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SAKARYA ÜNİVERSİTESİ

All solutions of the Diophantine equations $2F_n = 3^s \cdot y^b$ and $F_n \pm 1 = 3^s \cdot y^b$

İbrahim ERDURAN*[1], Zafer ŞİAR[1]

Abstract

The Fibonacci sequence (F_n) is defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. In this paper, we will give all solutions of the Diophantine equations $2F_n = 3^s \cdot y^b$ and $F_n \pm 1 = 3^s \cdot y^b$ in nonnegative integers $s \geq 0$, $y \geq 1$, $b \geq 2$, $n \geq 1$ and $(3, y) = 1$.

Keywords: Fibonacci and Lucas numbers, exponential Diophantine equations, elementary number theory

1. INTRODUCTION

The Fibonacci sequence (F_n) is defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The Lucas sequence (L_n), which is similar to the Fibonacci sequence, is defined by the same recursive pattern with initial conditions $L_0 = 2$, $L_1 = 1$. The terms of the Fibonacci and Lucas sequences are called Fibonacci and Lucas numbers, respectively. The Fibonacci and Lucas numbers for negative indices are defined by $F_{-n} = (-1)^{n+1} F_n$ and $L_{-n} = (-1)^n L_n$ for $n \geq 1$. For a brief history of Fibonacci and Lucas sequences one can consult [7]. The Fibonacci and Lucas sequences have many interesting properties and have been studied in the literature by many researchers. They specially have interested in square terms, perfect powers in these sequences and the exponential Diophantine equations including these sequences. Firstly, square terms and later perfect powers in the Fibonacci and Lucas sequences have attracted the attention of the researchers. As related these subjects, the authors gave the following theorems, which can be deduced from [4,5,6] and are useful to us.

Theorem 1. The only perfect powers in the Fibonacci sequence are $F_0 = 0, F_1 = 1, F_2 = 1, F_6 = 8$ and $F_{12} = 144$.

Theorem 2. The only perfect powers in the Lucas sequence are $L_1 = 1$ and $L_3 = 4$.

Theorem 3. If

$$F_n = 2^s \cdot y^b$$

for some integers $n \geq 1, y \geq 1, b \geq 2$ and $s \geq 0$ then $n \in \{1,2,3,6,12\}$. The solutions of the similar equation with F_n replaced by L_n have $n \in \{1,3,6\}$.

Theorem 4. If

$$F_n = 3^s \cdot y^b$$

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for some integers $n \geq 1, y \geq 1, b \geq 2$ and $s \geq 0$ then $n \in \{1,2,4,6,12\}$. The solutions of the similar equation with F_n replaced by L_n have $n \in \{1,2,3\}$.

Recently, many mathematicians have dealth with exponential Diophantine equations concerning Fibonacci and Lucas numbers. For example, the Diophantine equation $L_n + L_m = 2^a$ has been tackled in [2] by Bravo and Luca. Two years later, the same authors solved Diophantine equation $F_n + F_m = 2^a$ in [3]. Besides, in [5], Bugeaud et al. showed that if

$$F_n \pm 1 = y^a \quad (1)$$

for some nonnegative integers (n, y, a) with $a \geq 2$, then $n \in \{0,1,2,3,4,5,6\}$. In [1], the authors proved that if

$$F_n \pm 2 = y^a \quad (2)$$

for some nonnegative integers (n, y, a) with $a \geq 2$, then $n \in \{1,2,3,4,9\}$. Later, in [10], Luca and Patel handled the equation

$$F_n \pm F_m = y^p, \ p \geq 2, \quad (3)$$

which is general form of the equations (1) and (2). They found that the Diophantine equation (3) in integers (n, m, y, p) has solution either $max\{|n|, |m|\} \leq 36$ or $y = 0$ and $|n| = |m|$ if $n \equiv m(mod 2)$. This problem remain open for the case $n \not\equiv m(mod 2)$. But, in [8], the authors solved this equation by fixing y in the interval [2,1000].

