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Some Algebraic Structure on Figurate Numbers

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Abstract
In this study, some information about figurate numbers and centered polygonal numbers is given. Also, a general binary operator that includes all centered polygonal numbers is defined, and it is investigated whether the algebraic structures defined with the general binary operator specify a groupoid and semigroup or not. And finally, some examples are given on the subject.

1. Introduction

Figurate numbers are natural numbers that can be represented by regular geometric patterns with equally spaced points. The theory of figurate numbers is not one of the main subjects of mathematics, but the charm of these numbers has raised awareness among scientists for thousands of years. Many special numbers have been created by being inspired from figurate numbers. Pythagoras triples, Perfect numbers, Mersenne numbers, Cullen numbers, Woodall numbers, Fermat numbers, Fibonacci numbers, Pell numbers, Lucas numbers, Thabit numbers, etc. are examples of such numbers.

These special number classes which are derived from Figurate numbers have a long and rich history dating back to ancient Greek times. They were first introduced in the Pythagorean school in the 6th century BC to enable the connection between geometry and arithmetic. It is possible to give some of the mathematicians who worked in this field and whose works are still center of interest today; "Pythagoras of Samos (ca. 582 BC–ca. 507 BC), Diophantus of Alexandria (ca. 210–ca. 290), Leonardo of Pisa who was also known as Leonardo Fibonacci (ca. 1170–ca. 1250), Gerolamo Cardano (1501–1576), Pierre de Fermat (1601–1665), John Pell (1611–1685), Blaise Pascal (1623–1662), Leonhard Euler (1707–1783), Joseph Louis Lagrange (1736–1813), Carl Friedrich Gauss (1777–1855), Augustin-Louis Cauchy (1789–1857)." [1].

Figurate numbers have different names according to their form of shapes on the plane. If the geometric pattern is arranged with a regular polygon, then the figurate number is called a polygonal number. If the geometric pattern is arranged with a regular polyhedron, then the figurate number is called a polyhedral number. Also, there are many different forms of figurate numbers such as centered polygonal, pronic, oblong, L-shape, cross, pyramidal numbers, etc.

Polygonal numbers are examples of figurate numbers and are probably the most well-known. Polygonal numbers start from a fixed point and increase in numbers by constructing larger and larger regular polygons. In particular, triangular and square numbers are examples of polygonal numbers. Triangular numbers can be obtained by adding to a fixed point two, three, four, five etc. points and arranging them in the form of an equilateral triangle.

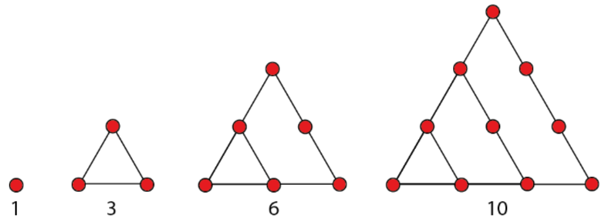

Figure 1. Some triangular numbers.

Similarly, square numbers can be obtained by adding to a fixed point three, five, seven, nine etc. points and arranging them in the form of a square.

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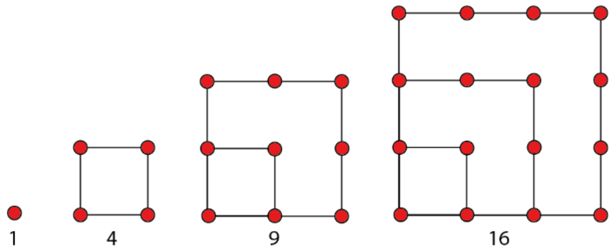

Figure 2. Some square numbers.

Mathematicians have worked on triangular and square numbers since ancient times. But studies on this field have been intensified for the last three centuries especially in this century. For example, in 1638, the famous mathematician Pierre de Fermat came up with Polygonal Number Theory but without proof. That is, "every positive integer can be written as the sum of three or fewer triangular numbers, and as the sum of four or fewer square numbers." In 1770 Joseph Louis Lagrange proved that polygonal number theory is valid for square case, Carl Friedrich Gauss proved the triangular case in 1796. And finally in 1818, Cauchy became the first person who proved the most general case of this problem that is, "every positive integer can be written as the sum of *n* or fewer *n*-gonal numbers." [2].

The other example of figurate numbers is Centered polygonal numbers. Centered polygonal numbers are regular polygons that are surrounded around a fixed point. By adding a point in the center in the plane, polygonal layers with a constant number of sides are constructed around this point. The number of points in these sides of the polygonal layers increases one more than the previous layer. Thus, the number of points in each polygonal layer of a centered *m*-gonal numbers increases m more points than the previous layer. For more detailed information on the subject, see [1], [2].

