^k}{k!}$. In literature, there are a lot of studies that involve Szàsz operators, Szàsz-Kantorovich operators and their generalizations. For instance, see [1]- [8] and [14]- [22]. Due to the rapid development

of the q-calculus, various generalizations of Szàsz Mirakyan operators involving q-integers have been introduced and approximation properties have been studied. Several researchers introduced and studied different generalizations of the q-Szász-Mirakjan operators in recent years ([28], [29], [19], [30], [41]). In [28], Mahmudov introduced and studied the following q-Szász-Mirakjan operators.

$$S_{n,q}(f;x)=\sum_{k=0}^{n}s_{n,k}(q;x)\ f\left(\frac{[k]_q}{q^{k-2}[n]_q}\right),$$

where $s_{n,k}(q;x)=\frac{1}{E_q\left([n]_q x\right)}q^{\frac{k(k-1)}{2}}\frac{[n]_q^k x^k}{[k]_q!}$.

About contributions on Kantorovich type modication modified many times q-Szász-Mirakjan operators, so we refer to the papers [31]- [34]. Recently, Fractional calculus and its applications have been paid more and more attention. fractional calculus deals with the study of fractional degree derivative and integral operators on complex or real fields and their applications (see [23]- [27]). Mahmudov and Kara, introduced and discussed the fractional integral of Riemann-Liouville integral type Szász Mirakyan-Kantorovich operators as follows:

$$K_n^{(\alpha)}(f;x)=\sum_{k=0}^{\infty}\alpha s_{n,k}(x)\int_0^1\frac{f\left(\frac{k+t}{n}\right)}{(1-t)^{1-\alpha}}dt, \tag{1}$$

where $s_{n,k}(x)=e^{-nx}\frac{(nx)^k}{k!}$.The aim of the present paper is to construct the frac-

tional q-integral of Riemann-Liouville type Szász-Mirakyan-Kantorovich operators and discuss their approximation properties. The fractional q-integral of Riemann-Liouville type ([35]) is given by $\left(I_q^0 f\right)(t)=f(t)$ and

$$(I_q^{\alpha}f)(x)=\frac{1}{\Gamma_q(\alpha)}\int_0^x\frac{f(t)}{(x-qt)^{(1-\alpha)}}d_qt \qquad (\ \alpha>0).$$

We start by reminding the basic concepts and notations about fractional q-calculus.

2. Preliminaries

For $q \in (0,1)$,

$$[m]_q = \frac{1-q^m}{1-q}, \quad m \in \mathbb{R}$$

The q-analog of the power function $(n-m)^{(k)}$ with $k \in \mathbb{N}_0 := \{0,1,2,...\}$ is

$$(n-m)^{(0)} = 1, \qquad (n-m)^{(k)} = \prod_{i=0}^{k-1} \left(n - mq^i\right), \qquad k \in \mathbb{N}, n, m \in \mathbb{R}.$$

More generally, if $\gamma \in \mathbb{R}$, then

$$(n-m)^{(\gamma)} = \prod_{i=0}^{\infty} \frac{n - mq^i}{n - mq^{\gamma+i}}, \quad n \neq 0.$$

Note if $m = 0$, then $(n)^{(\gamma)} = n^{\gamma}$. We also use the natation $0^{(\gamma)} = 0$ for $\gamma > 0$. The q-gamma function is defined by

$$\Gamma_q(t) = \frac{(1-q)^{(t-1)}}{(1-q)^{t-1}}, \quad t \in \mathbb{R} \setminus \{0, -1, -2, ...\}.$$

Obviously, $\Gamma_q(t+1) = [t]_q \Gamma_q(t)$.

For any $s, t > 0$, the q-beta function is defined by

$$B_q(s,t) = \int_0^1 u^{(s-1)} (1-qu)^{(t-1)} d_q u.$$

The q-beta function can be expressed by using the q-gamma function as follows:

$$B_q(s,t) = \frac{\Gamma_q(s)\Gamma_q(t)}{\Gamma_q(s+t)}.$$

The q-integral definition of the function h on the interval $[0,b]$ is given as:

$$(I_q h)(t) = \int_0^t h(s) d_q s = t(1-q) \sum_{i=0}^{\infty} h(tq^i) q^i, \quad t \in [0,b].$$

In q-calculus (see [36]) the following functions are well known as analogues of the exponential function:

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}, \qquad |x| < \frac{1}{1-q}, \quad |q| < 1,$$

and

$$E_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!} q^{\frac{k(k-1)}{2}}, \quad |q| < 1.$$

3. Riemann-Liouville Type Szász-Mirakyan-Kantorovich Operators

Lemma 1. *([28]) Let $0<q<1$. we have*

$$S_{n,q}(t^4;x)=\frac{x^4}{q^2}+\left(3q+2+\frac{1}{q}\right)\frac{x^3}{[n]_q}+\left(3q^3+3q^2+q\right)\frac{x^2}{[n]_q^2}+\frac{q^4x}{[n]_q^3}.$$

Definition 1. *Let $q\in(0,1)$ and $\alpha>0$. For $f\in C[0,\infty)$, Fractional q-integral of Riemann-Liouville type Szász-Mirakyan-Kantorovich operators can be defined by*

$$K_{n,q}^{(\alpha)}(f;x)=\sum_{k=0}^{\infty}[\alpha]_q\, s_{n,k}(q;x)\int_0^1 f\left(\frac{q^{1-k}[k]_q+t}{[n]_q}\right)(1-qt)^{(\alpha-1)}\,d_qt, \tag{2}$$

where $s_{n,k}(q;x)=\frac{1}{E_q\left([n]_qx\right)}q^{\frac{k(k-1)}{2}}\frac{[n]_q^kx^k}{[k]_q!}$ and $n\in\mathbb{N}$

if $\alpha=1$ and $q=1$, then the operator (2) reduces to classical Szász-Mirakyan - Kantorovich operators.

Due to the moments of the $K_{n,q}^{(\alpha)}$ operators plays significant role in our main results, we derive the following formula to obtain them.

Lemma 2. *Let $q\in(0,1)$ and $\alpha>0$. Then for $x\in[0,\infty)$, we have*

$$K_{n,q}^{(\alpha)}(t^m;x)=\sum_{j=0}^{m}\binom{m}{j}\frac{[\alpha]_q\,[n]_q^j\,B_q(m-j+1,\alpha)}{q^j\,[n]_q^m}S_{n,q}(t^j;x), \tag{3}$$

where

$$S_{n,q}(f;x)=\sum_{k=0}^{n}s_{n,k}(q;x)\ f\left(\frac{[k]_q}{q^{(k-2)}\,[n]_q}\right)$$

and

$$B_q\left(a,b\right)=\int_0^1 x^{a-1}\left(1-qx\right)^{b-1}d_qx,\qquad a,b>0.$$

Proof. From (2), we can write

$$\begin{aligned}K_{n,q}^{(\alpha)}(t^m;x)&=\sum_{k=0}^{\infty}[\alpha]_q\, s_{n,k}(q;x)\int_0^1\left(\frac{q^{1-k}[k]_q+t}{[n]_q}\right)^m(1-qt)^{(\alpha-1)}\,d_qt\\&=\sum_{k=0}^{\infty}[\alpha]_q\, s_{n,k}(q;x)\sum_{j=0}^{m}\binom{m}{j}\frac{q^{(1-k)j}\,[k]_q^j}{[n]_q^m}\int_0^1 t^{(m-j)}\left(1-qt\right)^{(\alpha-1)}d_qt\end{aligned}$$

$$= \sum_{j=0}^{m} \binom{m}{j} \frac{[\alpha]_q [n]_q^j B_q(m-j+1,\alpha)}{q^j [n]_q^m} \sum_{k=0}^{\infty} s_{n,k}(q;x) \frac{[k]_q^j}{q^{(k-2)j} [n]_q^j}$$

$$= \sum_{j=0}^{m} \binom{m}{j} \frac{[\alpha]_q [n]_q^j B_q(m-j+1,\alpha)}{q^j [n]_q^m} S_{n,q}(t^j;x).$$

□

For $j = 0,1,2,3,4$ $(K_{n,q}^{\alpha}(t^j;x))$,the following can be written immediately.

Lemma 3. *Let $q \in (0,1)$,$\alpha > 0$ and $n \in \mathbb{N}$. Then for $x \in [0,\infty)$, we have*

$$(i)\ K_{n,q}^{(\alpha)}(1;x) = 1,$$

$$(ii)\ K_{n,q}^{(\alpha)}(t;x) = x + \frac{1}{[n]_q [\alpha+1]_q}$$

$$(iii)\ K_{n,q}^{(\alpha)}(t^2;x) = \frac{[2]_q}{[\alpha+1]_q [\alpha+2]_q [n]_q^2} + \frac{\left(2+[\alpha+1]_q\right)}{[\alpha+1]_q [n]_q} x + \frac{x^2}{q},$$

$$(iv) K_{n,q}^{(\alpha)}(t^3;x) = \frac{[3]_q [2]_q}{[\alpha+1]_q [\alpha+2]_q [\alpha+3]_q [n]_q^3} + \left(\frac{3.[2]_q}{[\alpha+1]_q [\alpha+2]_q [n]_q^2} + \frac{3}{[\alpha+1]_q [n]_q^2} + 1 \right) x + \left(\frac{3}{q [n]_q [\alpha+1]_q} + \frac{2q^2+q}{q^3 [n]_q} \right) x^2 + \frac{x^3}{q^3},$$

$$(v) K_{n,q}^{(\alpha)}(t^4;x) = \frac{[4]_q!}{[\alpha+1]_q [\alpha+2]_q [\alpha+3]_q [\alpha+4]_q [n]_q^4} + \left(\frac{4[3]_q! + 6[\alpha+3]_q [2]_q! + [\alpha+2]_q [\alpha+3]_q \left\{4+[\alpha+1]_q\right\}}{[\alpha+1]_q [\alpha+2]_q [\alpha+3]_q [n]_q^3} \right) x + \left(\frac{6[2]_q!}{q[\alpha+1]_q [\alpha+2]_q [n]_q^2} + \frac{4\left(2q^2+q\right)}{q^3 [\alpha+1]_q [n]_q^2} + \frac{\left(3q^3+3q^2+q\right)}{q^4 [n]_q^2} \right) x^2 + \left(\frac{4}{q^3 [n]_q [\alpha+1]_q} + \frac{3q+2+\frac{1}{q}}{q^4 [n]_q} \right) x^3 + \frac{x^4}{q^6}.$$

Proof. Since they have the same proof technique, we only give for $K_{n,q}^{(\alpha)}(t^2;x)$. Using recurrence formula(3) and Lemma 1, we get

$$K_{n,q}^{(\alpha)}(t^2;x) = \frac{[\alpha]_q B_q(3,\alpha)}{[n]_q^2} S_{n,q}(1;x) + \frac{2[n]_q [\alpha]_q B_q(2,\alpha)}{q[n]_q^2} S_{n,q}(t;x)$$

$$+\frac{[n]_q^2\,[\alpha]_q\,B_q(1,\alpha)}{q^2\,[n]_q^2}S_{n,q}(t^2;x)$$
$$=\frac{[2]_q}{[\alpha+1]_q\,[\alpha+2]_q\,[n]_q^2}+\frac{2}{[\alpha+1]_q\,[n]_q}x+\left(\frac{x^2}{q}+\frac{x}{[n]_q}\right)$$
$$=\frac{[2]_q}{[\alpha+1]_q\,[\alpha+2]_q\,[n]_q^2}+\frac{\left(2+[\alpha+1]_q\right)}{[\alpha+1]_q\,[n]_q}x+\frac{x^2}{q}.$$

□

We are now ready to present the central moments of the operators $K_{n,q}^{(\alpha)}$.

