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Spinor Equations of Successor Curves

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Abstract

The aim of this study is to give spinor representation of successor curves in three-dimensional Euclidean space. In this study, the spinor representations of a curve with arc-length parameter and a successor curve of this curve with some arc-length parameter in three dimensional Euclidean space have been studied. For this, first of all, the curve with unit speed and its successor curve have been corresponded to two different spinors. Then, using the relationships between these curves, the relationships between the spinors corresponding to these curves have been given. Therefore, geometric interpretations of these curves and corresponding spinors have been made. In addition, different spinor equations of the mates and derivatives of these spinors have been examined and geometric interpretations of these spinor equations have been given. Then, spinor equations have been obtained in case the successor curves are helices. Consequently, two examples have been given.

1. Introduction

Curve theory is one of the most studied areas of differential geometry, with studies in different dimensions and spaces. Curve and curvature studies by Newton and Leibniz form the basis of curve theory. After giving the definition of curvature by Euler in the 18th century, space curves were defined by Monge. Later, in the 19th century, the equations, known today as Serret-Frenet frame and formulas, were studied by Serret and Frenet at different times. In general, the geometric locus of different positions taken by a moving point in space during motion is a curve. In this motion, the parameter range of the curve represents the time elapsed during the motion. In addition, the parameter of the curve has an important place in characterizing the curve. That is, the parameter of a given curve has many differences depending on whether the curve is parameterized in terms of arc length. The parametrization of the curve in terms of arc length provides great convenience in the characterization of the curve. On the other hand, considering any two curves in space, various special curves are defined by establishing some different relations between the Frenet vectors at the opposite points of these curves such as, in 1850, Bertrand curve pair was defined by establishing a special relationship between the normal vectors of any two curves. Another special curve pair is the Mannheim curve pair, which was given by A. Mannheim in 1878 and obtained by establishing a relationship between the normal of one of the two curves and the binormal of the other [23]. Another is the involute evolute curves obtained by establishing a special relationship between the tangents of any two curves [14]. There are a lot of studies about curve theory [4, 17]. The curves that form the basis of this study are the successor curves. The definition of successor curves as "Consider a unit speed curve (α) in three-dimensional Euclidean space. If the normal vector field of a curve (β) with the same arc-length parameter as the curve (α) is the tangent vector field of the curve (α), the curve (β) is called a successor curve of the curve (α). Every Frenet curve has a family of successor curves" was given by Menninger where both the curve (α) and the curve (β) are unit speed curves with the same arc-length parameter. Therefore, there are relationships between Frenet frames and Frenet curvatures of both curves [13]. Later, Masal obtained relations depending on the ground vectors of the successor curves by defining the successor planes and investigating the geometric meanings of the successor curvatures [13]. Before from that, the predecessor transformation, as opposed to the successor transformation, although not well defined in general, was given by Bilinski [2].

Spinors are physical structures used in many fields of applied sciences. It is used in physics, especially in quantum mechanics, applications of spinor theory, electron spin and theory of relativity. The wave function of a particle with a spin of $1/2$ is called a spinor. Also, the application of spinors in electromagnetic theory is very important. A spin structure in four-dimensional space is an extension of spinors to

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obtain Dirac spinors in physics [5, 11, 22]. Spinors are column vectors and act on Pauli spin matrices that are $2x2$ dimensional complex matrices. In quantum theory, spin is expressed in Pauli matrices. Cartan [3] first studied spinors geometrically. In that study by Cartan the spinor representations of basic geometric definitions were given. In Cartan's study, it was emphasized that the set of isotropic vectors in the vector space $\mathbb{C}^3$ creates a two-dimensional surface in the space $\mathbb{C}^2$. In that study of Cartan, it is seen that each isotropic vector in the complex vector space $\mathbb{C}^3$ corresponds to a vector with two components in the space $\mathbb{C}^2$. Cartan named these two-dimensional vectors as spinors [3]. The spinor algebra with two complex components acting on Pauli matrices representing $SU(2)$, a group of $2x2$ dimensional unitary matrices, gives a different representation of rotations in three-dimensional real vector space. Using this information, Vivarelli established a relationship between spinors and quaternions and obtained the quaternion representation of rotations in three-dimensional Euclidean space with spinors [21]. Then, the spinor representation of curves in three-dimensional Euclidean space was given by Torres del Castillo and Barrales [18]. They expressed the Frenet vectors and curvatures of curves in terms of spinors in that study [18]. Therefore, that study greatly has contributed to this our study. Based on this study, a spinor formulation of the Darboux frame of a curve on a surface of three-dimensional Euclidean space and the relationship between Frenet and Darboux frames was given by Kişi and Tosun [10]. After that, in another study, the spinor formulation of Bishop's frame of curves in three-dimensional Euclidean space was obtained [20]. In addition, Erişir and Kardağ obtained spinor equations of involute-evolute curve pairs in three-dimensional Euclidean space [7]. In another study, spinor formulation of Bertrand curves was given [8]. Moreover, the spinor formulations of some special curves in three-dimensional Minkowski space were obtained based on these studies in three-dimensional Euclidean space [1, 6, 9].
In this study, a curve (α) with unit speed and a successor curve (β) with the same arc-length parameter of the curve (α) have been considered. In addition, two spinor have been corresponded to the curve (α) and the successor curve (β) of the curve (α). After that, considering the relationships between these curves, the relationships between the spinors corresponding to these curves have been obtained. In addition, the geometric interpretations have been given for the angles between these spinors. Then, the spinor equations have been obtained for the mates and derivatives of these spinors corresponding to the curve (α) and the successor curve (β), and the geometric interpretations of these equations have been made. After that, considering that the successor curve (β) is helix, some theorems and results have been obtained for the spinor equations of this curve. Consequently, two examples have been given.

2. Preliminaries

In this section, firstly, the basic definitions and theorems about successor curves have been given. Then, the basic definitions and theorems about spinors introduced by Cartan [3], which is a fundamental study for spinors, have been mentioned. In addition, the spinor equations given by Torres del Castillo and Barrales have been expressed [18]. Consequently, the spinor formulation of curves in three-dimensional Euclidean space is given.