2. Material and Method

Binary operators have played an important role in many algebraic structures. It takes various names according to the binary operation defined on algebraic structures. Groupoid, semigroup and monoid are some of them. Specifically, it was proven by Sparavigna that it is a groupoid with binary operators defined on some polygonal numbers in [3]-[5]. Also, Emin studied semigroup construction on polygonal numbers in [6]. By using methods similar to those in these papers, we will give a general binary operator that includes all centered polygonal numbers. In addition, it will be investigated whether the algebraic structures defined with the general operator specify a groupoid and semigroup or not. By the way, maybe other algebraic properties, such as the studies in [7]-[10], can be studied by other mathematicians with the binary operation defined on this new algebraic structure.

The concept of groupoid, semigroup, monoid, and centered polygonal numbers will be explained and also their definitions and properties were given in this part of the study.

Definition 2.1. A groupoid (G, ∇) is defined as a non-empty set G on which a binary operation ∇ (by which we mean a map $\nabla: G \times G \longrightarrow G$) is defined. In other words, groupoid is an algebraic structure on a set with a binary operator. The only restriction on the operator is closure. It means that applying the binary operator on two elements of given set G returns with a value in which itself is a member of G. For more information, see [11], [12]. We say that (G, ∇) is a semigroup if the operation ∇ is associative, that is to say, if, for all $x, y, z \in G$

$$(x\nabla y)\nabla z = x\nabla(y\nabla z) \tag{1}$$

A semigroup is an associative groupoid; a semigroup with an identity is called a monoid.

Definition 2.2. For $m = 3, 4, 5, \ldots$ and $n \in \mathbb{N}, n-th$ centered m-gonal number formula is as follows [1];

$$CS_m(n) := \frac{mn^2 - mn + 2}{2} \tag{2}$$

Algebraically, for $n \in \mathbb{N}$ and $m \geq 3, n-th$ centered m-gonal number $CS_m(n)$ is obtained as the sum of the first n elements of the arithmetic progression $1, m, 2m, 3m, \ldots, (n-1)m$. So, it holds [1];

$$\begin{aligned} CS_m(n) &= 1 + m + 2m + 3m + \cdots + (n-1)m \\ &= 1 + m\big(1 + 2 + 3 + \cdots + (n-1)\big) \\ &= 1 + m\frac{(n-1)n}{2} \\ &= \frac{mn^2 - mn + 2}{2}. \end{aligned} \tag{3}$$

Example 2.1. In particular, for $m = 3, 4, 5, 6$ and $n \in \mathbb{N}, n - th$ centered m-gonal number formulas are as follows;

$$
\begin{aligned}
CS_3(n) &= 1 + 3\frac{n(n-1)}{2} = \frac{3n^2-3n+2}{2},\\
CS_4(n) &= 1 + 4\frac{n(n-1)}{2} = 2n^2-2n+1,\\
CS_5(n) &= 1 + 5\frac{n(n-1)}{2} = \frac{5n^2-5n+2}{2},\\
CS_6(n) &= 1 + 6\frac{n(n-1)}{2} = 3n^2-3n+1.
\end{aligned} \tag{4}
$$

The expression above implies the following recurrence formula for centered m-gonal numbers:

$$CS_m(n+1) = CS_m(n) + mn\,. \tag{5}$$

In particular, the result is;

$$
\begin{aligned}
CS_3(n+1) &= CS_3(n) + 3n,\\
CS_4(n+1) &= CS_4(n) + 4n,\\
CS_5(n+1) &= CS_5(n) + 5n,\\
CS_6(n+1) &= CS_6(n) + 6n,\\
CS_7(n+1) &= CS_7(n) + 7n,\\
CS_8(n+1) &= CS_8(n) + 8n.
\end{aligned} \tag{6}
$$

For some situations the value $CS_m(0) = 0$ can be accepted, where necessary, see [1], [6].

Example 2.2. For $n = 1, 2, 3, 4, 5,$ and $m = 3, 4, 5, 6$ some centered polygonal numbers are as follows [13, A005448, A001844, A005891, A003215];