Lemma 4. *Let $q\in(0,1)$ and $\alpha>0$. For every $x\in[0,\infty)$, there holds*

$$K_{n,q}^{(\alpha)}(t-x;x)=\frac{1}{[n]_q\,[\alpha+1]_q},$$
$$K_{n,q}^{(\alpha)}((t-x)^2;x)=\frac{[2]_q}{[\alpha+1]_q\,[\alpha+2]_q\,[n]_q^2}+\frac{x}{[n]_q}+x^2\left(\frac{1}{q}-1\right),$$
$$K_{n,q}^{(\alpha)}((t-x)^4;x)$$
$$=\frac{[4]_q!}{[\alpha+1]_q\,[\alpha+2]_q\,[\alpha+3]_q\,[\alpha+4]_q\,[n]_q^4}$$
$$+\left(\frac{6\,[2]_q}{[\alpha+1]_q\,[\alpha+2]_q\,[n]_q^3}+\frac{4}{[\alpha+1]_q\,[n]_q^3}+\frac{1}{[n]_q^3}\right)x$$
$$+\left(\begin{array}{c}\frac{6[2]_q}{q[\alpha+1]_q[\alpha+2]_q[n]_q^2}+\frac{4\left(2q^2+q\right)}{[\alpha+1]_q q^3[n]_q^2}\\+\frac{3q^3+3q^2+q}{q^4[n]_q^2}-\frac{6[2]_q-12[\alpha+2]_q}{[\alpha+1]_q[\alpha+2]_q[n]_q^2}-4\end{array}\right)x^2$$
$$+\left(\begin{array}{c}\frac{4}{q^3[\alpha+1]_q[n]_q}+\frac{3q+2+\frac{1}{q}}{q^4[n]_q}-\frac{12}{q[\alpha+1]_q[n]_q}\\-\frac{4\left(2q^2+q\right)}{q^3[n]_q}+\frac{6\left(2+[\alpha+1]_q\right)-4}{[\alpha+1]_q[n]_q}\end{array}\right)x^3$$
$$+\left(\frac{1}{q^6}-\frac{4}{q^3}+\frac{6}{q}-3\right)x^4.$$

Proof. Since they have the same proof technique, we only give for $K_{n,q}^{(\alpha)}((t-x)^2;x)$. From the linearity property of $K_{n,q}^{(\alpha)}(t;x)$and Lemma 3, we get

$$K_{n,q}^{(\alpha)}((t-x)^2;x)$$
$$=\frac{[2]_q}{[\alpha+1]_q\,[\alpha+2]_q\,[n]_q^2}+\frac{\left(2+[\alpha+1]_q\right)}{[\alpha+1]_q\,[n]_q}x+\frac{x^2}{q}-2x\left(x+\frac{1}{[n]_q\,[\alpha+1]_q}\right)+x^2.$$

□

Lemma 5. *Assume that the sequence (q_n) satisfy $0 < q_n \leq 1$ such that $q_n \to 1$ and $q_n^n \to b \in [0,1]$ as $n \to \infty$. For every $\alpha > 0$ and $x \in [0,\infty)$, there holds*

$$\lim_{n\to\infty} [n]_{q_n} K_{n,k}^{(\alpha)}((t-x);x) = \frac{1}{\alpha+1}, \tag{4}$$

$$\lim_{n\to\infty} [n]_{q_n} K_{n,k}^{(\alpha)}((t-x)^2;x) = x + x^2(1-b), \tag{5}$$

and

$$\lim_{n\to\infty} [n]_{q_n} K_{n,k}^{(\alpha)}((t-x)^4;x) = 0. \tag{6}$$

Proof. Using explicit formula for moments (Lemma 4), we obtain as

$$\lim_{n\to\infty} [n]_{q_n} K_{n,k}^{(\alpha)}((t-x);x) = \frac{1}{(\alpha+1)},$$

$$\begin{aligned} & \lim_{n\to\infty} [n]_{q_n} K_{n,k}^{(\alpha)}((t-x)^2;x) \\ = & \lim_{n\to\infty} [n]_{q_n} \left(\frac{[2]_{q_n}}{[\alpha+1]_{q_n}[\alpha+2]_{q_n}[n]_{q_n}^2} + \frac{x}{[n]_{q_n}} + x^2\left(\frac{1}{q_n}-1\right)\right) \\ = & x + x^2(1-b) \end{aligned}$$

and

$$\lim_{n\to\infty} [n]_{q_n} K_{n,k}^{(\alpha)}((t-x)^4;x) = 0.$$

□

In [28], Mahmudov gave the following formula for the moments of $S_{n,q}(t^m;x)$, which is a q-analogue of result of Beker [37].

Lemma 6. *[29]For $0 < q < 1$ and $m \in \mathbb{N}$, there holds*

$$S_{n,q}(t^m;x) = \sum_{j=1}^{m} a_{m,j}(q) \frac{x^j}{[n]_q^{m-j}} \tag{7}$$

where

$$a_{m+1,j}(q) = \frac{[j]_q\, a_{m,j}(q) + a_{m,j-1}(q)}{q^{j-2}}, \qquad m \geq 0, j \geq 1,$$
$$a_{0,0}(q) = 1, a_{m,0}(q) = 0, \quad m > 0, \quad a_{m,j}(q) = 0, \qquad m < j.$$

In particular $S_{n,q}(t^m;x)$ is a polynomial of degree m without a constant term.

Now we additionally need to give the following definitions for our main results:

1. $B_m[0,\infty) = \{f : [0,\infty) \to \mathbb{R}; |f(x)| \leq M_f(1+x^m)\}$,where M_f is constant depending on the function f.
2. $C_m[0,\infty) = B_m[0,\infty) \cap C[0,\infty)$.

3. $C_m^\star\left[0,\infty\right)=\left\{f:C_m\left[0,\infty\right):\lim\limits_{|x|\to\infty}\frac{|f(x)|}{1+x^m}<\infty\right\}$

The norm on the space $C_m^\star\left[0,\infty\right)$ is showed as $\|f(x)\|_m=\sup\limits_{x\in[0,\infty)}\frac{|f(x)|}{1+x^m}$.

Lemma 7. *Let $m\in\mathbb{N}\cup\{0\}$, $0<q<1$ and $\alpha>0$ be fixed. Then, we have*

$$\left\|K_{n,q}^{(\alpha)}(1+x^m;x)\right\|_m\leq C_{m,j}(q,\alpha),\ n\in\mathbb{N}, \tag{8}$$

where $C_{m,j}(q,\alpha)$ is a positive constant. Morever, we have

$$\left\|K_{n,q}^{(\alpha)}(f;x)\right\|_m\leq C_{m,j}(q,\alpha)\left\|f\right\|_m,\quad n\in\mathbb{N}, \tag{9}$$

where $f\in C_m^*\left[0,\infty\right)$. Thus, for any $m\in\mathbb{N}\cup\{0\}$, $K_{n,q}^{(\alpha)}:C_m^*\left[0,\infty\right)\to C_m^*\left[0,\infty\right)$ is a linear positive operator.

Proof. For $m=0$, inequality (8) is obvious.

For $m\geq1$, combining Lemma (3) and inequality (7), we obtain as

$$\begin{aligned}
&\frac{1}{x^m+1}K_{n,q}^{(\alpha)}(1+t^m;x)\\
&=\frac{1}{x^m+1}+\frac{1}{x^m+1}K_{n,q}^{(\alpha)}(t^m;x)\\
&\frac{1}{x^m+1}+\frac{1}{x^m+1}\sum_{j=0}^{m}\frac{[\alpha]_q\,[n]_q^j\,B_q(m-j+1,\alpha)}{q^j\,[n]_q^m}\sum_{j_0=1}^{j}a_{j,j_0}(q)\frac{x^{j_0}}{n^{j-j_0}}\\
&\leq1+k_{m,j}(q,\alpha)=C_{m,j}(q,\alpha).
\end{aligned}$$

$C_{m,j}(q,\alpha)$ is a positive constant with depend on q,m,j and α. Moreover,

$$\left\|K_{n,q}^{(\alpha)}(f;x)\right\|_m\leq\|f\|_m\left\|K_{n,q}^{(\alpha)}(1+t^m;x)\right\|_m \tag{10}$$

for every $f\in C_m^*\left[0,\infty\right)$. Therefore, from (8), we get

$$\left\|K_{n,q}^{(\alpha)}(f;x)\right\|_m\leq C_{m,j}(q,\alpha)\left\|f\right\|_m.$$

□

4. Direct Results

Let $C_B\left[0,\infty\right)$ denote the space of all real-valued continuous and bounded functions f on $[0,\infty)$. The norm on the space $C_B\left[0,\infty\right)$ is showed as

$$\|f\|_{C_B[0,\infty)}=\sup_{x\in[0,\infty)}|f(x)|.$$

Then, the modulus of continuity of $f\in C_B\left[0,\infty\right)$ is given by

$$w(f,\delta)=\sup_{0<h\leq\delta}\ \sup_{x\in[0,\infty)}|f(x+h)-f(x)|.$$

Further, Peetre's K-functional is defined by

$$K_2(f;\delta)=\inf_{g\in w^2}\left\{\|f-g\|+\delta\left\|g^{''}\right\|\right\}\qquad \delta>0,$$

where $w^2:=\left\{g\in C_B\,[0,\infty):g^{'},g^{''}\in C_B\,[0,\infty)\right\}$. By Theorem 2.4 in [11], there exists an absolute constant $L>0$ such that

$$K_2(f;\delta)\leq L\omega_2\left(f;\sqrt{\delta}\right). \tag{11}$$

where $\delta>0$ are absolute constant.