Definition 2.1. *Consider that the curves $\alpha : I \to \mathbb{E}^3$ and $\beta : I \to \mathbb{E}^3$ have the same arc-length parameter. In that case, if the tangent vector field of the curve (α) is the normal vector field of the curve (β), then the curve (β) is defined as a successor curve of the curve (α). Each Frenet curve has a family of successor curves [13].*

Theorem 2.2. *Let $\alpha, \beta : I \to \mathbb{E}^3$ be two curves which have same arc-length parameter and the curve (β) be the successor curve of the curve (α). In that case, we consider that the Frenet system of the curve (α) is $F = \{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}, \kappa, \tau\}$ and the successor system of the successor curve (β) is $F_1 = \{\boldsymbol{T}_1, \boldsymbol{N}_1, \boldsymbol{B}_1, \kappa_1, \tau_1\}$. Therefore, there are the relationships between of these systems as*

$$\begin{pmatrix} \boldsymbol{T}_1 \\ \boldsymbol{N}_1 \\ \boldsymbol{B}_1 \end{pmatrix} = \begin{pmatrix} 0 & -\cos\vartheta & \sin\vartheta \\ 1 & 0 & 0 \\ 0 & \sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} \boldsymbol{T} \\ \boldsymbol{N} \\ \boldsymbol{B} \end{pmatrix} \tag{2.1}$$

and

$$\begin{pmatrix} \kappa_1 \\ \tau_1 \end{pmatrix} = \kappa \begin{pmatrix} \cos\vartheta \\ \sin\vartheta \end{pmatrix} \tag{2.2}$$

where the angle ϑ is the angle between the binormal vectors $\boldsymbol{B}$ and $\boldsymbol{B}_1$. Moreover, for the torsion τ of the curve (α) the equation

$$\vartheta(s) = \vartheta_0 + \int \tau(s)ds$$

is hold where $\vartheta_0 = constant \in \mathbb{R}$ [13].

Theorem 2.3. *Suppose that the curve (β) is successor curve of the curve (α) and the vector $\boldsymbol{D}_1$ is Darboux vector of the successor curve (β). Therefore, Darboux vector is*

$$\boldsymbol{D}_1 = \kappa \boldsymbol{B} \tag{2.3}$$

where κ and $\boldsymbol{B}$ are the curvature and binormal vector field of the curve (α), respectively [13].

Theorem 2.4. *Consider that the curve (β) is successor curve of the curve (α) and the successor system of the successor curve (β) is $F_1 = \{\boldsymbol{T}_1, \boldsymbol{N}_1, \boldsymbol{B}_1, \kappa_1, \tau_1\}$. If the angle of intersection of tangent vector field $\boldsymbol{T}_1$ with a constant vector is always constant, then the successor curve (β) is defined helix [12, 13].*

Theorem 2.5. *Let the curve (β) be successor curve of the curve (α). In that case, if the successor curve (β) is helix, then the ratio of the curvatures of (β) $\frac{\kappa_1}{\tau_1}$ is constant [12].*

Theorem 2.6. *Consider that the curve (β) is successor curve of the curve (α). The successor curve (β) is helix, then the necessary and sufficient condition is that the curve (α) is planar curve [13].*

Spinors construct a vector space usually built on complex numbers with the aid of a linear group representation of the spin group. Cartan [3] expressed the spinors over the complex numbers geometrically. Therefore, Cartan gave that a vector γ with two complex components corresponds to an isotropic vector $\boldsymbol{x}=(x_1,\, x_2,\, x_3)$ in three dimensional complex vector space $\mathbb{C}^3$. In that case, the isotropic vectors in $\mathbb{C}^3$ form a surface with two dimensional in $\mathbb{C}^2$. Now, we consider that this surface is written by the parameters γ_1 and γ_2, therefore, $x_1={\gamma_1}^2-{\gamma_2}^2$, $x_2=i({\gamma_1}^2+{\gamma_2}^2)$, $x_3=-2\gamma_1\gamma_2$ and $\gamma_1=\pm\sqrt{\frac{x_1-ix_2}{2}}$, $\gamma_2=\pm\sqrt{\frac{-x_1-ix_2}{2}}$ [3]. Cartan called these complex vectors mentioned here as spinors

$$\gamma=\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}$$

[3]. With the aid of the study [3], in [18] it was matched the isotropic vector $\boldsymbol{a}+i\boldsymbol{b}\in\mathbb{C}^3$ with spinor $\gamma=(\gamma_1,\gamma_2)$ where $\boldsymbol{a},\, \boldsymbol{b}\in\mathbb{R}^3$. In that case, considering the Pauli matrices (P_1,P_2,P_3), the $2x2$ dimensional complex symmetric matrices $\boldsymbol{\sigma}$, can be created as

$$\sigma_1=CP_1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \sigma_2=CP_2=\begin{pmatrix}i & 0\\ 0 & i\end{pmatrix},\quad \sigma_3=CP_3=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}$$

where $C=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ [15, 16, 18]. In that case, in [18] the spinor equations are given by

$$\begin{aligned}\boldsymbol{a}+i\boldsymbol{b}&=\gamma^t\sigma\gamma,\\ \boldsymbol{c}&=-\hat{\gamma}^t\sigma\gamma\end{aligned}$$

where $\boldsymbol{a}+i\boldsymbol{b}$ is the isotropic vector in the space $\mathbb{C}^3$, $\boldsymbol{c}\in\mathbb{R}^3$ and the spinor mate $\hat{\gamma}$ of the spinor γ is

$$\hat{\gamma}=-\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\overline{\gamma}=-\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\overline{\gamma_1}\\ \overline{\gamma_2}\end{pmatrix}=\begin{pmatrix}-\overline{\gamma_2}\\ \overline{\gamma_1}\end{pmatrix}.$$

For the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ we know that these vectors have the same length $\|\boldsymbol{a}\|=\|\boldsymbol{b}\|=\|\boldsymbol{c}\|=\overline{\gamma}^t\gamma$ and are orthogonal to each other. In addition to that, the triads $\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}$, $\{\boldsymbol{b},\boldsymbol{c},\boldsymbol{a}\}$ and $\{\boldsymbol{c},\boldsymbol{a},\boldsymbol{b}\}$ correspond to different spinors [18].