Motivated by the above mentioned studies, in this paper, we consider the Diophantine equations

$$F_n + F_n = 2F_n = 3^s \cdot y^b \quad (4)$$

and

$$F_n \pm 1 = 3^s \cdot y^b \quad (5)$$

in nonnegative integers $s \geq 0, y \geq 1, b \geq 2, n \geq 1$ and $(3, y) = 1$.

2. PRELIMINARIES

In this section, we will give some idendities including Fibonacci and Lucas numbers, which will be used in the proofs of the main theorems. The following identites can be found in [9].

$$F_{n+1} + F_{n-1} = L_n. \quad (6)$$

If $m \geq 3$, then $F_m | F_n \Leftrightarrow m | n$. (7)

If $m \geq 2$, then $L_m | L_n \Leftrightarrow m | n$ and $\frac{n}{m}$ is odd. (8)

$$F_{3n} = F_n\left(5{F_n}^2 + 3(-1)^n\right). \quad (9)$$

The following theorem is given in [9].

Theorem 5. ([9], *Theorem* 10.9), *The following equalities hold.*

1. $F_{4k} + 1 = F_{2k-1} \cdot L_{2k+1}$,

2. $F_{4k+1} + 1 = F_{2k+1} \cdot L_{2k}$,

3. $F_{4k+2} + 1 = F_{2k+2} \cdot L_{2k}$,

4. $F_{4k+3} + 1 = F_{2k+1} \cdot L_{2k+2}$,

5. $F_{4k} - 1 = F_{2k+1} \cdot L_{2k-1}$,

6. $F_{4k+1} - 1 = F_{2k} \cdot L_{2k+1}$,

7. $F_{4k+2} - 1 = F_{2k} \cdot L_{2k+2}$,

8. $F_{4k+3} - 1 = F_{2k+2} \cdot L_{2k+1}$.

Using the property $(F_m, F_n) = F_{(m,n)}$ given in Theorem 10.3 of [9], it can be easily seen that the greatest common divisors of Fibonacci and Lucas numbers in the right side of the equalties in the above theorem are 1 or 3. Particularly,

$$\begin{aligned}(F_{2k-1}, L_{2k+1}) &= (F_{2k+1}, L_{2k}) = (F_{2k+1}, L_{2k+2}) \\ &= (F_{2k+1}, L_{2k-1}) = (F_{2k}, L_{2k+1}) \\ &= (F_{2k+2}, L_{2k+1}) = 1 \end{aligned} \quad (10)$$

and

$$(F_{2k+2}, L_{2k}) = (F_{2k}, L_{2k+2}) = 1 \text{ or } 3. \quad (11)$$

3. MAIN THEOREMS

Theorem 6. The only solutions of the Diophantine equation (4) in nonnegative integers $s \geq 0$, $y \geq 1$, $b \geq 2$, $n \geq 1$ and $(3, y) = 1$, are given by

$(n, s, y, b) =$
$(3,0,2,2), (6,0,2,4), (6,0,4,2), (12,2,2,5)$.

Proof. Assume that (n, s, y, b) is a solution of the equation (4). If $s = 0$, then we get $2F_n = y^b$ and therefore $F_n = 2^{b-1} \cdot \left(\frac{y}{2}\right)^b$. This equation has solutions only for $n = 3$ or $n = 6$ by Theorem 3. Thus, we can see by a simple computation that $(n, s, y, b) = (3,0,2,2), (6,0,2,4), (6,0,4,2)$.
From now on, assume that $s \geq 1$. On the other hand, it is clear that y is an even number. Say $y = 2^r x$ for some positive integers x and r such that $(x, 2) = 1$. Then we have the equation

$$F_n = 3^s \cdot 2^{rb-1} \cdot x^b. \qquad (12)$$

Let n be the smallest positive integer satisfying the equation (12). Since $b \geq 2$, it follows that 2^{rb-1} is an even number. Therefore $2|F_n$, which implies that $3|n$ by (7). Hence, $n = 3k$ for some positive integer k. Thus, we get the equation