Here, $\omega_2\,(f;\delta)$ is the second order modulus of smoothness of $f\in C_B\,[0,\infty)$ and defined as

$$\omega_2\,(f;\delta)=\sup_{0<h\leq\delta}\ \sup_{x\in[0,\infty)}|f(x+2h)-2f(x+h)+f(x)|$$

Lemma 8. *Let $f\in C_B\,[0,\infty)$, $0<q<1$ and $\alpha>0$. Consider the operators*

$$^{*}K_{n,q}^{(\alpha)}(f;x)=K_{n,q}^{(\alpha)}(f;x)+f(x)-f\left(x+\frac{1}{[n]_q\,[\alpha+1]_q}\right). \tag{12}$$

Then, for all $g\in w^2$, we have

$$\left|^{*}K_{n,q}^{(\alpha)}(g;x)-g(x)\right| \tag{13}$$
$$\leq\left(\frac{[2]_q}{[\alpha+1]_q\,[\alpha+2]_q\,[n]_q^2}+\frac{x}{[n]}+x^2\left(\frac{1}{q}-1\right)+\left(\frac{1}{[n]_q\,[\alpha+1]_q}\right)^2\right)\left\|g^{''}\right\|.$$

Proof. From (12) we have

$$\begin{aligned}^{*}K_{n,q}^{(\alpha)}((t-x)\,;x)&=K_{n,q}^{(\alpha)}((t-x)\,;x)-\left(x+\frac{1}{[n]_q\,[\alpha+1]_q}-x\right)\\&=K_{n,q}^{(\alpha)}(t;x)-xK_{n,q}^{(\alpha)}(1;x)-\left(x+\frac{1}{[n]_q\,[\alpha+1]_q}\right)+x=0.\end{aligned} \tag{14}$$

Let $x\in[0,\infty)$ and $g\in w^2$. Using the Taylor's formula,

$$g(t)-g(x)=(t-x)g^{'}(x)+\int_x^t(t-u)g^{''}(u)du, \tag{15}$$

Applying $^{*}K_{n,q}^{(\alpha)}$ and using (14), we can get

$$^{*}K_{n,q}^{(\alpha)}(g;x)-g(x)={}^{*}K_{n,q}^{(\alpha)}\left((t-x)g^{'}(x);x\right)+{}^{*}K_{n,q}^{(\alpha)}\left(\int_x^t(t-u)g^{''}(u)du;x\right)$$

$$= g^{'}(x)^{*}K_{n,q}^{(\alpha)}\left((t-x);x\right) + K_{n,q}^{(\alpha)}\left(\int_{x}^{t}(t-u)g^{''}(u)du;x\right)$$

$$- \int_{x}^{x+\frac{1}{[n]_q[\alpha+1]_q}}\left(x+\frac{1}{[n]_q\,[\alpha+1]_q}-u\right)g^{''}(u)du$$

$$= K_{n,q}^{(\alpha)}\left(\int_{x}^{t}(t-u)g^{''}(u)du;x\right)$$

$$- \int_{x}^{x+\frac{1}{[n]_q[\alpha+1]_q}}\left(x+\frac{1}{[n]_q\,[\alpha+1]_q}-u\right)g^{''}(u)du.$$

On the other hand, since

$$\int_{x}^{t}|t-u|\left|g^{''}(u)\right|du \le \left\|g^{''}\right\|\int_{x}^{t}|t-u|\,du \le (t-x)^2\left\|g^{''}\right\|$$

and

$$\left|\int_{x}^{x+\frac{1}{[n]_q[\alpha+1]_q}}\left(x+\frac{1}{[n]_q\,[\alpha+1]_q}-u\right)g^{''}(u)du\right|$$

$$\le \left(\frac{1}{[n]_q\,[\alpha+1]_q}\right)^2\left\|g^{''}\right\|,$$

we conclude that

$$\left|{}^{*}K_{n,q}^{(\alpha)}(g;x)-g(x)\right|$$

$$= \left|K_{n,q}^{(\alpha)}\left(\int_{x}^{t}(t-u)g^{''}(u)du;x\right) - \int_{x}^{x+\frac{1}{[n]_q[\alpha+1]_q}}\left(x+\frac{1}{[n]_q\,[\alpha+1]_q}-u\right)g^{''}(u)du\right|$$

$$\le \left\|g^{''}\right\|K_{n,q}^{(\alpha)}\left((t-x)^2;x\right) + \left(\frac{1}{[n]_q\,[\alpha+1]_q}\right)^2\left\|g^{''}\right\|.$$

Finally, from Lemma 4, we can write

$$\left|{}^{*}K_{n,q}^{(\alpha)}(g;x)-g(x)\right|$$

$$\leq \left(\frac{[2]_q}{[\alpha+1]_q [\alpha+2]_q [n]_q^2} + \frac{x}{[n]_q} + x^2\left(\frac{1}{q}-1\right) + \left(\frac{1}{[n]_q [\alpha+1]_q}\right)^2 \right) \left\| g^{''} \right\|.$$

□

Theorem 1. *Let $f \in C_B[0,\infty)$, $0 < q < 1$ and $\alpha > 0$. Then, for every $x \in [0,\infty)$, there exists a constant $M > 0$ such that*

$$\left| K_{n,q}^{(\alpha)}(f;x) - f(x) \right| \leq M\omega_2\left(f; \sqrt{\delta_n^{(\alpha)}(x)}\right) + \omega\left(f; \beta_n^{(\alpha)}(x)\right)$$

where

$$\delta_n^{(\alpha)}(x) = \left(\frac{[2]_q}{[\alpha+1]_q [\alpha+2]_q [n]_q^2} + \frac{x}{[n]_q} + x^2\left(\frac{1}{q}-1\right) + \left(\frac{1}{[n]_q [\alpha+1]_q}\right)^2 \right) \left\| g^{''} \right\|$$

and

$$\beta_n^{(\alpha)}(x) = \left| \frac{1}{[n]_q [\alpha+1]_q} \right|.$$

Proof. It follows from Lemma (8), that

$$\begin{aligned}
&\left| K_{n,q}^{(\alpha)}(f;x) - f(x) \right| \\
&\leq \left| {}^*K_{n,q}^{(\alpha)}(f;x) - f(x) \right| + \left| f(x) - f\left(x + \frac{1}{[n]_q [\alpha+1]_q}\right) \right| \\
&\leq \left| {}^*K_{n,q}^{(\alpha)}(f-g;x) - (f-g)(x) \right| \\
&+ \left| f(x) - f\left(x + \frac{1}{[n]_q [\alpha+1]_q}\right) \right| + \left| {}^*K_{n,q}^{(\alpha)}(g;x) - g(x) \right| \\
&\leq \left| {}^*K_{n,q}^{(\alpha)}(f-g;x) \right| + |(f-g)(x)| \\
&+ \left| f(x) - f\left(x + \frac{1}{[n]_q [\alpha+1]_q}\right) \right| + \left| {}^*K_{n,q}^{(\alpha)}(g;x) - g(x) \right|.
\end{aligned}$$

Since boundedness of the ${}^*K_{n,q}^{(\alpha)}$ and using inequality (13), we get

$$\begin{aligned}
&\left| K_{n,q}^{(\alpha)}(f;x) - f(x) \right| \\
&\leq 4\|f-g\| + \left| f(x) - f\left(x + \frac{1}{[n]_q [\alpha+1]_q}\right) \right| \\
&+ \left(\frac{[2]_q}{[\alpha+1]_q [\alpha+2]_q [n]_q^2} + \frac{x}{[n]_q} + x^2\left(\frac{1}{q}-1\right) + \left(\frac{1}{[n]_q [\alpha+1]_q}\right)^2 \right) \left\| g^{''} \right\|
\end{aligned}$$

$$\leq 4\|f-g\| + \omega\left(f; \left|\frac{1}{[n]_q [\alpha+1]_q}\right|\right) + \delta_n^{(\alpha)}(x) \left\|g^{''}\right\|.$$

Now, taking infimum on the right hand side over all $g \in w^2$ and using the property of Peetre's K-functional(11), we can get

$$\begin{aligned}\left|K_{n,q}^{(\alpha)}(f;x) - f(x)\right| &\leq 4K_2\left(f; \delta_n^{(\alpha)}(x)\right) + \omega\left(f; \beta_n^{(\alpha)}(x)\right) \\ &\leq M\omega_2\left(f; \sqrt{\delta_n^{(\alpha)}(x)}\right) + \omega\left(f; \beta_n^{(\alpha)}(x)\right).\end{aligned}$$

□

Corollary 1. *Let $0 < q_n < 1$, $\alpha > 0$. For any $A > 0$ and $f \in C_B[0,\infty)$, then $K_{n,q_n}^{(\alpha)}(f;x)$ converges to uniformly f on $[0,A]$ if and only if $q_n \to 1$ as $n \to \infty$.*

Theorem 2. *Let $K_{n,q}^{(\alpha)}$ be the operators defined by (2), $0 < q < 1$, $\alpha > 0$, $\rho \in (0,1]$ and D be any subset of the interval $[0,\infty)$. if $f \in C_B[0,\infty)$ is locally Lip(ρ) on D, i.e., if f satisfies the following inequality:*

$$|f(t) - f(x)| \leq C_{f,\rho} |t-x|^\rho, \qquad t \in D \text{ and } x \in [0,\infty), \tag{16}$$

then for each $x \in [0,\infty)$, we have

$$\left|K_{n,q}^{(\alpha)}(f;x) - f(x)\right| \leq C_{f,\rho}\left\{\left(K_{n,q}^{(\alpha)}\left((t-x)^2; x\right)\right)^{\frac{\rho}{2}} + 2d^\rho(x,D)\right\},$$

where $C_{f,\rho}$ is constant depending on f and ρ and $d(x,D)$ is the distance between x and D defined by

$$d(x,D) = \inf\{|t-x| : t \in D\}.$$

Proof. Let $\overline{D}$ denote the closure of D. Due to the features of infimum, there is at least a point $t_0 \in \overline{D}$ such that $d(x,D) = |x - t_0|$. By the triangle inequality

$$|f(t) - f(x)| \leq |f(t) - f(t_0)| + |f(x) - f(t_0)|.$$

Applying $K_{n,q}^{(\alpha)}$ to the above inequality and using (16), we can get

$$\begin{aligned}&\left|K_{n,q}^{(\alpha)}(f;x) - f(x)\right| \\ &\leq K_{n,q}^{(\alpha)}(|f(t) - f(t_0)|; x) + K_{n,q}^{(\alpha)}(|f(x) - f(t_0)|; x) \\ &\leq C_{f,\rho}\left\{K_{n,q}^{(\alpha)}(|t-t_0|^\rho; x) + |x-t_0|^\rho\right\} \\ &\leq C_{f,\rho}\left\{K_{n,q}^{(\alpha)}(|t-x|^\rho + |x-t_0|^\rho; x) + |x-t_0|^\rho\right\} \\ &= C_{f,\rho}\left\{K_{n,q}^{(\alpha)}(|t-x|^\rho; x) + 2|x-t_0|^\rho\right\}.\end{aligned}$$