Proposition 2.7. *Let two arbitrary spinors be γ and ψ. Then, the following statements hold;*

$$\begin{array}{ll} i) & \overline{\psi^t\sigma\gamma}=-\hat{\psi}^t\sigma\hat{\gamma}\\ ii) & \widehat{\lambda\psi+\mu\gamma}=\overline{\lambda}\hat{\psi}+\overline{\mu}\hat{\gamma}\\ iii) & \hat{\hat{\gamma}}=-\gamma\\ iv) & \psi^t\sigma\gamma=\gamma^t\sigma\psi\end{array}$$

where $\lambda,\, \mu\in\mathbb{C}$ and "$-$" is complex conjugate [18].

Now, let a curve parameterized by arc-length be $\alpha: I\to\mathbb{E}^3$, $(I\subseteq\mathbb{R})$. Therefore, $\|\alpha'(s)\|=1$ where s is the arc-length parameter of (α). In addition to that, consider that the Frenet frame of this curve $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$ and the spinor ξ represents Frenet frame $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$. Therefore, from equation (2.3) the following equations can be written as

$$\begin{aligned}\boldsymbol{N}+i\boldsymbol{B}&=\xi^t\sigma\xi=(\xi_1^2-\xi_2^2,\, i(\xi_1^2+\xi_2^2),\, -2\xi_1\xi_2),\\ \boldsymbol{T}&=-\widehat{\xi}^t\sigma\xi=(\xi_1\overline{\xi_2}+\overline{\xi_1}\xi_2,\, i(\xi_1\overline{\xi_2}-\overline{\xi_1}\xi_2),\, |\xi_1|^2-|\xi_2|^2)\end{aligned}$$

with $\overline{\xi}^t\xi=1$ [18]. Moreover, the following theorem can be given.

Theorem 2.8. *If the spinor ξ with two complex components represents Frenet frame $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$ of a curve (α) parameterized by its arc-length s, the Frenet curvatures are equivalent to the single spinor equation*

$$\frac{d\xi}{ds}=\frac{1}{2}(-i\tau\xi+\kappa\hat{\xi}) \tag{2.4}$$

where κ and τ are the curvature and torsion of (α), respectively [18].

3. Main Theorems and Proofs

3.1. Spinor Representation of Successor Curves

Let $\alpha: I\to\mathbb{E}^3$ be a curve with unit speed which has the arc-length parameter s and Frenet frame of this curve be $\{\boldsymbol{T},\boldsymbol{N},\boldsymbol{B}\}$. Therefore, we know that if the spinor corresponds to Frenet frame $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$ of the curve (α), then the spinor equations of these Frenet vectors can be written

$$\begin{aligned}\boldsymbol{N}+i\boldsymbol{B}&=\xi^t\sigma\xi,\\ \boldsymbol{T}&=-\hat{\xi}^t\sigma\xi\end{aligned}$$

[18]. In addition, the vectors of Frenet frame $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$ can be written as

$$\begin{aligned}\boldsymbol{T}&=-\hat{\xi}^t\sigma\xi,\\ \boldsymbol{N}&=\frac{1}{2}(\xi^t\sigma\xi-\hat{\xi}^t\sigma\hat{\xi}),\\ \boldsymbol{B}&=\frac{-i}{2}(\xi^t\sigma\xi+\hat{\xi}^t\sigma\hat{\xi})\end{aligned}\tag{3.1}$$

in terms of spinors, separately [7].

Now, consider that the curve (β), which has the same arc-length-parameter s with the curve (α), is a successor curve of the curve (α). Moreover, suppose that Frenet frame of the successor curve (β) is $\{\boldsymbol{T}_1,\boldsymbol{N}_1,\boldsymbol{B}_1\}$ and the spinor ϕ corresponds to Frenet frame $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$ of the successor curve (β) where we know that the different spinors correspond to Frenet frames $\{\boldsymbol{T}_1,\boldsymbol{N}_1,\boldsymbol{B}_1\}$, $\{\boldsymbol{N}_1,\boldsymbol{B}_1,\boldsymbol{T}_1\}$ and $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$. In that case, we can write

$$\begin{aligned}\boldsymbol{B}_1+i\boldsymbol{T}_1&=\phi^t\sigma\phi,\\ \boldsymbol{N}_1&=-\hat{\phi}^t\sigma\phi\end{aligned}\tag{3.2}$$

and give the following theorem.

Theorem 3.1. *Suppose that the curve (β) is a successor curve of the curve (α) and the curves (α) and (β) have the same arc-length parameter s. In that case, if the spinor ϕ corresponds to Frenet frame $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$ of the successor curve (β), then the spinor equation of the curvatures $\{\kappa_1,\tau_1\}$ of the successor curve (β) is*

$$\frac{d\phi}{ds}=\frac{\tau_1-i\kappa_1}{2}\hat{\phi}.\tag{3.3}$$

Proof. Let ϕ be the spinor corresponding to Frenet frame $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$ of the successor curve (β) of the curve (α). Therefore, since the spinor pair $\{\phi,\hat{\phi}\}$ forms a basis in the spinor space, the spinor $\frac{d\phi}{ds}$ can be written

$$\frac{d\phi}{ds}=f\phi+g\hat{\phi}\tag{3.4}$$

where f and g are two arbitrary complex functions. On the other hand, if we take derivative of the complex vector $\boldsymbol{B}_1+i\boldsymbol{T}_1$ in the equation (3.2) in terms of the arc-length parameter $s\in I$, we get

$$\frac{d\boldsymbol{B}_1}{ds}+i\frac{d\boldsymbol{T}_1}{ds}=\frac{d\phi}{ds}^t\sigma\phi+\phi^t\sigma\frac{d\phi}{ds}$$

and with the aid of the equation (3.4) we have

$$(-\tau_1+i\kappa_1)\boldsymbol{N}_1=f\phi^t\sigma\phi+g\hat{\phi}^t\sigma\phi+f\phi^t\sigma\phi+g\phi^t\sigma\hat{\phi}.$$

In addition to that, if we use the option $iv)$ in Proposition 2.7, we have

$$(-\tau_1+i\kappa_1)\,\boldsymbol{N}_1=2f(\phi^t\sigma\phi)+2g(\hat{\phi}^t\sigma\phi).$$

Therefore, we obtain $f=0$ and $g=\frac{\tau_1-i\kappa_1}{2}$. As a result, we get $\frac{d\phi}{ds}=\left(\frac{\tau_1-i\kappa_1}{2}\right)\hat{\phi}$. Consequently, the proof is completed by expressing the curvature κ_1 and torsion τ_1 of the successor curve (β) as a single spinor equation. □