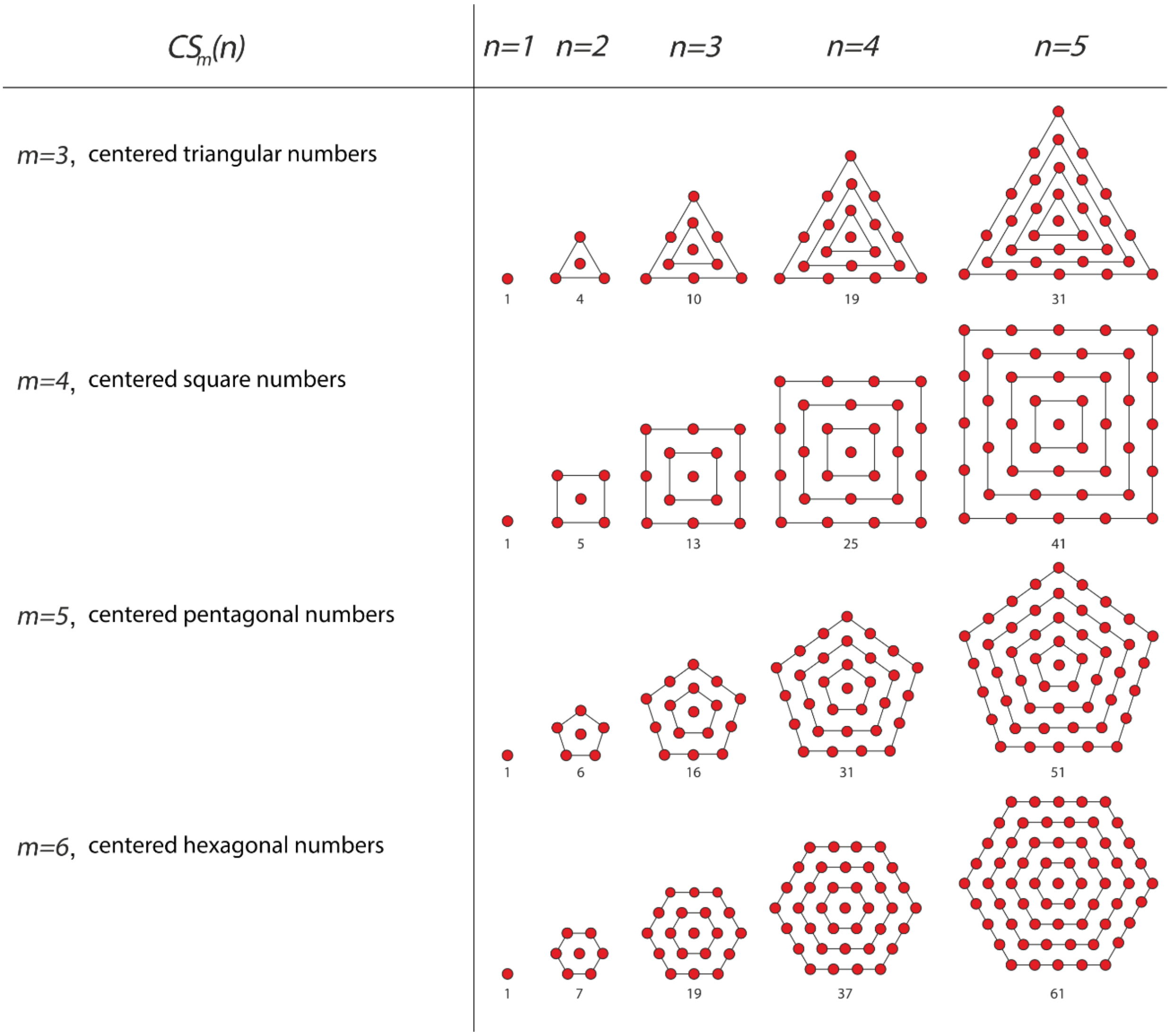

Figure 3. Some centered polygonal numbers

3. Findings and Discussion

In this section, we create a set consisting of elements of all centered polygonal numbers sequence $CS_m(n)$. After that we obtain an algebraic structure by defining binary operation on the defined set. And finally, we give a theorem and a corollary that show necessary conditions for this algebraic structure to be semigroup and monoid.

3.1. Construction of Algebraic Structure on Centered Polygonal Numbers

Before we can construct the theorem that yields the main result of this paper, we need to define a set and a binary operation on that set. So, let C denote the sequence of numbers $CS_m(n)$. That is, let

$$C=\left\{1,1+m,1+3m,1+6m,1+10m,\dots,1+m\frac{n(n-1)}{2},\dots\right\}. \qquad (7)$$

Now we can find a binary operation of given set of C since

$$\begin{aligned}\left(CS_m(n)-\frac{8-m}{8}\right)^{\frac{1}{2}}&=\left(\frac{mn^2-mn+2}{2}-\frac{8-m}{8}\right)^{\frac{1}{2}}\\&=\left(\frac{4mn^2-4mn+m}{8}\right)^{\frac{1}{2}}\\&=\left(\frac{m}{2}\left(\frac{4n^2-4n+1}{4}\right)\right)^{\frac{1}{2}}\\&=\sqrt{\frac{m}{2}}\left(\left(\frac{2n-1}{2}\right)^2\right)^{\frac{1}{2}}\\&=\sqrt{\frac{m}{2}}\left(n-\frac{1}{2}\right).\end{aligned} \qquad (8)$$

We define

$$\begin{aligned}M_n&=\left(CS_m(n)-\frac{8-m}{8}\right)^{\frac{1}{2}}=\sqrt{\frac{m}{2}}\left(n-\frac{1}{2}\right),\\M_k&=\left(CS_m(k)-\frac{8-m}{8}\right)^{\frac{1}{2}}=\sqrt{\frac{m}{2}}\left(k-\frac{1}{2}\right),\\M_{n+k}&=\left(CS_m(n+k)-\frac{8-m}{8}\right)^{\frac{1}{2}}=\sqrt{\frac{m}{2}}\left(n+k-\frac{1}{2}\right).\end{aligned} \qquad (9)$$

We use M_n for definition of binary operation.