Choosing $a_1 = \frac{2}{\rho}$ and $a_2 = \frac{2}{2-\rho}$ and applying Hölder inequality, we have:

$$\left|K_{n,q}^{(\alpha)}(f;x) - f(x)\right|$$

$$\leq C_{f,\rho}\left\{\left(K_{n,q}^{(\alpha)}\left(|t-x|^{a_1\rho};x\right)\right)^{\frac{1}{a_1}}\left(K_{n}^{(\alpha)}\left(1^{a_2};x\right)\right)^{\frac{1}{a_2}}+2d^{\rho}(x,D)\right\}$$
$$\leq C_{f,\rho}\left\{\left(K_{n,q}^{(\alpha)}\left((t-x)^2;x\right)\right)^{\frac{\rho}{2}}+2d^{\rho}(x,D)\right\}.$$

□

In [38], Lipcshitz type maximal function of the order ρ defined as

$$\phi_{\rho}(f;x)=\sup_{x,t\in[0,\infty),x\neq t}\frac{|f(t)-f(x)|}{|t-x|^{\rho}} \tag{17}$$

where $x\in[0,\infty)$ and $\rho\in(0,1]$. In the next theorem we obtain local direct estimate of the operators $K_{n,q}^{(\alpha)}$ by using (17).

Theorem 3. *Let $f\in C_B[0,\infty)$, $0<q<1$, $\alpha>0$ and $\rho\in(0,1]$. Then, for all $x\in[0,\infty)$, we have*

$$\left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right|\leq\phi_{\rho}(f;x)\left(K_{n,q}^{(\alpha)}\left((t-x)^2;x\right)\right)^{\frac{\rho}{2}}.$$

Proof. From the equation (17), we have

$$\left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right|\leq\phi_{\rho}(f;x)\,K_{n,q}^{(\alpha)}\left(|t-x|^{\rho};x\right)$$

Applying the Hölder inequality with $a_1=\frac{2}{\rho}$ and $a_2=\frac{2}{2-\rho}$, we get

$$\left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right|\leq\phi_{\rho}(f;x)\left(K_{n,q}^{(\alpha)}\left((t-x)^2;x\right)\right)^{\frac{\rho}{2}}.$$

□

Theorem 4. *For $0<q<1$, $\alpha>0$, $f\in C_2[0,\infty)$, $w_{a+1}(f;\delta)$ is the modulus of continuity of f on the interval $[0,a+1]\subset[0,\infty)$, $a>0$. Then, we have*

$$\left\|K_{n,q}^{(\alpha)}(f;x)-f(x)\right\|_{C[0,a]}\leq 4N_f\left(1+a^2\right)\delta_n(x)+2w_{a+1}\left(f;\sqrt{\delta_n(x)}\right).$$

where $\sqrt{K_{n,q}^{(\alpha)}((t-x)^2;x)}$ given by Lemma 4 and $\|f\|_{C[0,a]}=\sup_{x\in[0,a]}|f(x)|$.

Proof. For $0\leq x\leq a$ and $a+1<t$, since $1<t-x$, we have

$$\begin{aligned}|f(t)-f(x)|&\leq M_f\left(x^2+t^2+2\right)\\&\leq M_f\left(2(t-x)^2+2+3x^2\right)\\&\leq M_f(t-x)^2\left(4+3x^2\right)\\&\leq 4M_f(t-x)^2\left(1+a^2\right).\end{aligned} \tag{18}$$

Also, for $0\leq x\leq a$ and $a+1\geq t$, we have

$$|f(t)-f(x)|\leq w_{a+1}(f;|t-x|)$$

$$\leq \left(1+\frac{|t-x|}{\delta}\right) w_{a+1}(f;\delta), \tag{19}$$

with $\delta > 0$.

For $0 \leq x \leq a$ and $t \geq 0$, combining (18) and (19) gives

$$\begin{aligned} & |f(t)-f(x)| \\ \leq \quad & 4M_f(t-x)^2\left(1+a^2\right)+\left(1+\frac{|t-x|}{\delta}\right) w_{a+1}(f;\delta), \end{aligned} \tag{20}$$

Applying Cauchy-Schwarz's inequality to the above inequality(20), we get

$$\begin{aligned} & \left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right| \\ & \leq K_{n,q}^{(\alpha)}(f;x)\left(|f(t)-f(x)|;x\right) \\ & \leq 4M_f\left(1+a^2\right) K_{n,q}^{(\alpha)}\left((t-x)^2;x\right)+\left(1+\frac{\sqrt{K_{n,q}^{(\alpha)}\left((t-x)^2;x\right)}}{\delta}\right) w_{a+1}(f;\delta) \\ & \leq 4M_f\left(1+a^2\right) K_{n,q}^{(\alpha)}\left((t-x)^2;x\right)+2w_{a+1}\left(f;\delta_n(x)\right) \end{aligned}$$

on choosing $\delta := \delta_n(x) = \sqrt{K_{n,q}^{(\alpha)}\left((t-x)^2;x\right)}$. □

5. Weighted Approximation

Theorem 5. *Let $q = q_n \in (0,1]$ such that $q_n \to 1$ as $n \to \infty$ and $\alpha > 0$. Then for each $f \in C_2^*[0,\infty)$, we have:*

$$\lim_{n\to\infty}\left\|K_{n,q_n}^{(\alpha)}(f;x)-f(x)\right\|_2 = 0.$$

Proof.

- Since the Korovkin type theorem on the weighted approximation([12]), we need to verify

$$\lim_{n\to\infty}\left\|K_{n,q_n}^{(\alpha)}(t^m;x)-x^m\right\|_2 = 0, \quad m=0,1,2. \tag{21}$$

- For $m = 0$, obvious.
- For $m = 1$ and $m = 2$, using Lemma 3, we can write:

$$\begin{aligned} \lim_{n\to\infty}\left\|K_{n,q_n}^{(\alpha)}(t;x)-x\right\|_2 &= \sup_{x\geq 0}\frac{\left|K_{n,q_n}^{(\alpha)}(t;x)-x\right|}{1+x^2} \\ &= \sup_{x\geq 0}\frac{1}{1+x^2}\left|\frac{1}{[n]_{q_n}[\alpha+1]_{q_n}}\right| \\ &= \frac{1}{[n]_{q_n}[\alpha+1]_{q_n}}\sup_{x\geq 0}\frac{1}{1+x^2} \\ &\leq \frac{1}{[n]_{q_n}[\alpha+1]_{q_n}} \to 0,\ n\to\infty \end{aligned}$$

and

$$\lim_{n\to\infty}\left\|K_{n,q_n}^{(\alpha)}(t^2;x)-x^2\right\|_2$$
$$=\sup_{x\geq 0}\frac{\left|K_{n,q_n}^{(\alpha)}(t^2;x)-x^2\right|}{1+x^2}$$
$$=\sup_{x\geq 0}\frac{1}{1+x^2}\left|\frac{[2]_q}{[\alpha+1]_{q_n}[\alpha+2]_{q_n}[n]_{q_n}^2}+\frac{\left(2+[\alpha+1]_q\right)}{[\alpha+1]_{q_n}[n]_{q_n}}x+\frac{x}{q_n}^2-x^2\right|$$
$$\leq\left(\frac{1}{q_n}-1\right)\sup_{x\geq 0}\frac{x^2}{1+x^2}+\frac{\left(2+[\alpha+1]_{q_n}\right)}{[\alpha+1]_{q_n}[n]_{q_n}}\sup_{x\geq 0}\frac{x}{1+x^2}$$
$$+\frac{[2]_q}{[\alpha+1]_{q_n}[\alpha+2]_{q_n}[n]_{q_n}^2}\sup_{x\geq 0}\frac{1}{1+x^2}$$
$$\leq\left(\frac{1}{q_n}-1\right)+\frac{\left(2+[\alpha+1]_{q_n}\right)}{[\alpha+1]_{q_n}[n]_{q_n}}+\frac{[2]_q}{[\alpha+1]_{q_n}[\alpha+2]_{q_n}[n]_{q_n}^2}\to 0,\ n\to\infty,$$

which implies that

$$\lim_{n\to\infty}\left\|K_{n,q}^{(\alpha)}(t^m;x)-x^m\right\|_2=0,\quad m=0,1,2.$$

□

In the next theorem, we present a weighted approximation theorem for $f\in C_2^*[0,\infty)$, where Doğru studied for classical Szász operators in [13].