Theorem 3.2. *Let (α) and (β) be two curves which have the same arc-length parameter in Euclidean space $\mathbb{E}^3$ and the curve (β) be the successor curve of the curve (α). Therefore, the spinor equations of the Frenet frame $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$ of the successor curve (β) are*

$$\begin{aligned}\boldsymbol{B}_1&=\frac{1}{2}(\phi^t\sigma\phi-\hat{\phi}^t\sigma\hat{\phi}),\\ \boldsymbol{T}_1&=\frac{-i}{2}(\phi^t\sigma\phi+\hat{\phi}^t\sigma\hat{\phi}),\\ \boldsymbol{N}_1&=-\hat{\phi}^t\sigma\phi.\end{aligned}\tag{3.5}$$

Proof. Consider that the curve (β) is the successor curve of the curve (α) and the spinor ϕ corresponds to Frenet frame $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$ of the successor curve (β). Therefore, considering the spinor equation $\boldsymbol{B}_1+i\boldsymbol{T}_1=\phi^t\sigma\phi$ in the equation (3.2) we see that

$$\begin{aligned}\boldsymbol{T}_1&=\mathrm{Im}(\phi^t\sigma\phi),\\ \boldsymbol{B}_1&=\mathrm{Re}(\phi^t\sigma\phi).\end{aligned}$$

In that case, we obtain $\boldsymbol{T}_1=\frac{-i}{2}(\phi^t\sigma\phi-\overline{\phi^t\sigma\phi})$, $\boldsymbol{B}_1=\frac{1}{2}(\phi^t\sigma\phi+\overline{\phi^t\sigma\phi})$ and, consequently, considering the option $i)$ in Proposition 2.7 we have

$$\begin{aligned}\boldsymbol{T}_1&=\frac{-i}{2}(\phi^t\sigma\phi+\hat{\phi}^t\sigma\hat{\phi}),\\ \boldsymbol{B}_1&=\frac{1}{2}(\phi^t\sigma\phi-\hat{\phi}^t\sigma\hat{\phi}).\end{aligned}$$

We also know the spinor equation of the normal vector field $\boldsymbol{N}_1=-\hat{\phi}^t\sigma\phi$ in the equation (3.2). □

In addition to Theorem 3.2, we see that Frenet frame $\{\boldsymbol{B}_1, \boldsymbol{T}_1, \boldsymbol{N}_1\}$ of the successor curve (β) can be written

$$\begin{aligned}
\boldsymbol{B}_1 &= \frac{1}{2}\left(\phi_1^2 - \phi_2^2 - \overline{\phi_2}^2 + \overline{\phi_1}^2, i(\phi_1^2 + \phi_2^2 - \overline{\phi_1}^2 - \overline{\phi_2}^2), -2\phi_1\phi_2 - 2\overline{\phi_1}\,\overline{\phi_2}\right), \\
\boldsymbol{T}_1 &= \frac{-i}{2}\left(\phi_1^2 - \phi_2^2 + \overline{\phi_2}^2 - \overline{\phi_1}^2, i(\phi_1^2 + \phi_2^2 + \overline{\phi_1}^2 + \overline{\phi_2}^2), -2\phi_1\phi_2 + 2\overline{\phi_1}\,\overline{\phi_2}\right), \\
\boldsymbol{N}_1 &= \left(\phi_1\overline{\phi_2} + \overline{\phi_1}\phi_2, i(\phi_1\overline{\phi_2} - \overline{\phi_1}\phi_2), |\phi_1|^2 - |\phi_2|^2\right)
\end{aligned} \tag{3.6}$$

in terms of the components of the spinor $\phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$ with easy calculations.

Now, we give the relationship between the spinors ξ and ϕ with following theorem.

Theorem 3.3. *Suppose that (α) and (β) have the same arc-length parameter in Euclidean space $\mathbb{E}^3$ and the curve (β) is the successor curve of the curve (α). In addition, the spinor pair (ξ, ϕ) corresponds to the Frenet frames $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$ and $\{\boldsymbol{B}_1, \boldsymbol{T}_1, \boldsymbol{N}_1\}$ of the curves (α, β), respectively. Therefore, there is the relationship between these spinors ξ and ϕ as*

$$\xi = \pm e^{i\left(\frac{\pi}{4} - \frac{\vartheta}{2}\right)}\phi \tag{3.7}$$

where the angle ϑ is the angle between the binormal vector fields $\boldsymbol{B}$ and $\boldsymbol{B}_1$.

Proof. Let ξ be the spinor corresponding to the Frenet frame $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$ of the curve (α) and ϕ be the spinor corresponding to the Frenet frame $\{\boldsymbol{B}_1, \boldsymbol{T}_1, \boldsymbol{N}_1\}$ of the successor curve (β). In that case, we can write for the complex vector $\boldsymbol{B}_1 + i\boldsymbol{T}_1 \in \mathbb{C}^3$

$$\boldsymbol{B}_1 + i\boldsymbol{T}_1 = -ie^{i\vartheta}(\boldsymbol{N} + i\boldsymbol{B})$$

and

$$\xi^t \sigma \xi = e^{i\left(\frac{\pi}{2} - \vartheta\right)}\phi^t \sigma \phi \tag{3.8}$$

with the aid of the equation (2.1). Therefore, we obtain that

$$\begin{aligned}
\xi_1 &= \pm\, e^{i\left(\frac{\pi}{4} - \frac{\vartheta}{2}\right)}\phi_1, \\
\xi_2 &= \pm\, e^{i\left(\frac{\pi}{4} - \frac{\vartheta}{2}\right)}\phi_2
\end{aligned}$$

and, consequently

$$\xi = \pm\, e^{i\left(\frac{\pi}{4} - \frac{\vartheta}{2}\right)}\phi$$

where the spinors ξ and $-\xi$ correspond to the same Frenet frame $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$ and, similarly, the spinors ϕ and $-\phi$ correspond to the same Frenet frame $\{\boldsymbol{B}_1, \boldsymbol{T}_1, \boldsymbol{N}_1\}$. □

Now, we give a geometric interpretation of spinor representations of the successor curves with following corollary.