$$\begin{aligned}M_{n+k}=M_n\nabla M_k&=M_n+M_k+\sqrt{\frac{m}{8}}\\&=\sqrt{\frac{m}{2}}\left(n-\frac{1}{2}\right)+\sqrt{\frac{m}{2}}\left(k-\frac{1}{2}\right)+\sqrt{\frac{m}{8}}\\&=\sqrt{\frac{m}{2}}\left(n+k-\frac{1}{2}\right).\end{aligned} \qquad (10)$$

Therefore, we have the binary operation:

$$\begin{aligned}&\left(CS_m(n+k)-\frac{8-m}{8}\right)^{\frac{1}{2}}=M_{n+k}\\&=M_n\nabla M_k=M_n+M_k+\sqrt{\frac{m}{8}}\\&=\left(CS_m(n)-\frac{8-m}{8}\right)^{\frac{1}{2}}+\left(CS_m(k)-\frac{8-m}{8}\right)^{\frac{1}{2}}+\frac{\sqrt{m}}{2\sqrt{2}}\,.\end{aligned} \qquad (11)$$

As a result, from (11), we can rewrite the defined binary operation as follows:

$$\begin{aligned}&CS_m(n)\nabla CS_m(k)=CS_m(n+k)\\&=CS_m(n)+CS_m(k)+\frac{m}{4}-1\\&+2\left(CS_m(n)-\frac{8-m}{8}\right)^{\frac{1}{2}}\left(CS_m(k)-\frac{8-m}{8}\right)^{\frac{1}{2}}\\&+\frac{\sqrt{m}}{\sqrt{2}}\left(CS_m(n)-\frac{8-m}{8}\right)^{\frac{1}{2}}+\frac{\sqrt{m}}{\sqrt{2}}\left(CS_m(k)-\frac{8-m}{8}\right)^{\frac{1}{2}}.\end{aligned}$$

In the following lemma, we give a necessary condition for the algebraic structure (C,∇) to be a groupoid.

Lemma 3.1.1. Let C be the set of sequence of numbers $CS_m(n)$, that is, let

$$C=\left\{1,1+m,1+3m,1+6m,1+10m,\dots,1+m\frac{n(n-1)}{2},\dots\right\}.$$

Also let ∇ be a binary operation on C such that

$$\begin{aligned}&CS_m(n)\nabla CS_m(k)=CS_m(n+k)\\&=CS_m(n)+CS_m(k)+\frac{m}{4}-1\\&+2\left(CS_m(n)-\frac{8-m}{8}\right)^{\frac{1}{2}}\left(CS_m(k)-\frac{8-m}{8}\right)^{\frac{1}{2}}\\&+\frac{\sqrt{m}}{\sqrt{2}}\left(CS_m(n)-\frac{8-m}{8}\right)^{\frac{1}{2}}+\frac{\sqrt{m}}{\sqrt{2}}\left(CS_m(k)-\frac{8-m}{8}\right)^{\frac{1}{2}}.\end{aligned} \qquad (12)$$

where $n, k \in \mathbb{N}$ and $m \geq 3$. Then the algebraic structure (C, ∇) is a groupoid.

Proof. From binary operation (12), we can have recursive relation for $k = 1$:

$$CS_m(n)\nabla CS_m(1) = CS_m(n+1)$$
$$= CS_m(n) + CS_m(1) + \frac{m}{4} - 1$$
$$+2\left(CS_m(n) - \frac{8-m}{8}\right)^{\frac{1}{2}}\left(CS_m(1) - \frac{8-m}{8}\right)^{\frac{1}{2}}$$
$$+\frac{\sqrt{m}}{\sqrt{2}}\left(CS_m(n) - \frac{8-m}{8}\right)^{\frac{1}{2}} + \frac{\sqrt{m}}{\sqrt{2}}\left(CS_m(1) - \frac{8-m}{8}\right)^{\frac{1}{2}}.$$

Thus, $CS_m(n+1) = CS_m(n)\nabla CS_m(1)$

$$= CS_m(n) + \frac{m}{2} + \sqrt{2m}\left(CS_m(n) - \frac{8-m}{8}\right)^{\frac{1}{2}}.$$

And so, we have

$$CS_m(n+1) = CS_m(n) + \sqrt{2m}\left(CS_m(n) - \frac{8-m}{8}\right)^{\frac{1}{2}} + \frac{m}{2}. \quad (13)$$

We prove this part of the lemma using the method of mathematical induction on n. For $n = 2, 3, 4, 5$ and starting from number $CS_m(1) = 1$, we have $1 + m, 1 + 3m, 1 + 6m, 1 + 10m$ which are the elements of the set of C. Indeed, for $m \geq 3$,

$$CS_m(2) = CS_m(1) + \sqrt{2m}\left(CS_m(1) - \frac{8-m}{8}\right)^{\frac{1}{2}} + \frac{m}{2}$$
$$= 1 + \sqrt{2m}\left(1 - \frac{8-m}{8}\right)^{\frac{1}{2}} + \frac{m}{2}$$
$$= 1 + \sqrt{2m}\frac{\sqrt{m}}{2\sqrt{2}} + \frac{m}{2} = 1 + m.$$