Theorem 6. *Let $q=q_n\in(0,1]$ such that $q_n\to 1$ as $n\to\infty$ and $\alpha>0$. For each $f\in C_2^*[0,\infty)$ and $\beta>0$, we have*

$$\lim_{n\to\infty}\sup_{x\geq 0}\frac{\left|K_{n,q_n}^{(\alpha)}(f;x)-f(x)\right|}{(1+x^2)^{1+\beta}}=0.$$

Proof. Let $x_0\in[0,\infty)$ be arbitrary but fixed. Then

$$\sup_{x\in[0,\infty)}\frac{\left|K_{n,q_n}^{(\alpha)}(f;x)-f(x)\right|}{(1+x^2)^{1+\beta}}$$
$$=\sup_{x\in[0,x_0]}\frac{\left|K_{n,q_n}^{(\alpha)}(f;x)-f(x)\right|}{(1+x^2)^{1+\beta}}+\sup_{x\in(x_0,\infty)}\frac{\left|K_{n,q_n}^{(\alpha)}(f;x)-f(x)\right|}{(1+x^2)^{1+\beta}}$$
$$\leq\left\|K_{n,q_n}^{(\alpha)}(f)-f\right\|_{C[0,x_0]}+\|f\|_2\sup_{x\in(x_0,\infty)}\frac{\left|K_{n,q_n}^{(\alpha)}\left(1+t^2;x\right)\right|}{(1+x^2)^{1+\beta}}$$

$$+\sup_{x\in(x_0,\infty)}\frac{|f(x)|}{(1+x^2)^{1+\beta}}$$
$$=H_1+H_2+H_3.$$

Since $|f(x)|\leq N_f(1+x^2)$, we have

$$H_3=\sup_{x\in(x_0,\infty)}\frac{|f(x)|}{\left(1+x^2\right)^{1+\beta}}\leq\sup_{x\in(x_0,\infty)}\frac{N_f}{\left(1+x^2\right)^{\beta}}\leq\frac{N_f}{\left(1+x_0^2\right)^{\beta}}.$$

Firstly, From Theorem (4), we have

$$H_1 \text{ goes to zero as } n\to\infty.$$

Secondly, by Theorem 5,

$$\begin{aligned}H_2&=\|f\|_2\lim_{n\to\infty}\sup_{x\in(x_0,\infty)}\frac{\left|K_{n,q}^{(\alpha)}\left(1+t^2;x\right)\right|}{\left(1+x^2\right)^{1+\beta}}\\&=\sup_{x\in(x_0,\infty)}\frac{\left(1+x^2\right)}{\left(1+x^2\right)^{1+\beta}}\|f\|_2\\&=\sup_{x\in(x_0,\infty)}\frac{\|f\|_2}{\left(1+x^2\right)^{\beta}}\leq\frac{\|f\|_2}{\left(1+x_0^2\right)^{\beta}}.\end{aligned}$$

Moreover, if we choose $x_0>0$ large enough, we can see that

$$H_2\to 0 \text{ and } H_3\to 0 \text{ as } n\to\infty,$$

Combining, H_1, H_2 and H_3, we get desired result. □

In the next theorem we obtain direct estimation in terms of weighted modulus of continuity. For every $f\in C_m^*[0,\infty)$ the weighted modulus of continuity defined as

$$\Omega_m(f,\delta)=\sup_{x\geq 0,\ 0<h\leq\delta}\frac{|f(x+h)-f(x)|}{1+(x+h)^m}, \tag{22}$$

Lemma 9. *[39]If $f\in C_m^*[0,\infty), m\in\mathbb{N}$, then*

(i) $\Omega_m(f,\delta)$ is a monotone increasing function of δ,
(ii) $\lim_{\delta\to 0^+}\Omega_m(f,\delta)=0$,
(iii) for any $\rho\in[0,\infty), \Omega_m(f,\rho\delta)\leq(1+\rho)\Omega_m(f,\delta)$.

In the next theorem, we express the approximation error of $K_{n,q}^{(\alpha)}$ by using Ω_m.

Theorem 7. *For $f\in C_m^*[0,\infty)$, we have*

$$\left\|K_{n,q}^{(\alpha)}(f)-f\right\|_{m+1}\leq N\Omega_m(f,(1/\sqrt{qn})),$$

where N is a constant independent of f and n.

Proof. From (22) and Lemma 9, we can write

$$|f(t)-f(x)| \leq (1+(x+|t-x|)^m)\left(\frac{|t-x|}{\delta}+1\right)\Omega_m(f,\delta)$$
$$\leq (1+(2x+t)^m)\left(\frac{|t-x|}{\delta}+1\right)\Omega_m(\varphi,\delta).$$

Then, we have

$$\begin{aligned}
&\left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right| \\
&\leq K_{n,q}^{(\alpha)}\left|(f(t)-f(x)\right|;x) \\
&\leq \Omega_m(f,\delta)\left(K_{n,q}^{(\alpha)}((1+(2x+t)^m);x)+K_{n,q}^{(\alpha)}\left((1+(2x+t)^m)\frac{|t-x|}{\delta};x\right)\right). \\
&= \Omega_m(f,\delta)\left(K_{n,q}^{(\alpha)}\left(1+(2x+t)^m;x\right)+I_1\right).
\end{aligned}$$

Applying Cauchy-Schwartz inequality to the I_1, we get

$$I_1 \leq (K_{n,q}^{(\alpha)}\left((1+(2x+t)^m)^2;x\right))^{1/2}\left(K_{n,q}^{(\alpha)}\left(\frac{|t-x|^2}{\delta^2};x\right)\right)^{1/2}.$$

Therefore,

$$\begin{aligned}
&\left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right| && (23)\\
\leq\ &\Omega_m(f,\delta)K_{n,q}^{(\alpha)}((1+(2x+t)^m);x) \\
&+\Omega_m(f,\delta)(K_{n,q}^{(\alpha)}\left((1+(2x+t)^m)^2;x\right))^{1/2}\left(K_{n,q}^{(\alpha)}\left(\frac{|t-\tau|^2}{\delta^2};x\right)\right)^{1/2}. && (24)
\end{aligned}$$

By Lemma 7 and Lemma 4,

$$K_{n,q}^{(\alpha)}(1+(2x+t)^m;x) \leq C_{m,j}(q,\alpha)\left(1+x^m\right),$$
$$(K_{n,q}^{(\alpha)}\left((1+(2x+t)^m)^2;x\right))^{1/2} \leq C_{m,j}^1(q,\alpha)\left(1+x^m\right). \qquad (25)$$

and

$$\left(K_{n,q}^{(\alpha)}\left(\frac{|t-x|^2}{\delta^2};x\right)\right)^{1/2} \leq \frac{1}{\delta}\sqrt{\frac{[2]_q}{[\alpha+1]_q\,[\alpha+2]_q\,[n]_q^2}+\frac{x}{[n]}+x^2\left(\frac{1}{q}-1\right)}$$
$$\leq \frac{(2+x)}{\delta\sqrt{qn}}. \qquad (26)$$

Combining 23, 25 and 26, we have

$$\begin{aligned}
&\left|K_{n,q}^{(\alpha)}(f;x)-f(x)\right| \\
&\leq \Omega_m(f,\delta)\left(C_{m,j}(q,\alpha)\left(1+x^m\right)+C_{m,j}^1(q,\alpha)\frac{\left(1+x^m\right)(2+x)}{\delta\sqrt{qn}}\right)
\end{aligned}$$

$$= \Omega_m(f,\delta)\left(C_{m,j}(q,\alpha)\left(1+x^m\right)+C^1_{m,j}(q,\alpha)C_1\frac{\left(1+x^{m+1}\right)}{\delta\sqrt{qn}}\right),$$

where

$$C_1=\sup_{x\geq 0}\frac{\left(2+2x^m+x+2x^{m+1}\right)}{1+x^{m+1}}.$$

if we take $\delta=\left(1/\sqrt{q\left[n\right]_q}\right)$ in the above inequality, we obtain the desired result. $\square$

Next result is a Voronovskaja type formula for the operators $K^{(\alpha)}_{n,q}(f;x)$.

6. Voronovskaja Type

Theorem 8. *Let $q=q_n\in(0,1]$ such that $q_n\to 1$, $q^n_n\to b$ as $n\to\infty$ and $\alpha>0$. For any $f\in C^*_2\left[0,\infty\right)$ such that $f^{\,\prime}$, $f^{\,\prime\prime}\in C^*_2\left[0,\infty\right)$ the following equality holds*

$$\begin{aligned}&\lim_{n\to\infty}\left[n\right]_{q_n}\left[K^{(\alpha)}_{n,q_n}(f;x)-f(x)\right]\\ =\;&\frac{1}{(\alpha+1)}f^{\,\prime}(x)+\frac{1}{2}\left(x+x^2\left(1-b\right)\right)f^{\,\prime\prime}(x).\end{aligned}$$

Proof. By the Taylor's formula, we can write

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+r(t,x)(t-x)^2 \tag{27}$$

where $r(t,x)$ is Peano form of remainder, $r(.,x)\in C^*_2\left[0,\infty\right)$ and $\lim_{t\to x}r(t,x)=0$.

Applying $K^{(\alpha)}_{n,q}$ to the both sides of (27), we get

$$\begin{aligned}&\left[n\right]_{q_n}\left[K^{(\alpha)}_{n,q_n}(f;x)-f(x)\right]\\ =\;&f^{\,\prime}(x)\left[n\right]_{q_n}K^{(\alpha)}_{n,q_n}\left((t-x);x\right)+\frac{1}{2}f^{\,\prime\prime}(x)\left[n\right]_{q_n}K^{(\alpha)}_{n,q_n}\left((t-x)^2;x\right)\\ &+\left[n\right]_{q_n}K^{(\alpha)}_{n,q_n}\left(r(t,x)(t-x)^2;x\right).\end{aligned}$$

By Cauchy-Schwarz inequality, we have

$$K^{(\alpha)}_{n,q_n}\left(r(t,x)(t-x)^2;x\right)\leq\sqrt{K^{(\alpha)}_{n,q_n}\left(r^2(t,x);x\right)}\sqrt{K^{(\alpha)}_{n,q_n}\left((t-x)^4;x\right)}. \tag{28}$$

Observe that $r^2(t,x)=0$ and $r^2(.,x)\in C^*_2\left[0,\infty\right)$.

Then, it follows from that Corollary (1),

$$\lim_{n\to\infty}\left[n\right]_{q_n}K^{(\alpha)}_{n,q_n}\left(r^2(t,x);x\right)=r^2(x,x)=0. \tag{29}$$

Moreover, from (6), (28) and (29), we can obtain

$$\lim_{n\to\infty}K^{(\alpha)}_{n,q_n}\left(r(t,x)(t-x)^2;x\right)=0 \tag{30}$$

Hence, combining (4), (5) and (30), we get

$$\lim_{n\to\infty}[n]_{q_n}\left[K_{n,q_n}^{(\alpha)}(f;x)-f(x)\right]$$
$$=\frac{1}{(\alpha+1)}f^{\,'}(x)+\frac{1}{2}\left(x+x^2(1-b)\right)f^{\,''}(x).$$

□

7. Bivariate Fractional q-Integral

In this section, we introduce the bivariate fractional q-integral of Riemann-Liouville integral type $K_{n,q}^{(\alpha)}(f;x)$ (2)as follows:

$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f;x,y)$$
$$=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}[\alpha_1]_{q_1}[\alpha_2]_{q_2}s_{n_1,k_1}(q_1;x)s_{n_2,k_2}(q_2;y)$$
$$\int_0^1\int_0^1 f\left(\frac{q_1^{1-k_1}[k_1]_{q_1}+t_1}{[n_1]_{q_1}},\frac{q_2^{1-k_2}[k_2]_{q_2}+t_2}{[n_2]_{q_2}}\right)(1-t_1)^{\alpha_1-1}(1-t_2)^{\alpha_2-1}d_{q_1}t_1d_{q_2}t_2$$

where $(x,y)\in I^2=[0,\infty)\times[0,\infty)$ and $\alpha_1,\alpha_2>0$.