Corollary 3.4. *Consider that the curve (β) is the successor curve of the curve (α) and the spinor pair (ξ, ϕ) corresponds to the Frenet frames $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$ and $\{\boldsymbol{B}_1, \boldsymbol{T}_1, \boldsymbol{N}_1\}$ of the curves (α, β), respectively. Therefore, the angle between the spinors ξ and ϕ is $\left(\frac{\pi}{4} - \frac{\vartheta}{2}\right)$ where the angle ϑ is the angle between the binormal vector fields $\boldsymbol{B}$ and $\boldsymbol{B}_1$.*

In addition to that, similar to Theorem 3.3 we can give a relationship between the mates of spinors ξ and ϕ with following corollary.

Corollary 3.5. *Suppose that the spinors ξ and ϕ correspond to the Frenet frames $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$ of the curve (α) and $\{\boldsymbol{B}_1, \boldsymbol{T}_1, \boldsymbol{N}_1\}$ of the successor curve (β), respectively. In that case, there is the relationship between the mates of spinors ξ and ϕ as*

$$\hat{\phi} = \pm e^{i\left(\frac{\pi}{4} - \frac{\vartheta}{2}\right)}\hat{\xi}$$

where the angle ϑ is the angle between the binormal vector fields $\boldsymbol{B}$ and $\boldsymbol{B}_1$.

Proof. Let ξ and ϕ be two spinors corresponding to the Frenet frames of the (α) and the successor curve (β). In that case, if the operation of spinor mate is applied to both sides of the equation (3.7), we get $\hat{\xi} = \pm\left(\widehat{e^{i\left(\frac{\pi}{4} - \frac{\vartheta}{2}\right)}\phi}\right)$. We know *ii)* in Proposition 2.7, therefore, we have $\hat{\xi} = \pm\overline{e^{i\left(\frac{\pi}{4} - \frac{\vartheta}{2}\right)}}\hat{\phi}$ and, consequently,

$$\hat{\phi} = \pm e^{i\left(\frac{\pi}{4} - \frac{\vartheta}{2}\right)}\hat{\xi}.$$

□

Therefore, we can obtain a geometric interpretation of spinor representations of the successor curves below.

Corollary 3.6. *Suppose that the spinors ξ and ϕ correspond to the Frenet frames $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$ of the curve (α) and $\{\boldsymbol{B}_1, \boldsymbol{T}_1, \boldsymbol{N}_1\}$ of the successor curve (β), respectively. In this case, while the spinor ϕ rotates at an angle $\left(\frac{\pi}{4} - \frac{\vartheta}{2}\right)$ to the spinor ξ, the spinor $\hat{\phi}$ makes a rotation in the* ***negative*** *direction with the* ***same*** *angle to the spinor $\hat{\xi}$.*

Corollary 3.7. *There is the relationship between the derivative spinors $\frac{d\xi}{ds}$ and $\frac{d\phi}{ds}$ that*

$$\frac{d\xi}{ds} = \pm e^{i\left(\frac{\pi}{4} - \frac{\vartheta}{2}\right)}\left(\frac{d\phi}{ds} - \frac{i}{2}\frac{d\vartheta}{ds}\phi\right)$$

where $\frac{d\vartheta}{ds} = \tau$ is the torsion of the curve (α).

3.2. Some Applications

In this section, firstly, we give the spinor equations of the Darboux vector of the successor curve (β). After that, we assume that the successor curve (β) of the curve (α) is helix and we give the spinor equations in that case. Therefore, we can express the following theorems and corollaries.

Theorem 3.8. *Let the curve (β) be the successor curve of the curve (α) and (α), (β) be the curves which have the same arc-length parameter s. If we consider that the spinor ϕ corresponds to Frenet frame $\{\boldsymbol{B}_1, \boldsymbol{T}_1, \boldsymbol{N}_1\}$ of the successor curve (β) and Darboux vector of the successor curve (β) is $\boldsymbol{D}_1$, then the spinor equation of Darboux vector of the successor curve is*

$$\boldsymbol{D}_1 = \frac{\kappa}{2}\left[e^{-i\vartheta}\phi^t\sigma\phi - e^{i\vartheta}\hat{\phi}^t\sigma\hat{\phi}\right]$$

where ϑ is the angle between the binormal vector fields $\boldsymbol{B}$ of the curve (α) and $\boldsymbol{B}_1$ of the successor curve (β).

Proof. Suppose that the successor curve of the curve (α) is (β), the spinor ϕ corresponds to Frenet frame $\{\boldsymbol{B}_1, \boldsymbol{T}_1, \boldsymbol{N}_1\}$ of the successor curve (β) and Darboux vector of the successor curve (β) is $\boldsymbol{D}_1$. On the other hand, we know that Darboux vector of the successor curve is $\boldsymbol{D}_1 = \tau_1\boldsymbol{T}_1 + \kappa_1\boldsymbol{B}_1$. Therefore, considering the equations (2.2) and (3.6) we have

$$\boldsymbol{D}_1 = \frac{1}{2}\left[(\kappa_1 - i\tau_1)\phi^t\sigma\phi - (\kappa_1 + i\tau_1)\hat{\phi}^t\sigma\hat{\phi}\right]$$

and consequently,

$$\boldsymbol{D}_1 = \frac{\kappa}{2}\left[e^{-i\vartheta}\phi^t\sigma\phi - e^{i\vartheta}\hat{\phi}^t\sigma\hat{\phi}\right]$$

where the angle ϑ is the angle between the binormal vector fields $\boldsymbol{B}$ and $\boldsymbol{B}_1$. □

We can give the following corollary with the aid of the equations (2.3) and (3.1), and Theorem 3.3.

Corollary 3.9. *Consider that the curve (β) is the successor curve of the curve (α) and (α, β) have the same arc-length parameter s. Now, we assume that the spinor ϕ corresponds to Frenet frame $\{\boldsymbol{B}_1, \boldsymbol{T}_1, \boldsymbol{N}_1\}$ of the successor curve (β) and Darboux vector of the successor curve (β) is $\boldsymbol{D}_1$. In that case, Darboux vector of the successor curve (β) can be written as*

$$\boldsymbol{D}_1 = -\frac{i\kappa}{2}\left(\xi^t\sigma\xi + \hat{\xi}^t\sigma\hat{\xi}\right)$$

in terms of the spinor ξ corresponding to the curve (α).

Now, we consider that the successor curve (β) of the curve (α) is helix and we give the spinor equations in that case with following theorems and corollaries.