$$CS_m(3) = CS_m(2) + \sqrt{2m}\left(CS_m(2) - \frac{8-m}{8}\right)^{\frac{1}{2}} + \frac{m}{2}$$
$$= 1 + m + \sqrt{2m}\left(1 + m - \frac{8-m}{8}\right)^{\frac{1}{2}} + \frac{m}{2}$$
$$= 1 + m + \sqrt{2m}\frac{\sqrt{9m}}{2\sqrt{2}} + \frac{m}{2} = 1 + 3m.$$

$$CS_m(4) = CS_m(3) + \sqrt{2m}\left(CS_m(3) - \frac{8-m}{8}\right)^{\frac{1}{2}} + \frac{m}{2}$$
$$= 1 + 3m + \sqrt{2m}\left(1 + 3m - \frac{8-m}{8}\right)^{\frac{1}{2}} + \frac{m}{2}$$
$$= 1 + 3m + \sqrt{2m}\frac{\sqrt{25m}}{2\sqrt{2}} + \frac{m}{2}$$
$$= 1 + 3m + \frac{6m}{2} = 1 + 6m.$$

$$CS_m(5) = CS_m(4) + \sqrt{2m}\left(CS_m(4) - \frac{8-m}{8}\right)^{\frac{1}{2}} + \frac{m}{2}$$
$$= 1 + 6m + \sqrt{2m}\left(1 + 6m - \frac{8-m}{8}\right)^{\frac{1}{2}} + \frac{m}{2}$$
$$= 1 + 6m + \sqrt{2m}\frac{\sqrt{49m}}{2\sqrt{2}} + \frac{m}{2}$$
$$= 1 + 6m + \frac{8m}{2} = 1 + 10m.$$

Now we suppose that the recursive formula (13) is true for n. That is, for $m \geq 3$,

$$CS_m(n) = CS_m(n-1) + \sqrt{2m}\left(CS_m(n-1) - \frac{8-m}{8}\right)^{\frac{1}{2}} + \frac{m}{2}$$
$$= \frac{m(n-1)^2 - m(n-1) + 2}{2}$$
$$+\sqrt{2m}\left(\frac{m(n-1)^2 - m(n-1) + 2}{2} - \frac{8-m}{8}\right)^{\frac{1}{2}} + \frac{m}{2}$$
$$= \frac{mn^2 - 3mn + 2m + 2}{2}$$
$$+\sqrt{2m}\left(\frac{mn^2 - 3mn + 2m + 2}{2} - \frac{8-m}{8}\right)^{\frac{1}{2}} + \frac{m}{2}$$
$$= \frac{mn^2 - 3mn + 2m + 2}{2} + \frac{m(2n-3)}{2} + \frac{m}{2}$$

$$= \frac{mn^2 - mn + 2}{2}$$

is the element of the set of C. Also, from (3) and (9), we have

$$CS_m(n) = CS_m(n-1) + \sqrt{2m}\left(CS_m(n-1) - \frac{8-m}{8}\right)^{\frac{1}{2}} + \frac{m}{2}$$
$$= 1 + m + 2m + 3m + \cdots + (n-3)m + (n-2)m$$
$$+\sqrt{2m}\sqrt{\frac{m}{2}}\left(n - 1 - \frac{1}{2}\right) + \frac{m}{2}$$
$$= 1 + m + 2m + 3m + \cdots + (n-3)m + (n-2)m$$
$$+(n-1)m - \frac{m}{2} + \frac{m}{2}$$
$$= 1 + m + 2m + 3m + \cdots + (n-3)m + (n-2)m$$
$$+(n-1)m. \quad (14)$$

We need to show that the recursive formula (13) is true for $n + 1$. From (9) and (14),

$$CS_m(n) + \sqrt{2m}\left(CS_m(n) - \frac{8-m}{8}\right)^{\frac{1}{2}} + \frac{m}{2}$$
$$= 1 + m + 2m + 3m + \cdots + (n-3)m + (n-2)m$$
$$+(n-1)m + \sqrt{2m}\sqrt{\frac{m}{2}}\left(n - \frac{1}{2}\right) + \frac{m}{2}$$
$$= 1 + m + 2m + 3m + \cdots + (n-3)m + (n-2)m$$
$$+(n-1)m + nm - \frac{m}{2} + \frac{m}{2}$$

$= 1 + m + 2m + 3m + \cdots + (n-3)m + (n-2)m$
$+(n-1)m + nm$
$= 1 + m(1 + 2 + 3 + \cdots + (n-2) + (n-1) + n)$
$= 1 + m\dfrac{n(n+1)}{2}$
$= \dfrac{mn^2 + mn + 2}{2}$
$= CS_m(n+1).$

Therefore $CS_m(n+1)$ is element of the set of C. That means, the algebraic structure (C, ∇) satisfies the properties of closure which gives us the lemma 3.1.1. that (C, ∇) is a groupoid. Hence the result.

In the following theorem, we give a necessary condition for the algebraic structure (C, ∇) to be a semigroup which is the main conclusion of this study.

