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Araştırma Makalesi / Research Article

A generalization of multinomial expansion

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Abstract
In this paper, a generalization of multinomial expansion in terms of permanent is given. Then, it is obtained some results related to this generalization

Keywords: Multinomial expansion, permanent

1.Introduction

It is possible to expand any power of $\sum_{\ell=1}^{m} x^{(\ell)}$ which is known as *multinomial expansion*. It can be used the permanent in order to express a generalization of multinomial expansion.
The book "Permanents" by Minc[1] and the survey papers by Minc[2,3] provide an excellent source of reference on permanents.
If $\mathrm{a}_1, \mathrm{a}_2, \ldots$ are defined as column vectors, then matrix obtained by taking k_1 copies of a_1, k_2 copies of $\mathrm{a}_2, \ldots$ can be denoted as $[\underset{k_1}{\mathrm{a}_1} \quad \underset{k_2}{\mathrm{a}_2} \ldots]$ and *per*A denotes permanent of a square matrix A, which is defined as similar to determinants except that all terms in expansion have a positive sign.
From now on, we write $\sum_{k_1,k_2,\ldots,k_{m-1}}$, $\sum_{i_1,i_2,\ldots,i_n}$, and C instead of $\sum_{k_1=0}^{n}\sum_{k_2=0}^{n-k_1}\sum_{k_3=0}^{n-k_1-k_2}\cdots\sum_{k_{m-1}=0}^{n-k_1-k_2-\ldots-k_{m-2}}$, $\sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_n=1}^{m_n}$ and $\prod_{l=1}^{m}\frac{1}{k_l!}$, respectively.

2. Multinomial expansion

The following theorem can be expressed for an expansion of $\prod_{j=1}^{n}\sum_{i=1}^{m_j} x_j^{(i)}$.

Theorem.

$$\prod_{j=1}^{n}\sum_{i=1}^{m_j} x_j^{(i)} = \sum_{k_1,k_2,\ldots,k_{m-1}} \mathrm{C}\, per[\underset{k_1}{\mathrm{X}_1} \quad \underset{k_2}{\mathrm{X}_2} \quad \ldots \quad \underset{k_m}{\mathrm{X}_m}], \tag{1}$$

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where $m = \max\{m_1, m_2, ..., m_n\}$, $X_l = (x_1^{(l)}, x_2^{(l)}, ..., x_n^{(l)})'$ $(l = 1, 2, ..., m)$ is column vector, and $\sum_{\ell=1}^{m} k_\ell = n$.

Proof.

The theorem will be proved by induction on *m*.

For $m = 1$, $\prod_{j=1}^{n} x_j^{(1)} = \frac{1}{n!} per[\underset{n}{X_1}]$. Thus, it is true for $m = 1$. Suppose that (1) is true for $m = w$. Then, the proof is obvious for $m = w + 1$.

Thus, the proof is completed. Upon using properties of permanent in (1), we get

$$\prod_{j=1}^{n} \sum_{i=1}^{m_j} x_j^{(i)} = \sum_{k_1, k_2, ..., k_{m-1}} C \sum_{n_{s_1}, n_{s_2}, ..., n_{s_{m-1}}} \prod_{\ell=1}^{m} per[\underset{k_\ell}{X_\ell}][s_\ell /.) \quad , \tag{2}$$

Where $\sum_{n_{s_1}, n_{s_2}, ..., n_{s_{m-1}}}$ denotes sum over $\bigcup_{l=1}^{m-1} s_l$ for which $s_\upsilon \cap s_\vartheta = \phi$ for $\upsilon \neq \vartheta$, $\bigcup_{l=1}^{m} s_l = \{1, 2, ..., n\}$ and $n_{s_l} = k_l$ is cardinality of s_l, where $s_\ell = \{s_\ell^{(1)}, s_\ell^{(2)}, \ldots, s_\ell^{(k_\ell)}\}$. Here, $A[s_\ell /.)$ is matrix obtained from A by taking rows whose indices are in s_ℓ.Note that it is obtained (2) by expansion of permanents. Upon using expansion of permanent in (1), we get

$$\prod_{j=1}^{n} \sum_{i=1}^{m_j} x_j^{(i)} = \sum_{k_1, k_2, ..., k_{m-1}} C \sum_{P} \left(\prod_{\ell_1=1}^{k_1} x_{i_{\ell_1}}^{(1)} \right) \left(\prod_{\ell_2=k_1+1}^{k_1+k_2} x_{i_{\ell_2}}^{(2)} \right) \cdots \prod_{\ell_m=n-k_m+1}^{n} x_{i_{\ell_m}}^{(m)} \tag{3}$$