$$F_n = F_{3k} = F_k\left(5{F_k}^2 + 3(-1)^k\right)$$
$$= 3^s \cdot 2^{rb-1} \cdot x^b \qquad (13)$$

by (9). Since $s \geq 1$, it follows that $3|F_n$ and thus it can be seen that $3|F_k$ and $3|\left(5{F_k}^2 + 3(-1)^k\right)$. Therefore, it should be $s \geq 2$. Then, the equation (13) can be written

$$\left(\frac{F_k}{3}\right)\left(\frac{5{F_k}^2+3(-1)^k}{3}\right) = 3^{s-2} \cdot 2^{rb-1} \cdot x^b. \qquad (14)$$

Also, it is obvious that $\left(\frac{F_k}{3}, \frac{5{F_k}^2+3(-1)^k}{3}\right) = 1$ and thus $3 \nmid \left(\frac{5{F_k}^2+3(-1)^k}{3}\right)$. Hence, we have

$$\frac{F_k}{3} = 3^{s-2}u^b \text{ and } \frac{5{F_k}^2+3(-1)^k}{3} = 2^{rb-1}v^b$$

or

$$\frac{F_k}{3} = 3^{s-2} \cdot 2^{rb-1} \cdot u^b \text{ and } \frac{5{F_k}^2+3(-1)^k}{3} = v^b$$

for some positive integers u and v such that $(u, v) = 1$, $uv = x$. In the first case, the equation $F_k = 3^{s-1}u^b$ has solution only for $k = 4$ or $k = 12$ by Theorem 4 since $s \geq 2$. If $k = 4$, then we get $s = u = 1$ and $b = 5$, $r = v = 1$ from the equality

$$\frac{5{F_k}^2+3(-1)^k}{3} = \frac{5{F_4}^2+3(-1)^4}{3} = 16 = 2^{rb-1} \cdot v^b .$$

Thus $(n, s, y, b) = (12,2,2,5)$ is a solution. In the second case, we have the equation $F_k = 3^{s-1} \cdot 2^{rb-1} \cdot u^b$, which is in the form (12). But, since $k < n$, this contradicts our assumption that n is the smallest positive integer satisfiying the equation $F_k = 3^{s-1} \cdot 2^{rb-1} \cdot u^b$.

As a result, the solutions (n, s, y, b) satisfying (4) are $(3,0,2,2), (6,0,2,4), (6,0,4,2)$ and $(12,2,2,5)$. Thus, the proof is completed.

Theorem 7. Let $s \geq 0, y \geq 1, b \geq 2, n \geq 2$ and $(3, y) = 1$. Then all solutions of the equation

$$F_n \pm 1 = 3^s \cdot y^b \qquad (15)$$

are

$F_4 + 1 = 4 = 3^0 \cdot 2^2$, $F_6 + 1 = 9 = 3^2 \cdot 1^b$,
$F_3 + 1 = 3 = 3^1 \cdot 1^b$

and

$F_5 - 1 = 4 = 3^0 \cdot 2^2$, $F_3 - 1 = 1 = 3^0 \cdot 1^b$,
$F_7 - 1 = 12 = 3^1 \cdot 2^2$.

Proof. Assume that (n, s, y, b) is a solution of the equation (15). If we divide n by 4, we can write $n = 4k + r$ with $0 \leq r \leq 3$ for some integers k, r. Thus, considering the equation (15) together with Theorem 5, we have the following cases:

i) $F_{4k} + 1 = F_{2k-1} \cdot L_{2k+1} = 3^s y^b$,

ii) $F_{4k+1} + 1 = F_{2k+1} \cdot L_{2k} = 3^s y^b$,

iii) $F_{4k+2} + 1 = F_{2k+2} \cdot L_{2k} = 3^s y^b$,

iv) $F_{4k+3} + 1 = F_{2k+1} \cdot L_{2k+2} = 3^s y^b$,

v) $F_{4k} - 1 = F_{2k+1} \cdot L_{2k-1} = 3^s y^b$,

vi) $F_{4k+1} - 1 = F_{2k} \cdot L_{2k+1} = 3^s y^b$,

vii) $F_{4k+2} - 1 = F_{2k} \cdot L_{2k+2} = 3^s y^b$,

viii) $F_{4k+3} - 1 = F_{2k+2} \cdot L_{2k+1} = 3^s y^b$.