Theorem 3.10. *Suppose that the curve (β) is the successor curve of the curve (α) and the successor curve is helix. In addition, the spinor ϕ corresponds to the Frenet frame $\{\boldsymbol{B}_1, \boldsymbol{T}_1, \boldsymbol{N}_1\}$ of the successor curve (β). Therefore, the spinor $\frac{d\phi}{ds}$ is*

$$\frac{d\phi}{ds} = \frac{\tau_1}{2\cos\theta}e^{-i\theta}\hat{\phi} \tag{3.9}$$

where s is the arc-length parameter of both the curve (α) and the successor curve (β), τ_1 is the torsion of the successor curve (β), and $\theta = \arccos\langle\boldsymbol{T}_1, \boldsymbol{U}\rangle = constant$.

Proof. Let the successor curve (β) of the curve (α) be helix and ϕ be corresponds to the Frenet frame $\{\boldsymbol{B}_1, \boldsymbol{T}_1, \boldsymbol{N}_1\}$ of the successor curve (β). In that case, if the equation $\frac{\kappa_1}{\tau_1} = \tan\theta = constant$ is considered since the successor curve is helix, the equation (3.3) can be written as

$$\frac{d\phi}{ds} = \frac{\tau_1 - i\tan\theta\,\tau_1}{2}\hat{\phi} = \frac{\tau_1}{2\cos\theta}(\cos\theta - i\sin\theta)\hat{\phi}$$

and consequently,

$$\frac{d\phi}{ds} = \frac{\tau_1}{2\cos\theta}e^{-i\theta}\hat{\phi}.$$

□

We know that if the successor curve (β) of the curve (α) is helix, then the curve (α) is planar curve and $\tau = 0$ in Theorem 2.6. Therefore with the aid of the equation (2.4) we can give the following corollary.

Corollary 3.11. *Consider that the curve (β) is the successor curve of the curve (α) and the successor curve is helix. In addition, the spinor ξ corresponds to the Frenet frame $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$ of the curve (α). In that case, the spinor $\frac{d\xi}{ds}$ can be written that*

$$\frac{d\xi}{ds} = \frac{\kappa}{2}\hat{\xi}$$

where s is the arc-length parameter of both the curve (α) and the successor curve (β) and κ is the curvature of the curve (α).

Corollary 3.12. *Consider that the spinors ξ and ϕ correspond to the Frenet frames $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$ of the curve (α) and $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$ of the successor curve (β), selected as helix, respectively. Therefore, there is the relationship between the spinors $\frac{d\xi}{ds}$ and $\frac{d\phi}{ds}$ as*

$$\frac{d\phi}{ds}=\pm\frac{\sin\vartheta}{\cos\theta}e^{i\left(\frac{\pi}{4}-\frac{\vartheta}{2}-\theta\right)}\frac{d\xi}{ds}$$

where the angle ϑ is the angle between the binormal vector fields $\boldsymbol{B}$ and $\boldsymbol{B}_1$ and $\theta=\arccos\langle\boldsymbol{T}_1,\boldsymbol{U}\rangle=constant$.

Corollary 3.13. *Suppose that the curve (β) is the successor curve of the curve (α) and the spinor ϕ corresponds to the Frenet frame $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$ of the successor curve (β), respectively. In that case, the successor curve (β) is helix with constant angle θ and axis $\boldsymbol{U}$, then the necessary and sufficient condition is that the constant vector $\boldsymbol{U}$ can be written*

$$\boldsymbol{U}=\frac{1}{2}\left(e^{i\left(\theta-\frac{\pi}{2}\right)}\phi^t\sigma\phi+e^{-i\left(\theta-\frac{\pi}{2}\right)}\hat{\phi}^t\sigma\hat{\phi}\right). \tag{3.10}$$

Proof. $(\Rightarrow)$: Let the successor curve (β) of the curve (α) be helix. In that case, there is a constant vector $\boldsymbol{U}$ making a constant angle θ with tangent vector $\boldsymbol{T}_1$ at all points of the curve (β) and it can be written $\boldsymbol{U}=\cos\theta\boldsymbol{T}_1+\sin\theta\boldsymbol{B}_1$. On the other hand, consider that the spinor ϕ corresponds to the Frenet frame $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$ of the successor curve (β). Therefore, if we use the equation (3.5) in the equation $\boldsymbol{U}$, then we get

$$\boldsymbol{U}=\frac{1}{2}\left(e^{i\left(\theta-\frac{\pi}{2}\right)}\phi^t\sigma\phi+e^{-i\left(\theta-\frac{\pi}{2}\right)}\hat{\phi}^t\sigma\hat{\phi}\right).$$

Now, if we take the derivative of the vector $\boldsymbol{U}$ with respect to the arc-length parameter s, then we have

$$\boldsymbol{U}'=\frac{1}{2}\left[e^{i\left(\theta-\frac{\pi}{2}\right)}\left(\left(\frac{d\phi}{ds}\right)^t\sigma\phi+\phi^t\sigma\left(\frac{d\phi}{ds}\right)\right)+e^{-i\left(\theta-\frac{\pi}{2}\right)}\left(\left(\widehat{\frac{d\phi}{ds}}\right)^t\sigma\hat{\phi}+\hat{\phi}^t\sigma\left(\widehat{\frac{d\phi}{ds}}\right)\right)\right].$$

and with the aid of the equation (3.9) we get

$$\begin{aligned}\boldsymbol{U}'&=\frac{i}{2}\left[-\frac{\tau_1}{2\cos\theta}\left(\hat{\phi}^t\sigma\phi+\phi^t\sigma\hat{\phi}\right)+\frac{\tau_1}{2\cos\theta}\left(\hat{\phi}^t\sigma\phi+\phi^t\sigma\hat{\phi}\right)\right],\\ \boldsymbol{U}'&=\boldsymbol{0}\end{aligned}$$

and consequently, the vector $\boldsymbol{U}$ is constant.