Similarly, using expansion of permanent in (2), we simply obtain

$$\prod_{j=1}^{n} \sum_{i=1}^{m_j} x_j^{(i)} = \sum_{k_1, k_2, ..., k_{m-1}} \sum_{n_{s_1}, n_{s_2}, ..., n_{s_{m-1}}} \prod_{\ell=1}^{m} \prod_{i_l=1}^{k_\ell} x_{s_\ell^{(i_l)}}^{(\ell)} \tag{4}$$

Also, (4) can be written as

$$\prod_{j=1}^{n} \sum_{i=1}^{m_j} x_j^{(i)} = \sum_{i_1, i_2, ..., i_n} x_1^{(i_1)} x_2^{(i_2)} \ldots x_n^{(i_n)} \tag{5}$$

In (1), if $n = 3$, $m_1 = 2$, $m_2 = 3$, $m_3 = 4$, and $X_1 = (x_1^{(1)}, x_2^{(1)}, x_3^{(1)})'$, $X_2 = (x_1^{(2)}, x_2^{(2)}, x_3^{(2)})'$, $X_3 = (0, x_2^{(3)}, x_3^{(3)})'$, $X_4 = (0, 0, x_3^{(4)})'$, the following identity can be written

$$\left(x_1^{(1)}+x_1^{(2)}\right)\left(x_2^{(1)}+x_2^{(2)}+x_2^{(3)}\right)\left(x_3^{(1)}+x_3^{(2)}+x_3^{(3)}+x_3^{(4)}\right)$$

$$=\sum_{k_1=0}^{3}\sum_{k_2=0}^{3-k_1}\sum_{k_3=0}^{3-k_1-k_2}\frac{1}{k_1!k_2!k_3!(3-k_1-k_2-k_3)!}\,per[\underset{k_1}{X_1}\ \underset{k_2}{X_2}\ \underset{k_3}{X_3}\ \underset{3-k_1-k_2-k_3}{X_4}]$$

$$=\sum_{k_2=0}^{3}\sum_{k_3=0}^{3-k_2}\frac{1}{k_2!k_3!(3-k_2-k_3)!}\,per[\underset{k_2}{X_2}\ \underset{k_3}{X_3}\ \underset{3-k_2-k_3}{X_4}]+$$

$$\sum_{k_2=0}^{2}\sum_{k_3=0}^{2-k_2}\frac{1}{1!k_2!k_3!(2-k_2-k_3)!}\,per[\underset{1}{X_1}\ \underset{k_2}{X_2}\ \underset{k_3}{X_3}\ \underset{2-k_2-k_3}{X_4}]+$$

$$\sum_{k_2=0}^{1}\sum_{k_3=0}^{1-k_2}\frac{1}{2!k_2!k_3!(1-k_2-k_3)!}\,per[\underset{2}{X_1}\ \underset{k_2}{X_2}\ \underset{k_3}{X_3}\ \underset{1-k_2-k_3}{X_4}]$$

$+$

$$\frac{1}{3!}\,per[\underset{3}{X_1}]$$

$$=\left[\sum_{k_3=0}^{3}\frac{1}{k_3!(3-k_3)!}\,per[\underset{k_3}{X_3}\ \underset{3-k_3}{X_4}]\right.$$

$$+\sum_{k_3=0}^{2}\frac{1}{1!k_3!(2-k_3)!}\,per[\underset{1}{X_2}\ \underset{k_3}{X_3}\ \underset{2-k_3}{X_4}]$$

$$+\sum_{k_3=0}^{1}\frac{1}{1!1!k_3!(1-k_3)!}\,per[\underset{1}{X_1}\ \underset{1}{X_2}\ \underset{k_3}{X_3}\ \underset{1-k_3}{X_4}]$$

$$\left.+\frac{1}{3!}\,per[\underset{3}{X_2}]\right]+\left[\sum_{k_3=0}^{2}\frac{1}{1!k_3!(2-k_3)!}\,per[\underset{1}{X_1}\ \underset{k_3}{X_3}\ \underset{2-k_3}{X_4}]\right.$$

$$+\sum_{k_3=0}^{1}\frac{1}{1!1!k_3!(1-k_3)!}\,per[\underset{1}{X_1}\ \underset{1}{X_2}\ \underset{k_3}{X_3}\ \underset{1-k_3}{X_4}]$$

$$\left.+\frac{1}{1!2!}\,per[\underset{1}{X_1}\ \underset{2}{X_2}]\right]+\left[\sum_{k_3=0}^{1}\frac{1}{2!k_3!(1-k_3)!}\,per[\underset{2}{X_1}\ \underset{k_3}{X_3}\ \underset{1-k_3}{X_4}]+\frac{1}{2!1!}\,per[\underset{2}{X_1}\ \underset{1}{X_2}]\right]++\frac{1}{3!}\,per[\underset{3}{X_1}]$$