Theorem 3.1.1. Let C be the set of sequence of numbers $CS_m(n)$, that is, let

$$C = \left\{1, 1+m, 1+3m, 1+6m, 1+10m, \ldots, 1 + m\frac{n(n-1)}{2}, \ldots\right\}.$$

Also let ∇ be a binary operation on C such that

$$CS_m(n)\nabla CS_m(k) = CS_m(n+k)$$
$$= CS_m(n) + CS_m(k) + \frac{m}{4} - 1$$
$$+2\left(CS_m(n) - \frac{8-m}{8}\right)^{\frac{1}{2}}\left(CS_m(k) - \frac{8-m}{8}\right)^{\frac{1}{2}}$$
$$+ \frac{\sqrt{m}}{\sqrt{2}}\left(CS_m(n) - \frac{8-m}{8}\right)^{\frac{1}{2}} + \frac{\sqrt{m}}{\sqrt{2}}\left(CS_m(k) - \frac{8-m}{8}\right)^{\frac{1}{2}}.$$

where $n, k \in \mathbb{N}$ and $m \geq 3$. Then the algebraic structure (C, ∇) is a semigroup.

Proof. From lemma 3.1.1. the algebraic structure (C, ∇) is a groupoid. Now we need to show that (C, ∇) satisfies the properties of associativity. From (9) and (10), we know

$M_{n+k} = M_n \nabla M_k = M_n + M_k + \sqrt{\frac{m}{8}}$ where $n, k \in \mathbb{N}$, $m \geq 3$ and

$$M_n = \left(CS_m(n) - \frac{8-m}{8}\right)^{\frac{1}{2}} = \sqrt{\frac{m}{2}}\left(n - \frac{1}{2}\right),$$
$$M_k = \left(CS_m(k) - \frac{8-m}{8}\right)^{\frac{1}{2}} = \sqrt{\frac{m}{2}}\left(k - \frac{1}{2}\right),$$
$$M_{n+k} = \left(CS_m(n+k) - \frac{8-m}{8}\right)^{\frac{1}{2}} = \sqrt{\frac{m}{2}}\left(n + k - \frac{1}{2}\right).$$

For $n, k, p \in \mathbb{N}$ and $m \geq 3$,

$$(M_n \nabla M_k)\nabla M_p = M_{n+k} + M_p + \sqrt{\frac{m}{8}}$$
$$= \sqrt{\frac{m}{2}}\left(n + k - \frac{8-m}{8}\right) + \sqrt{\frac{m}{2}}\left(p - \frac{8-m}{8}\right) + \sqrt{\frac{m}{8}}$$
$$= \sqrt{\frac{m}{2}}\left(n + k + p - \frac{8-m}{4}\right) + \sqrt{\frac{m}{8}} \tag{15}$$

And

$$M_n \nabla (M_k \nabla M_p) = M_n + M_{k+p} + \sqrt{\frac{m}{8}}$$
$$= \sqrt{\frac{m}{2}}\left(n - \frac{8-m}{8}\right) + \sqrt{\frac{m}{2}}\left(k + p - \frac{8-m}{8}\right) + \sqrt{\frac{m}{8}}$$
$$= \sqrt{\frac{m}{2}}\left(n + k + p - \frac{8-m}{4}\right) + \sqrt{\frac{m}{8}} \tag{16}$$

So, with the results of equations (15) and (16), we obtain, $(M_n \nabla M_k)\nabla M_p = M_n \nabla (M_k \nabla M_p)$ which gives us the theorem 3.1.1. that (C, ∇) satisfies the properties of associativity. Hence the result.

As seen in Figure 3, centered polygonal numbers start from $CS_m(1) = 1$. However, in some studies, as you can see in [1] and [6], centered polygonal numbers start from the number $CS_m(0) = 0$. Now by considering the start point as the number $CS_m(0) = 0$ and theorem 3.1.1., we can give the following corollary which gives the conditions for (C, ∇) to be a monoid.

Corollary 3.1.1. Let C be the set of sequence of numbers $CS_m(n)$ and let ∇ be a binary operation on C(defined in (7)) such that,

$$CS_m(n)\nabla CS_m(k) = CS_m(n+k)$$
$$= CS_m(n) + CS_m(k) + \frac{m}{4} - 1$$
$$+2\left(CS_m(n) - \frac{8-m}{8}\right)^{\frac{1}{2}}\left(CS_m(k) - \frac{8-m}{8}\right)^{\frac{1}{2}}$$
$$+ \frac{\sqrt{m}}{\sqrt{2}}\left(CS_m(n) - \frac{8-m}{8}\right)^{\frac{1}{2}} + \frac{\sqrt{m}}{\sqrt{2}}\left(CS_m(k) - \frac{8-m}{8}\right)^{\frac{1}{2}}.$$

where $n, k \in \mathbb{N}$ and $m \geq 3$. If $CS_m(0) = 0 \in C$, then the algebraic structure (C, ∇) is a monoid.