$(\Leftarrow)$: Let the successor curve of the curve (α) be (β). Moreover, we suppose that a constant vector $\boldsymbol{U}$ as

$$\boldsymbol{U}=\frac{1}{2}\left(e^{i\left(\theta-\frac{\pi}{2}\right)}\phi^t\sigma\phi+e^{-i\left(\theta-\frac{\pi}{2}\right)}\hat{\phi}^t\sigma\hat{\phi}\right).$$

Therefore, with the aid of the equation (3.6) we can obtain

$$\langle\boldsymbol{T}_1,\boldsymbol{U}\rangle=-\frac{1}{4}\left[\begin{array}{l}\left(e^{i\theta}\left(\phi_1^2-\phi_2^2\right)+e^{-i\theta}\left(-\overline{\phi_1}^2+\overline{\phi_2}^2\right)\right)\left(\phi_1^2-\phi_2^2+\overline{\phi_2}^2-\overline{\phi_1}^2\right)\\ -\left(e^{i\theta}\left(\phi_1^2+\phi_2^2\right)+e^{-i\theta}\left(\overline{\phi_1}^2+\overline{\phi_2}^2\right)\right)\left(\phi_1^2+\phi_2^2+\overline{\phi_1}^2+\overline{\phi_2}^2\right)\\ +\left(-2\phi_1\phi_2+\overline{\phi_1}\,\overline{\phi_2}\right)\left(-2e^{i\theta}\phi_1\phi_2+2e^{-i\theta}\overline{\phi_1}\,\overline{\phi_2}\right)\end{array}\right]$$

and

$$\langle\boldsymbol{T}_1,\boldsymbol{U}\rangle=\frac{1}{2}\left(|\phi_1|^2+|\phi_2|^2\right)\left(e^{i\theta}+e^{-i\theta}\right)=\cos\theta\left(|\phi_1|^2+|\phi_2|^2\right).$$

We know that the spinor ϕ corresponds to the unit vectors, Frenet frame $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$, therefore, $\overline{\phi}^t\phi=|\phi_1|^2+|\phi_2|^2=1$ and $\langle\boldsymbol{T}_1,\boldsymbol{U}\rangle=\cos\theta$. On the other hand, since the vector $\boldsymbol{U}$ is constant as per the theorem, $\boldsymbol{U}'=\boldsymbol{0}$ is hold. Therefore, we obtain

$$\boldsymbol{U}'=\frac{1}{2}\left[i\theta'\left(e^{i\left(\theta-\frac{\pi}{2}\right)}\phi^t\sigma\phi-e^{-i\left(\theta-\frac{\pi}{2}\right)}\hat{\phi}^t\sigma\hat{\phi}\right)+\hat{\phi}^t\sigma\phi\left((\tau_1-i\kappa_1)e^{i\left(\theta-\frac{\pi}{2}\right)}-(\tau_1+i\kappa_1)e^{-i\left(\theta-\frac{\pi}{2}\right)}\right)\right]$$

and if we make necessary adjustments in last equation, then we get

$$\boldsymbol{U}'=\boldsymbol{0}=i\left[\theta'\left(\cos(\theta-\frac{\pi}{2})\boldsymbol{B}_1-\sin(\theta-\frac{\pi}{2})\boldsymbol{T}_1\right)-\left(\tau_1\sin(\theta-\frac{\pi}{2})+\kappa_1\cos(\theta-\frac{\pi}{2})\right)\boldsymbol{N}_1\right].$$

As a result, we have

$$\begin{aligned}\theta'\sin(\theta-\frac{\pi}{2})&=0,\\ \theta'\cos(\theta-\frac{\pi}{2})&=0,\end{aligned}$$

and $\theta'=0$ and $\theta=constant$. Consequently, $\langle\boldsymbol{T}_1,\boldsymbol{U}\rangle=cos\theta=constant$ and the successor curve (β) is helix. □

Corollary 3.14. *Let the curve (β), selected as helix, be the successor curve of the curve (α). In addition, the spinor ξ corresponds to the Frenet frame $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$ of the curve (α). Therefore, the spinor equation of axis $\boldsymbol{U}$ of the helix successor curve can be written*

$$\boldsymbol{U}=\frac{1}{2}\left[-e^{i(\theta+\vartheta)}\xi^t\sigma\xi+e^{-i(\theta+\vartheta)}\hat{\xi}^t\sigma\hat{\xi}\right].$$

Proof. Consider that the successor curve (β) of the curve (α) is helix. Therefore, there is the constant angle θ such as $\langle \boldsymbol{T}_1, \boldsymbol{U}\rangle = \cos\theta = constant$. If we use the equations (3.8) and (3.10) we get easily

$$\boldsymbol{U} = \frac{1}{2}\left[-e^{i(\theta+\vartheta)}\xi^t\sigma\xi + e^{-i(\theta+\vartheta)}\hat{\xi}^t\sigma\hat{\xi}\right]$$

where the successor curve (β) is helix, therefore, the curve (α) is planar curve. In that case, $\tau = 0 = \vartheta' = 0$ and the angles θ, ϑ are constant angles. □

Now, we give two examples.

Example 3.15. *Let $\alpha : I \to \mathbb{E}^3$ be curve with arc-length parameter s such as*

$$\alpha(s) = \left(\frac{2}{\sqrt{5}}\cos s, \frac{2}{\sqrt{5}}\sin s, \frac{s}{\sqrt{5}}\right).$$

Therefore, we obtain that Frenet apparatus $\{\boldsymbol{T},\boldsymbol{N},\boldsymbol{B},\kappa,\tau\}$ of (α) are

$$\begin{aligned}
\boldsymbol{T}(s) &= \frac{1}{\sqrt{5}}(-2\sin s, 2\cos s, 1),\\
\boldsymbol{N}(s) &= (-\cos s, -\sin s, 0),\\
\boldsymbol{B}(s) &= \frac{1}{\sqrt{5}}(\sin s, -\cos s, 2)
\end{aligned}$$

and

$$\kappa = \frac{2}{\sqrt{5}}, \quad \tau = \frac{1}{\sqrt{5}}.$$

Now, we assume that the spinor ξ corresponds to Frenet frame $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$ of the curve (α). In that case, we obtain

$$\begin{aligned}
\xi_1 &= \pm\sqrt{\frac{\sqrt{5}+1}{2\sqrt{5}}}\, ie^{\frac{-is}{2}},\\
\xi_2 &= \pm\sqrt{\frac{\sqrt{5}-1}{2\sqrt{5}}}\, e^{\frac{is}{2}}
\end{aligned}$$