Fractional q-integral of Riemann-Liouville integral type $K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(\cdot;x,y)$ can be rewritten as

$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(\cdot;x,y)=K_{n_1,q_1}^{(\alpha_1)}(\cdot;x)\times K_{n_2,q_2}^{(\alpha_2)}(\cdot;y).$$

Lemma 10. *Let $e_{ij}(x,y)=x^iy^j$, $0<q_1,q_2<1$, $0\le i+j\le 2$ and $\alpha_1,\alpha_2>0$. For $(x,y)\in I^2=[0,\infty)\times[0,\infty)$, we have*

$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{00};x,y)=1,$$
$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{10};x,y)=x+\frac{1}{[n_1]_{q_1}[\alpha_1+1]_{q_1}},$$
$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{01};x,y)=y+\frac{1}{[n_2]_{q_2}[\alpha_2+1]_{q_2}},$$
$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{20};x,y)=\frac{[2]_{q_1}}{[\alpha_1+1]_{q_1}[\alpha_1+2]_{q_1}[n_1]_{q_1}^2}+\frac{\left(2+[\alpha_1+1]_{q_1}\right)}{[\alpha_1+1]_{q_1}[n_1]_{q_1}}x+\frac{x}{q_1}^2,$$
$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{02};x,y)=\frac{[2]_{q_2}}{[\alpha_2+1]_{q_1}[\alpha_2+2]_{q_2}[n_2]_{q_2}^2}+\frac{\left(2+[\alpha_2+1]_{q_2}\right)}{[\alpha_2+1]_{q_2}[n_2]_{q_2}}y+\frac{y}{q_2}^2.$$

Remark 1. *According to above Lemma (10), we get*

$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{10}-x;x,y)=\frac{1}{[n_1]_{q_1}[\alpha_1+1]_{q_1}},$$

$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{01}-y;x,y) = \frac{1}{[n_2]_{q_2}[\alpha_2+1]_{q_2}},$$

$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}((e_{10}-x)^2;x,y) = \frac{[2]_{q_1}}{[\alpha_1+1]_{q_1}[\alpha_1+2]_{q_1}[n_1]_{q_1}^2} + \frac{x}{[n_1]_{q_1}} + x^2\left(\frac{1}{q_1}-1\right)$$
$$= \delta_{n_1}^{(\alpha_1)}(q_1;x),$$

$$K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}((e_{01}-y)^2;x,y) = \frac{[2]_{q_2}}{[\alpha_2+1]_{q_2}[\alpha_2+2]_{q_2}[n_2]_{q_2}^2} + \frac{y}{[n_2]_{q_2}} + y^2\left(\frac{1}{q_2}-1\right)$$
$$= \delta_{n_2}^{(\alpha_2)}(q_2;y).$$

In the next theorem, we obtain the uniform convergence of the bivariate q-Riemann-Liouville fractional integral type of q-Szász-Mirakyan-Kantorovich operators to the bivariate functions defined on $I^2 = [0,\infty)\times[0,\infty)$.

Theorem 9. *Let $C(I^2)$ be the space of continuous bivariate function on $I^2 = [0,\infty)\times[0,\infty)$ and $\alpha_1,\alpha_2 > 0$. Then for any $f \in C(I^2)$, we have*

$$\lim_{n_1,n_2\to\infty}\left\|K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}f - f\right\| = 0.$$

Proof. Using lemma 1, we get

$$\left\|K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}e_{00} - e_{00}\right\| = 0, \left\|K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}e_{10} - e_{10}\right\| \to 0$$
$$\left\|K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}e_{01} - e_{01}\right\| \to 0, \left\|K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(e_{20}+e_{02}) - (e_{20}+e_{02})\right\| \to 0$$
$$\text{as } n_1,n_2\to\infty$$

As a result, by Volkov's theorem [40], we get

$$\lim_{n_1,n_2\to\infty}\left\|K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}f - f\right\| = 0.$$

□

For bivariate real functions, modulus of continuity defined as

$$w(f;\delta_n,\delta_m) = \sup\left\{|f(t,s)-f(x,y)| : (t,s), x,y \in I^2, |t-x| \le \delta_n, |s-y| \le \delta_m\right\}.$$

Theorem 10. *Let $f \in C(I^2)$, $0 < q_1,q_2 < 1$and $\alpha_1,\alpha_2 > 0$. Then for all $(x,y) \in I^2$, the inequality*

$$\left|K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f;x,y) - f(x,y)\right| \le 4w\left(f;\delta_{n_1}^{(\alpha_1)}(q_1;x),\delta_{n_2}^{(\alpha_2)}(q_2;y)\right)$$

holds, where $\delta_{n_1}^{(\alpha_1)}(q_1;x), \delta_{n_2}^{(\alpha_2)}(q_2;y)$ are as in Remark 1.

Proof. By the positivity and linearity properties of the $K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}$, we can write

$$\left|K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(f;x,y) - f(x,y)\right| \le K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}(|f(t,s)-f(x,y)|;x,y)$$

$$\leq w(f;\delta_1,\delta_2)\left(K_{n_1,q_1}^{(\alpha_1)}(1;x)+\frac{1}{\delta_1}K_{n_1,q_1}^{(\alpha_2)}(|t-x|;x)\right)$$
$$\times\left(K_{n_2,q_2}^{(\alpha_2)}(1;y)+\frac{1}{\delta_2}K_{n_2,q_2}^{(\alpha_2)}(|s-y|;y)\right)$$

Applying Cauchy-Schwarz inequality, we obtain

$$K_{n_1,q_1}^{(\alpha_1)}(|t-x|;x)\leq K_{n_1,q_1}^{(\alpha_1)}\left((t-x)^2;x\right)^{\frac{1}{2}}$$
$$K_{n_2,q_2}^{(\alpha_2)}(|s-y|;y)\leq K_{n_2,q_2}^{(\alpha_2)}\left((s-y)^2;y\right)^{\frac{1}{2}}$$

Choosing $\delta_1=\delta_{n_1}^{(\alpha_1)}(q_1;x)$ and $\delta_2=\delta_{n_2}^{(\alpha_2)}(q_2;y)$, we have desired result. □

Now, we are present some graphs and numerical results for $K_{n,q}^{(\alpha)}$ and $K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}$ obtained by using Matlab.

8. Graphical Simulations

Example 1. *Consider* $f(x)=x^3-9x^2-15x+9$ *with* $x\in[0,6]$. *Here we take the value of* $q\in\{0.75,0.85,0.95\}$,*for* $K_{100,q}^{(2)}$. *The Figure 1 demonstrate the convergence of operators* $K_{100,q}^{(2)}$ *to* $f(x)$ *for increasing values of* q *and fixed* α,n. *Moreover, absolute error function* $E_{100,q}^{(2)}(f;x)=\left|K_{100,q}^{(2)}(f;x)-f(x)\right|$ *is illustrated in Figure 2. Then, numerical values of* $E_{100,q}^{(2)}(f;x)$ *at some points on the interval* $[0,6]$ *for* $\{q\in 0.75,0.85,0.95\}$ *are given in Table 1.*

Table 1. Estimation of the absolute error function $E_{100,q}^{(2)}$ with $f(x)=x^3-9x^2-15x+9$ for some values of x in $[0,6]$ and $q\in\{0.75,0.85,0.95\}$.

x	$E_{100,0.75}^{(2)}$	$E_{100,0.85}^{(2)}$	$E_{100,0.95}^{(2)}$
0	0.479	0.170	0.056
1	1.830	0.761	0.178
2	1.013	0.283	0.692
3	16.273	6.732	3.551
4	52.172	22.357	9.397
5	116.932	50.926	19.228
6	218.776	96.211	34.043

As we increase the value of q *and fixed* α *and* n, *the approximation is good, i.e for the largest value of* q, *the error is minumum.*

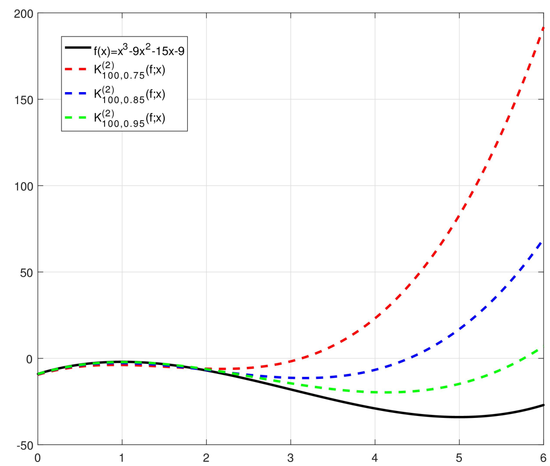

FIGURE 1. Approximation to $f(x) = x^3 - 9x^2 - 15x + 9$ by $K^{(2)}_{100,q}(f;x)$ for $q \in \{0.75, 0.85, 0.95\}$.

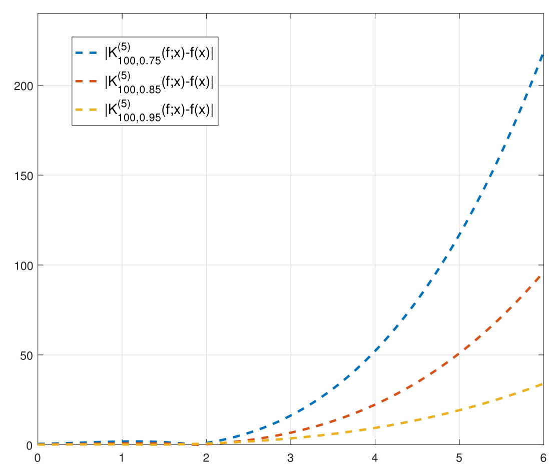

FIGURE 2. $E^{(2)}_{100,q}(f;x)$ for $f(x) = x^3 - 9x^2 - 15x + 9$ and $q = \{0.75, 0.85, 0.95\}$.