Proof. In theorem 3.1.1., we have proved that (C, ∇) is a semigroup. To show that the algebraic structure (C, ∇) is a monoid, it must be shown that it has an identity element. Let $CS_m(0) = 0 \in C$, then, from the binary operation ∇ we have,

$$CS_m(n) \nabla CS_m(0) = CS_m(n+0) = CS_m(n)$$

and

$$CS_m(0) \nabla CS_m(n) = CS_m(0+n) = CS_m(n).$$

Thus, we have,

$CS_m(n) \nabla CS_m(0) = CS_m(0) \nabla CS_m(n) = CS_m(n)$

which gives us the corollary 3.1.1. that (C, ∇) satisfies the properties of identity. Hence the result.

Example 3.1.1. [13, A005448] known as centered triangular numbers. Centered triangular numbers are integer having the following form below:

$$\begin{aligned} CS_3(n) &= 1 + 3 + 6 + 9 + 12 + \cdots + 3(n-1) \\ &= 1 + 3\big(1 + 2 + 3 + 4 + \cdots + (n-1)\big) \\ &= 1 + 3\frac{(n-1)n}{2} \\ &= \frac{3n^2 - 3n + 2}{2}. \end{aligned}$$

Let A denote the sequence of numbers $CS_3(n)$. That is, let

$A = \{1,4,10,19,31,46,64,85,109,136,166,199,235, 274,316,\ldots\}$

Now we can find a binary operation of given set of A. Since

$$\begin{aligned} \left(CS_3(n) - \frac{5}{8}\right)^{\frac{1}{2}} &= \left(\frac{3n^2 - 3n + 2}{2} - \frac{5}{8}\right)^{\frac{1}{2}} \\ &= \left(\frac{12n^2 - 12n + 3}{8}\right)^{\frac{1}{2}} \\ &= \left(\frac{3}{2}\left(\frac{4n^2 - 4n + 1}{4}\right)\right)^{\frac{1}{2}} \\ &= \sqrt{\frac{3}{2}}\left(\left(\frac{2n-1}{2}\right)^2\right)^{\frac{1}{2}} \\ &= \sqrt{\frac{3}{2}}\left(n - \frac{1}{2}\right). \end{aligned}$$

we define:

$$T_n = \left(CS_3(n) - \frac{5}{8}\right)^{\frac{1}{2}} = \sqrt{\frac{3}{2}}\left(n - \frac{1}{2}\right),$$

$$T_k = \left(CS_3(k) - \frac{5}{8}\right)^{\frac{1}{2}} = \sqrt{\frac{3}{2}}\left(k - \frac{1}{2}\right),$$

$$T_{n+k} = \left(CS_3(n+k) - \frac{5}{8}\right)^{\frac{1}{2}} = \sqrt{\frac{3}{2}}\left(n + k - \frac{1}{8}\right).$$

We use T_n for definition of binary operation:

$$\begin{aligned} T_{n+k} = T_n \nabla T_k &= T_n + T_k + \frac{\sqrt{3}}{2\sqrt{2}} \\ &= \sqrt{\frac{3}{2}}\left(n - \frac{1}{2}\right) + \sqrt{\frac{3}{2}}\left(k - \frac{1}{2}\right) + \frac{\sqrt{3}}{2\sqrt{2}} \\ &= \sqrt{\frac{3}{2}}\left(n + k - \frac{1}{2}\right). \end{aligned}$$

As a result, we can rewrite the defined binary operation as follows:

$$\begin{aligned} &CS_3(n) \nabla CS_3(k) = CS_3(n+k) \\ &= CS_3(n) + CS_3(k) - \frac{1}{4} \\ &+2\left(CS_3(n) - \frac{5}{8}\right)^{\frac{1}{2}}\left(CS_3(k) - \frac{5}{8}\right)^{\frac{1}{2}} \\ &+\sqrt{\frac{3}{2}}\left(CS_3(n) - \frac{5}{8}\right)^{\frac{1}{2}} + \sqrt{\frac{3}{2}}\left(CS_3(k) - \frac{5}{8}\right)^{\frac{1}{2}} \end{aligned} \tag{17}$$

From binary operation (17), we can have recursive relation for $k = 1$:

$$\begin{aligned} CS_3(n+1) &= CS_3(n) \nabla CS_3(1) \\ &= CS_3(n) + 2\sqrt{\frac{3}{2}}\left(CS_3(n) - \frac{5}{8}\right)^{\frac{1}{2}} + \frac{3}{2}. \end{aligned}$$

And so, we obtain

$$CS_3(n+1) = CS_3(n) + \frac{2\sqrt{3}}{\sqrt{2}}\left(CS_3(n) - \frac{5}{8}\right)^{\frac{1}{2}} + \frac{3}{2}.$$

Starting from number $CS_3(1) = 1$, we have $4, 10, 19, 31, 46, 64, 85, 109, 136, 166, 199, 235, 274, 316, \ldots$ which are the elements of the set of A. From lemma 3.1.1. and theorem 3.1.1., one can say that the algebraic structure (A, ∇) is a groupoid and semigroup. Also, if the $CS_3(0) = 0 \in A$ then the algebraic structure (A, ∇) is a monoid from corollary 3.1.1.