Case i) In this case, since $3 \nmid F_{2k-1}$ and $3 \nmid L_{2k+1}$ by (7) and (8), it follows that $s = 0$. Then, using (10), we have the equations $F_{2k-1} = u^b$ and $L_{2k+1} = v^b$ for some integers u and v such that $(u, v) = 1$ and $y = uv$. By Theorem 1, it is seen that $k = 1, v = b = 2$ and therefore $u = 1$. Thus, $(n, s, y, b) = (4,0,2,2)$.

Case ii) Since $3 \nmid F_{2k+1}$ by (7), we have $F_{2k+1} = u^b, L_{2k} = 3^s v^b$ for some integers u and v such that $(u, v) = 1$ and $y = uv$. It can be seen that these equations have no solution by Theorem 1 and Theorem 4.

Case iii) We know that $(F_{2k+2}, L_{2k}) = 1$ or 3 by (11). Firstly, let $(F_{2k+2}, L_{2k}) = 1$. If k is odd, then $3 \mid F_{2k+2}$ and $3 \mid L_{2k}$ by (7) and (8) and thus $3 \mid (F_{2k+2}, L_{2k})$, which contradicts the fact that $(F_{2k+2}, L_{2k}) = 1$. Therefore k is even. Thus, since $3 \nmid F_{2k+2}$ and $3 \nmid L_{2k}$ by (7) and (8), it follows that $s = 0$. Then we have the equations $F_{2k+2} = u^b$ and $L_{2k} = v^b$ for some integers u and v such that $(u, v) = 1$ and $y = uv$. But, the equation $L_{2k} = v^b$ has no solutions by Theorem 2. Secondly, let $(F_{2k+2}, L_{2k}) = 3$. Then, it is obvious that $s \geq 2$, $\left(\frac{F_{2k+2}}{3}, \frac{L_{2k}}{3}\right) = 1$ and k is odd. In this case, we have the following equations

$$\frac{F_{2k+2}}{3} = u^b, \frac{L_{2k}}{3} = 3^{s-2} v^b$$

and

$$\frac{F_{2k+2}}{3} = 3^{s-2}, \frac{L_{2k}}{3} = v^b$$

for some integers u and v such that $(u, v) = 1$ and $y = uv$. In both cases, we get $k = 1$ by Theorem 4. Thus, making necessary calculation we obtain $(n, s, y, b) = (6,2,1, b)$.

Case iv) In this case, since $3 \nmid F_{2k+1}$ by (7), we have the equation $F_{2k+1} \cdot \frac{L_{2k+2}}{3^s} = y^b$. Using (10), this equation implies that there exist positive integers u and v such that $(u, v) = 1$, $y = uv$, $F_{2k+1} = u^b$ and $\frac{L_{2k+2}}{3^s} = v^b$. By Theorem 1, the equation $F_{2k+1} = u^b$ has a solution only for $k = 0$. Thus, we get $(n, s, y, b) = (3,1,1, b)$.

Since the proof of the last four cases is similar to that of the first four cases, we omit them. Considering a similar argument, we see that the solutions of the equation (15) is $(n, s, y, b) = (5,0,2,2), (3,0,1, b), (7,1,2,2)$. Thus, the proof is completed.

4. CONCLUDING REMARK

It is an open problem to find all solutions of the equation $F_n \pm F_m = 3^s \cdot y^b$ in nonnegative integers $s \geq 0, y \geq 1, b \geq 2, n \geq m \geq 1$ and $(3, y) = 1$. Theorem 6 and Theorem 7 can be useful to find the solutions of the equation $F_n \pm F_m = 3^s \cdot y^b$.

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