Example 2. *Let* $f(x) = x^6$ *with* $x \in [0, 6]$. *Here we take the value of* $n \in \{10, 100\}$, $\alpha = 5$ *and* $q = 0.95$. *The Figure 3 demonstrate the convergence of operators*

$K^{(5)}_{n,0.95}$*to* $f(x)$ *for increasing values of* n*. Secondly, The absolute error function* $E^{(5)}_{n,0.95}(f;x) = \left|K^{(5)}_{n,0.95}(f;x) - f(x)\right|$ *is illustrated in Figure 4. Finally, numerical values of* $E^{(5)}_{n,0.95}(f;x)$ *at some points on the interval* $[0,6]$ *for* $n \in \{10,100\}$ *are given in Table 2.*

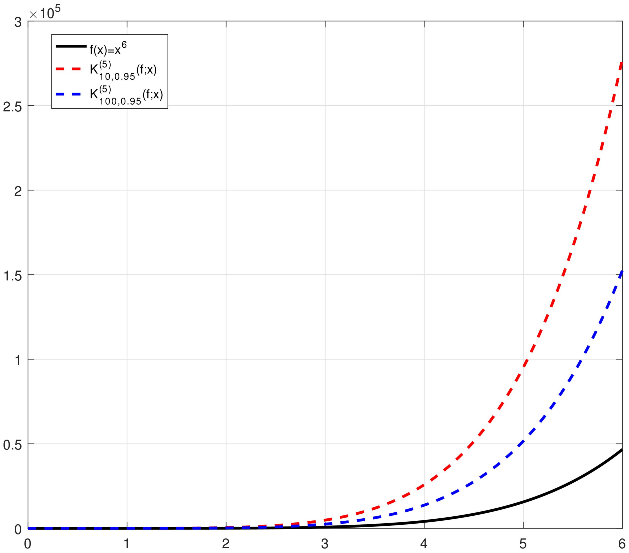

FIGURE 3. Approximation to $f(x) = x^6$ by $K^{(5)}_{n,0.95}(f;x)$ for $n \in \{10,100\}$.

TABLE 2. Estimation of the absolute error function $E^{(5)}_{n,0.95}$ with $f(x) = x^6$ for some values of x in $[0,6]$ and $n \in \{10,100\}$

x	$E^{(5)}_{10,0.95}$	$E^{(5)}_{100,0.95}$
1	8.57	3.06
2	401.68	164.39
3	4067.76	1758.63
4	21481.26	9566.43
5	78862.08	35778.98
6	229422.04	105421.79

As we increase the value of n *and fixed* α *and* q*, the approximation is good, i.e for the largest value of* n*, the error is minumum.*

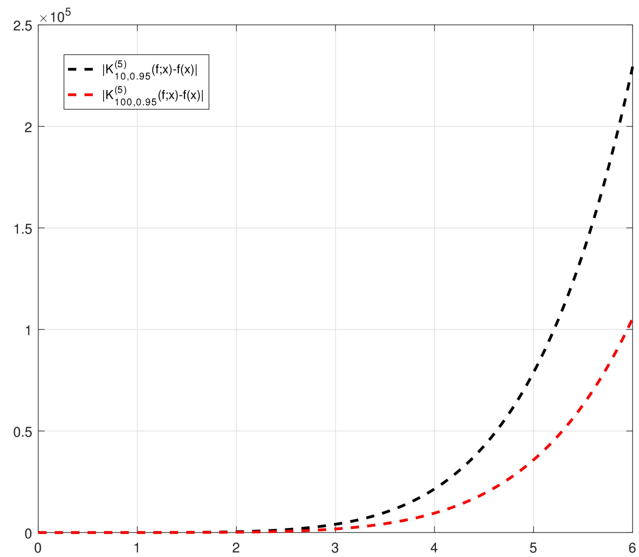

FIGURE 4. $E^{(5)}_{n,0.95}(f;x)$ for $n = \{10, 100\}$, $f(x) = x^6$.

Example 3. *Let* $f(x) = x^3 - 4x^2 + 2$ *with* $x \in [0, 5]$. *Here we take the value of* $\alpha \in \{0.1, 10\}$, $n = 150$ *and* $q = 0.95$. *The Figure 5 demonstrate the convergence of operators* $K^{(\alpha)}_{150,0.95}$ *to* $f(x)$ *for increasing values of* α. *Secondly, The absolute error function* $E^{(\alpha)}_{150,0.95}(f;x) = \left|K^{(\alpha)}_{n,q}(f;x) - f(x)\right|$ *is illustrated in Figure 6. Finally, numerical values of* $E^{(\alpha)}_{150,0.95}$ *at some points on the interval* $[3, 5]$ *for* $\alpha \in \{0.1, 10\}$ *are given in Table 3.*

TABLE 3. Estimation of the absolute error function $E^{(\alpha)}_{150,0.95}$ with $f(x) = x^3 - 4x^2 + 2$ for some values of x in $[3, 5]$ and $\alpha \in \{0.1, 10\}$

x	$E^{(0.1)}_{150,0.95}$	$E^{(10)}_{150,0.95}$
3	6.682	6.472
3.5	9.853	9.388
4	13.944	13.161
4.5	19.081	17.917
5	25.388	23.780

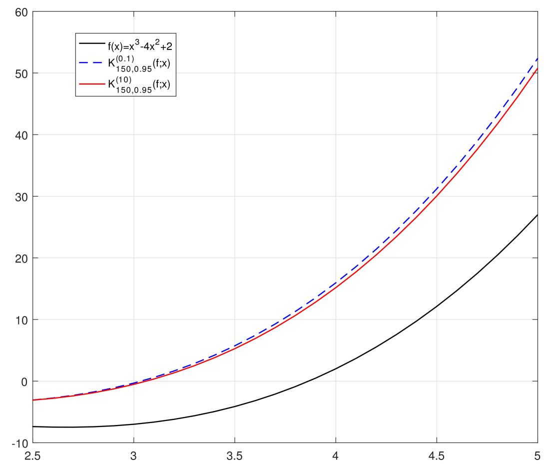

FIGURE 5. Approximation to $f(x) = x^3 - 4x^2 + 2$ by $K_{150,0.95}^{(\alpha)}(f;x)$ for $\alpha \in \{0.1, 10\}$.

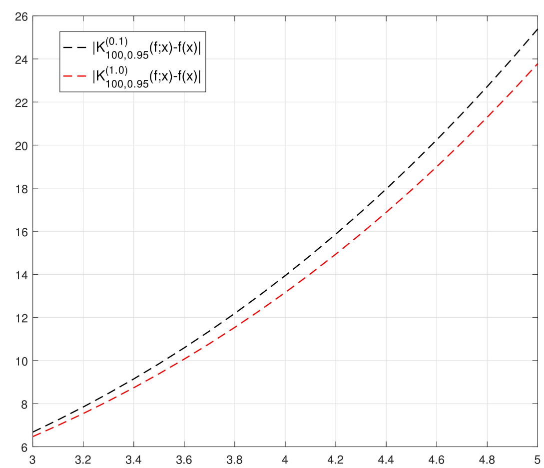

FIGURE 6. $E_{150,0.95}^{(\alpha)}(f;x)$ for $f(x) = x^3 - 4x^2 + 2$ and $\alpha \in \{0.1, 10\}$.

Now, we are present some graphs and numerical results for the convergence of bivariate fractional q-integral Riemann-Liouville integral type Szász-Mirakyan-Kantorovich operators $K_{n_1,n_2,q_1,q_2}^{(\alpha_1,\alpha_2)}$ by considering the function $f(x,y) = x + y$.

Example 4. *Consider,* $f(x,y) = x + y$ *with* $(x,y) \in [0,4]\times [0,4]$*. Here we take the value of* $n_1, n_2 \in \{5, 150\}$ *,* $q_1 = q_2 = 0.75$ *and* $\alpha_1 = \alpha_2 = 0.1$ *. The Figure 7 explains the convergence of the operators* $K^{(0.1,0.1)}_{n_1,n_2,0.75,0.75}$ *towards the function* $f(x,y)$ *for increasing values of* n_1, n_2*. Secondly, The absolute error function* $E^{(0.1,0.1)}_{n_1,n_2,0.75,0.75}(f;x,y) = \left|K^{(0.1,0.1)}_{n_1,n_2,0.75,0.75}(f;x,y) - f(x,y)\right|$ *is illustrated Figure 8. Finally numerical values of* $E^{(0.1,0.1)}_{n_1,n_2,0.75,0.75}$ *at some points on the interval* $[0,4]\times[0,4]$ *for* $n_1, n_2 \in \{5, 150\}$ *are given in Table 4.*

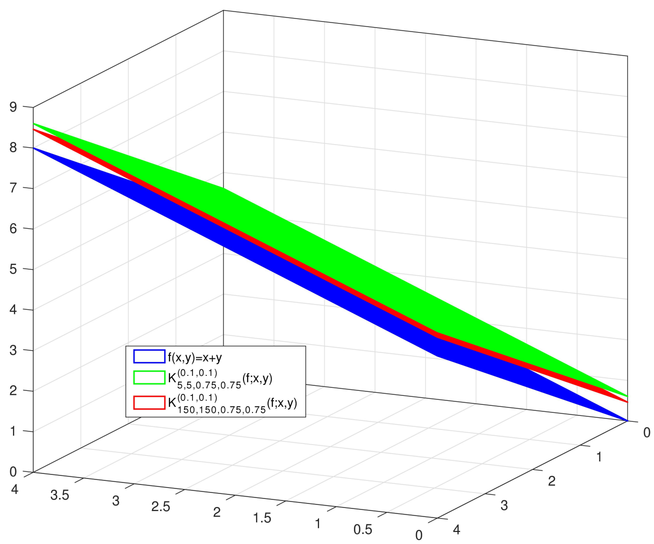

FIGURE 7. Convergence of the operators $K^{(0.1,0.1)}_{n_1,n_2,0.75,0.75}$ to the function $f(x,y) = x + y$.

As we increase the value of n_1 *and* n_2 *and fixed* α_1*,* α_2, q_1 *and* q_2*, the approximation is good, i.e for the largest value of* n_1 *and* n_2 *and fixed* α_1*,* α_2, q_1 *and* q_2*, the error is minumum.*

Example 5. *Consider* $f(x,y) = x + y$ *with* $(x,y) \in [0,4]\times [0,4]$*. Here we take the value of* $\alpha_1, \alpha_1 \in \{0.1, 10\}$*,* $q_1 = q_2 = 0.75$ *and* $n_1 = n_2 = 5$*. The Figure 9 explains the convergence of the operators* $K^{(\alpha_1,\alpha_2)}_{5,5,0.75,0.75}$ *towards the function* $f(x,y)$ *for increasing values of* $\alpha_1, \alpha_2 \in \{0.1, 10\}$ *. Secondly, absolute error function*

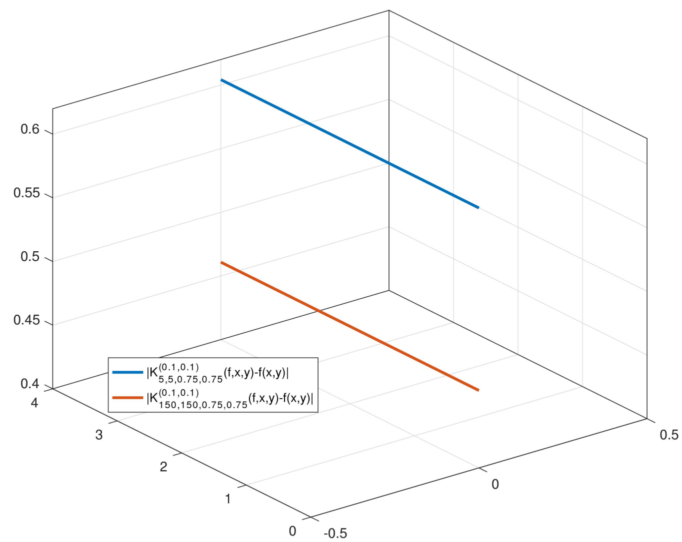

FIGURE 8. $\mathrm{E}^{(0.1,0.1)}_{n_1,n_2,0.75,0.75}$ with $f(x,y)=x+xy+12y^2$ for $n_1,n_2\in\{5,150\}$ on the interval $[0,4]\times[0,4]$.