Example 3.1.2. [13, A005891] known as centered pentagonal numbers. Centered pentagonal numbers are integer having the following form below:

$$\begin{aligned} CS_5(n) &= 1+5+10+15\ +20+\cdots+5(n-1) \\ &= 1+5\big(1+2+3+4+\cdots+(n-1)\big) \\ &= 1+5\frac{(n-1)n}{2} \\ &= \frac{5n^2-5n+2}{2}. \end{aligned}$$

Let B denote the sequence of numbers $CS_5(n)$. That is, let
$B\ = \{1, 6, 16, 31, 51, 76, 91, 106, 141, 181, 226, 276, 331, 391, \dots\}$
Now we can find a binary operation of given set of B. Since

$$\begin{aligned} \left(CS_5(n)-\frac{3}{8}\right)^{\frac{1}{2}} &= \left(\frac{5n^2-5n+2}{2}-\frac{3}{8}\right)^{\frac{1}{2}} \\ &= \left(\frac{20n^2-20n+5}{8}\right)^{\frac{1}{2}} \\ &= \left(\frac{5}{2}\left(\frac{4n^2-4n+1}{4}\right)\right)^{\frac{1}{2}} \\ &= \sqrt{\frac{5}{2}}\left(\left(\frac{2n-1}{2}\right)^2\right)^{\frac{1}{2}} \\ &= \sqrt{\frac{5}{2}}\left(n-\frac{1}{2}\right). \end{aligned}$$

we define:

$$P_n = \left(CS_5(n)-\frac{3}{8}\right)^{\frac{1}{2}} = \sqrt{\frac{5}{2}}\left(n-\frac{1}{2}\right),$$

$$P_k = \left(CS_5(k)-\frac{3}{8}\right)^{\frac{1}{2}} = \sqrt{\frac{5}{2}}\left(k-\frac{1}{2}\right),$$

$$P_{n+k} = \left(CS_5(n+k)-\frac{3}{8}\right)^{\frac{1}{2}} = \sqrt{\frac{5}{2}}\left(n+k-\frac{1}{2}\right).$$

We use P_n for definition of binary operation:

$$\begin{aligned} P_{n+k} = P_n \nabla P_k &= P_n + P_k + \frac{\sqrt{5}}{2\sqrt{2}} \\ &= \sqrt{\frac{5}{2}}\left(n-\frac{1}{2}\right)+\sqrt{\frac{5}{2}}\left(k-\frac{1}{2}\right)+\frac{\sqrt{5}}{2\sqrt{2}} \\ &= \sqrt{\frac{5}{2}}\left(n+k-\frac{1}{2}\right). \end{aligned}$$

As a result, we can rewrite the defined binary operation as follows:

$$\begin{aligned} &CS_5(n)\nabla CS_5(k) = CS_5(n+k) \\ &= CS_5(n)+CS_5(k)+\frac{1}{4} \\ &+2\left(CS_5(n)-\frac{3}{8}\right)^{\frac{1}{2}}\left(CS_5(k)-\frac{3}{8}\right)^{\frac{1}{2}} \\ &+\sqrt{\frac{5}{2}}\left(CS_5(n)-\frac{3}{8}\right)^{\frac{1}{2}}+\sqrt{\frac{5}{2}}\left(CS_5(k)-\frac{3}{8}\right)^{\frac{1}{2}}. \end{aligned} \tag{18}$$

From binary operation (18), we can have recursive relation for $k=1$:

$$\begin{aligned} CS_5(n+1) &= CS_5(n)\nabla CS_5(1) \\ &= CS_5(n)+\frac{2\sqrt{5}}{\sqrt{2}}\left(CS_5(n)-\frac{3}{8}\right)^{\frac{1}{2}}+\frac{5}{2}. \end{aligned}$$

And so, we obtain

$$CS_5(n+1) = CS_5(n)+\frac{2\sqrt{5}}{\sqrt{2}}\left(CS_5(n)-\frac{3}{8}\right)^{\frac{1}{2}}+\frac{5}{2}.$$

Starting from number $CS_5(1)=1$, we have $6, 16, 31, 51, 76, 91, 106, 141, 181, 226, 276, 331, 391, \dots$ which are the elements of the set of B. From lemma 3.1.1. and theorem 3.1.1., one can say that the algebraic structure (B,∇) is a groupoid and semigroup. Also, if the $CS_5(0)=0\in B$, then the algebraic structure (B,∇) is a monoid from Corollary 3.1.1.

4. Conclusion and Suggestions

Firstly, in this paper, the definition of algebraic structure and, in particular, the definitions of groupoid, semigroup, and monoid were made. Also, figurate numbers and centered polygonal numbers were introduced. Later, the algebraic structure was proved to be a groupoid and a semigroup with the binary operation defined on centered polygonal numbers. And finally, some examples were given on the subject.

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Statement of Research and Publication Ethics

In this paper, research and publication ethics were followed.

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