where s is the arc-length parameter of the curve (α). Moreover, we get

$$\frac{d\xi}{ds} = \frac{1}{2\sqrt{5}}(-i\xi + 2\hat{\xi}).$$

On the other hand, consider that the curve (β) be a representation of the family of successor curves of the curve (α). In that case, if we take Frenet frame $\{\boldsymbol{T}_1,\boldsymbol{N}_1,\boldsymbol{B}_1\}$ of the successor curve (β) and the relationship $\boldsymbol{T} = \boldsymbol{N}_1$, we have

$$\begin{aligned}
\boldsymbol{T}_1 &= \begin{pmatrix}
\cos\left(\frac{1}{\sqrt{5}}s+c\right)\cos s + \frac{1}{\sqrt{5}}\sin\left(\frac{1}{\sqrt{5}}s+c\right)\sin s,\\
\cos\left(\frac{1}{\sqrt{5}}s+c\right)\sin s - \frac{1}{\sqrt{5}}\sin\left(\frac{1}{\sqrt{5}}s+c\right)\cos s,\\
\frac{2}{\sqrt{5}}\sin\left(\frac{1}{\sqrt{5}}s+c\right)
\end{pmatrix},\\
\boldsymbol{N}_1 &= \frac{1}{\sqrt{5}}\left(-2\sin s, 2\cos s, 1\right),\\
\boldsymbol{B}_1 &= \begin{pmatrix}
-\sin\left(\frac{1}{\sqrt{5}}s+c\right)\cos s + \frac{1}{\sqrt{5}}\cos\left(\frac{1}{\sqrt{5}}s+c\right)\sin s,\\
-\sin\left(\frac{1}{\sqrt{5}}s+c\right)\sin s - \frac{1}{\sqrt{5}}\cos\left(\frac{1}{\sqrt{5}}s+c\right)\cos s,\\
\frac{2}{\sqrt{5}}\cos\left(\frac{1}{\sqrt{5}}s+c\right)
\end{pmatrix}
\end{aligned}$$

where $\tau = \frac{d\vartheta}{ds} = \frac{1}{\sqrt{5}}$ and as a result, $\vartheta = \frac{1}{\sqrt{5}}s + c$, and $c = constant$. In addition to that, the curvatures κ_1 and τ_1 of the successor curve (β) can be obtained

$$\begin{aligned}
\kappa_1 &= \frac{2}{\sqrt{5}}\cos\left(\frac{1}{\sqrt{5}}s+c\right),\\
\tau_1 &= \frac{2}{\sqrt{5}}\sin\left(\frac{1}{\sqrt{5}}s+c\right).
\end{aligned}$$

Therefore, if we assume that the spinor ϕ corresponds to Frenet frame $\{\boldsymbol{B}_1,\boldsymbol{T}_1,\boldsymbol{N}_1\}$, then the spinor components of ϕ can be given

$$\begin{aligned}
\phi_1 &= \pm\sqrt{\frac{\sqrt{5}+1}{2\sqrt{5}}}\, e^{i\left(\frac{\pi}{4}+\frac{\vartheta-s}{2}\right)},\\
\phi_2 &= \pm\sqrt{\frac{\sqrt{5}-1}{2\sqrt{5}}}\, e^{i\left(\frac{\pi}{4}-\frac{\vartheta+s}{2}\right)}
\end{aligned}$$

where $\vartheta = \frac{1}{\sqrt{5}}s + c$ *and consequently, we have*

$$\frac{d\phi}{ds} = \frac{1}{2\sqrt{5}}\left(\sin\left(\frac{1}{\sqrt{5}}s+c\right) - 2i\cos\left(\frac{1}{\sqrt{5}}s+c\right)\right)\hat{\phi}.$$

Example 3.16. *Consider that the helix* $\beta(s) = \left(\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$ *be a successor curve of the curve* (α). *In that case, the successor system* $\{\boldsymbol{T}_1, \boldsymbol{N}_1, \boldsymbol{B}_1, \kappa_1, \tau_1\}$ *can be calculated as*

$$\begin{aligned}
\boldsymbol{T}_1 &= \frac{1}{\sqrt{2}}\left(-\sin\frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}, 1\right),\\
\boldsymbol{N}_1 &= \left(-\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, 0\right),\\
\boldsymbol{B}_1 &= \frac{1}{\sqrt{2}}\left(\sin\frac{s}{\sqrt{2}}, -\cos\frac{s}{\sqrt{2}}, 1\right)
\end{aligned}$$

and

$$\kappa_1 = \frac{1}{2}, \quad \tau_1 = \frac{1}{2}$$

where we can take $\theta = \frac{\pi}{4}$ *since* $\frac{\tau_1}{\kappa_1} = \cot\theta = 1$. *Moreover, the torsion of the curve* (α) *is* $\tau = \vartheta' = 0$ *as per Theorem 2.6 and the curvature of the curve* (α) *is* $\kappa = \frac{1}{\sqrt{2}}$ *since* $\kappa_1 = \kappa\cos\vartheta$.

On the other hand, let ϕ *be the spinor corresponding to Frenet frame of the successor curve* (β). *In that case, we obtain that the components of this spinor are*

$$\begin{aligned}
\phi_1 &= \pm\sqrt{\frac{1+i}{2\sqrt{2}}}e^{-i\frac{s}{2\sqrt{2}}},\\
\phi_2 &= \pm\sqrt{\frac{1+i}{2\sqrt{2}}}e^{i\frac{s}{2\sqrt{2}}}.
\end{aligned}$$

Now, we give the curve (α). *We know that* $\boldsymbol{T} = \boldsymbol{N}_1$ *therefore,*

$$\boldsymbol{T} = \left(-\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, 0\right)$$

and, as a result, we can take

$$\alpha(s) = \sqrt{2}\left(-\sin\frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}, 1\right).$$

In that case, we have

$$\begin{aligned}
\boldsymbol{N} &= \left(\sin\frac{s}{\sqrt{2}}, -\cos\frac{s}{\sqrt{2}}, 0\right),\\
\boldsymbol{B} &= (0,0,1).
\end{aligned}$$

Consequently, if we assume the spinor ξ *corresponding to Frenet frame of the curve* (α), *then we have*

$$\begin{aligned}
\xi_1 &= \pm\frac{1}{\sqrt{2}}e^{i\left(\frac{\pi}{4} - \frac{s}{2\sqrt{2}}\right)},\\
\xi_2 &= \pm\frac{1}{\sqrt{2}}e^{i\left(\frac{\pi}{4} + \frac{s}{\sqrt{2}}\right)}.
\end{aligned}$$

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