TABLE 4. Estimation of the absolute error function $E^{(0.1,0.1)}_{n_1,n_2,0.75,0.75}$ with $f(x,y)=x+y$ for some values of (x,y) in $[0,4]\times[0,4]$ and $n_1,n_2\in\{5,150\}$.

x	y	$E^{(0.1,0.1)}_{5,5,0.75,0.75}$	$E^{(0.1,0.1)}_{150,150,0.75,0.75}$
0	0	0.604	0.461
0	0.5	0.604	0.461
0	1	0.604	0.461
0	1.5	0.604	0.461
0	2	0.604	0.461
0	2.5	0.604	0.461
0	3	0.604	0.461
0	3.5	0.604	0.461
0	4	0.604	0.461

$E^{(\alpha_1,\alpha_2)}_{5,5,0.75,0.75}(f;x,y)=\left|K^{(\alpha_1,\alpha_2)}_{n_1,n_2,,q_1,q_2}(f;x,y)-f(x,y)\right|$ *is illustrated Figure 10. Finally, numerical values of* $E^{(\alpha_1,\alpha_2)}_{5,5,0.75,0.75}$ *at some points on the interval* $[0,4]\times[0,4]$ *for* $\alpha_1,\alpha_2\in\{0.1,10\}$ *are given in Table 5.*

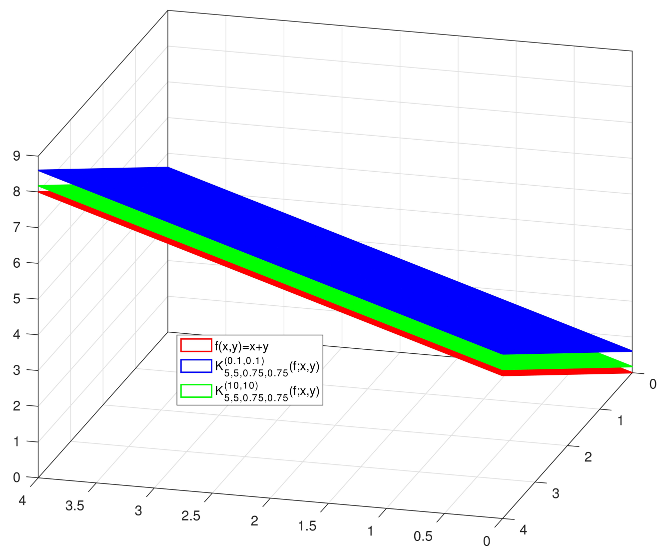

FIGURE 9. Convergence of the operators $K^{(\alpha_1,\alpha_2)}_{5,5,0.75,0.75}(f;x,y)$ to the function $f(x,y)=x+y$.

TABLE 5. Estimation of the absolute error function $E^{(\alpha_1,\alpha_2)}_{50,50,0.75,0.75}$ with $f(x,y)=x+y$ for some values of (x,y) in $[0,4]\times[0,4]$ and $\alpha_1,\alpha_2\in\{0.1,10\}$.

x	y	$E^{(0.1,0.1)}_{50,50,0.75,0.75}$	$E^{(10,10)}_{50,50,0.75,0.75}$
0.1	0.1	0.461	0.131
0.1	0.5	0.461	0.131
0.1	1	0.461	0.131
0.1	1.5	0.461	0.131
0.1	2	0.461	0.131
0.1	2.5	0.461	0.131
0.1	3	0.461	0.131
0.1	3.5	0.461	0.131
0.1	4	0.461	0.131

As we increase the value of α_1 and α_2 and fixed q_1,q_2,n_1 and n_2, the approximation is good, i.e for the largest value of α_1 and α_2 and fixed q_1,q_2,n_1 and n_2, the error is minumum.

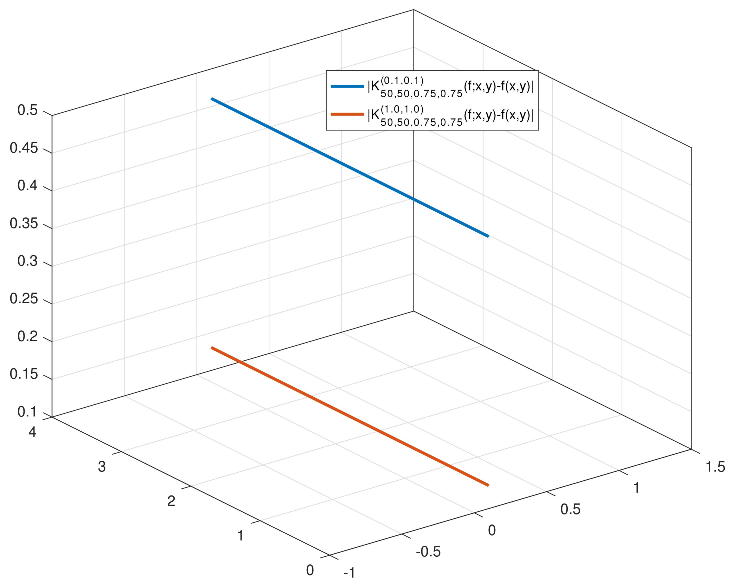

FIGURE 10. For some (x, y) points, error function $E^{(\alpha_1,\alpha_2)}_{50,50,0.75,0.75}$ with $f(x, y) = x + y$.

Example 6. *Consider $f(x) = x + y$ with $(x, y) \in [0, 4] \times [0, 4]$. Here we take the value of $q \in \{0.35, 0.75\}$, $n_1 = n_2 = 10$ and $\alpha_1 = \alpha_1 = 5$ for $K^{(\alpha_1,\alpha_2)}_{n_1,n_2,,q_1,q_2}$. The Figure 11 demonstrate the convergence of operators $K^{(5,5)}_{10,10,q_1,q_2}$ to $f(x, y)$ for increasing values of q_1 and q_2. Moreover, function of absolute error $E^{(5,5)}_{10,10,q_1,q_2}(f; x, y) = \left|K^{(\alpha_1,\alpha_2)}_{n_1,n_2,,q_1,q_2}(f; x, y) - f(x, y)\right|)$ in is illustrated Figure 12. Then, numerical values of $E^{(5,5)}_{10,10,q_1,q_2}$ at some points on the interval $[0, 4] \times [0, 4]$ for $q_1, q_2 \in \{0.35, 0.75\}$ are given in Table 6.*

As we increase the value of q_1 and q_2 and fixed α_1, α_2, n_1 and n_2, the approximation is good, i.e for the largest value of q_1 and q_2 and fixed α_1, α_2, n_1 and n_2, the error is minumum.

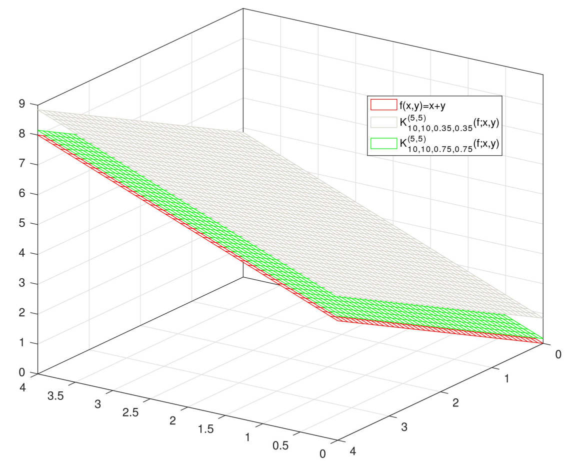

FIGURE 11. Approximation to $f(x,y) = x + y$ by $K^{(5,5)}_{10,10,q_1,q_2}$ $q_1, q_2 \in \{0.35, 0.75\}$.

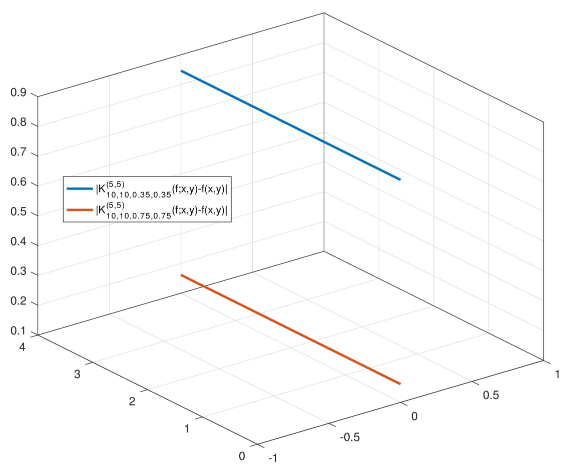

FIGURE 12. $E^{(2,2)}_{10,10,q_1,q_2}(f;x)$ for $f(x,y) = x + y$ and $q_1, q_2 = \{0.35, 0.75\}$.

Declaration of Competing Interests The authors declare that they have no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

Table 6. Estimation of the absolute error function $E^{(5,5)}_{10,10,q_1,q_2}$ with $f(x,y) = x + y$ for some values of x in $[0,4] \times [0,4]$ and $q_1, q_2 \in \{0.35, 0.75\}$.

x	y	$E^{(5,5)}_{10,10,0.35,0.35}$	$E^{(5,5)}_{10,10,0.75,0.75}$
0	0	0.847	0.161
0	0.5	0.847	0.161
0	1	0.847	0.161
0	1.5	0.847	0.161
0	2	0.847	0.161
0	2.5	0.847	0.161
0	3	0.847	0.161
0	3.5	0.847	0.161
0	4	0.847	0.161

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