$$=\left[\frac{1}{3!}\,per[\underset{3}{X_4}]+\frac{1}{1!2!}\,per[\underset{1}{X_3}\ \underset{2}{X_4}]+\frac{1}{2!1!}\,per[\underset{2}{X_3}\ \underset{1}{X_4}]+\frac{1}{3!}\,per[\underset{3}{X_3}]+\frac{1}{1!2!}\,per[\underset{1}{X_2}\ \underset{2}{X_4}]+\frac{1}{1!1!1!}\right.$$

$$per[\underset{1}{X_2}\ \underset{1}{X_3}\ \underset{1}{X_4}]+\frac{1}{1!2!}\,per[\underset{1}{X_2}\ \underset{2}{X_3}]+\frac{1}{2!1!}\,per[\underset{2}{X_2}\ \underset{1}{X_4}]$$

$$\left.+\frac{1}{2!1!}\,per[\underset{2}{X_2}\ \underset{1}{X_3}]+\frac{1}{3!}\,per[\underset{3}{X_2}]\right]+\left[\frac{1}{1!2!}\,per[\underset{1}{X_1}\ \underset{2}{X_4}]+\frac{1}{1!1!1!}\,per[\underset{1}{X_1}\ \underset{1}{X_3}\ \underset{1}{X_4}]\right.$$

$$+\frac{1}{1!2!}\,per[\underset{1}{X_1}\ \underset{2}{X_3}]+\frac{1}{1!1!1!}\,per[\underset{1}{X_1}\ \underset{1}{X_2}\ \underset{1}{X_4}]+\frac{1}{1!1!1!}\,per[\underset{1}{X_1}\ \underset{1}{X_2}\ \underset{1}{X_3}]$$

$$+\ \frac{1}{1!2!}\,per[\underset{1}{X_1}\ \ \underset{2}{X_2}]\Bigg]$$

$$+\left[\frac{1}{2!1!}\ per[\underset{2}{X_1}\ \ \underset{1}{X_4}]+\frac{1}{2!1!}\ per[\underset{2}{X_1}\ \ \underset{1}{X_3}]+\frac{1}{2!1!}\,per[\underset{2}{X_1}\ \ \underset{1}{X_2}]\right]+\frac{1}{3!}\ \ per[\underset{3}{X_1}]$$

$$= x_1^{(1)}x_2^{(1)}x_3^{(1)} + x_1^{(1)}x_2^{(1)}x_3^{(2)} + x_1^{(1)}x_2^{(1)}x_3^{(3)} + x_1^{(1)}x_2^{(1)}x_3^{(4)} + x_1^{(1)}x_2^{(2)}x_3^{(1)}$$

$$+ x_1^{(1)}x_2^{(2)}x_3^{(2)} + x_1^{(1)}x_2^{(2)}x_3^{(3)} + x_1^{(1)}x_2^{(2)}x_3^{(4)} + x_1^{(1)}x_2^{(3)}x_3^{(1)} + x_1^{(1)}x_2^{(3)}x_3^{(2)}$$

$$+ x_1^{(1)}x_2^{(3)}x_3^{(3)} + x_1^{(1)}x_2^{(3)}x_3^{(4)} + x_1^{(2)}x_2^{(1)}x_3^{(1)} + x_1^{(2)}x_2^{(1)}x_3^{(2)} + x_1^{(2)}x_2^{(1)}x_3^{(3)}$$

$$+ x_1^{(2)}x_2^{(1)}x_3^{(4)} + x_1^{(2)}x_2^{(2)}x_3^{(1)} + x_1^{(2)}x_2^{(2)}x_3^{(2)} + x_1^{(2)}x_2^{(2)}x_3^{(3)} + x_1^{(2)}x_2^{(2)}x_3^{(4)}$$

$$+ x_1^{(2)}x_2^{(3)}x_3^{(1)} + x_1^{(2)}x_2^{(3)}x_3^{(2)} + x_1^{(2)}x_2^{(3)}x_3^{(3)} + x_1^{(2)}x_2^{(3)}x_3^{(4)}\ .$$

3. Results

In this section, we give five expressions for multinomial expansion.

Result 1.,

$$(x^{(1)}+x^{(2)}+\ldots+x^{(m)})^n = \sum_{k_1,k_2,\ldots,k_{m-1}} C\ n!\prod_{\ell=1}^{m}(x^{(\ell)})^{k_\ell} \qquad (6)$$

Proof. In (1) - (5), if $m_1 = m_2 = \ldots = m_n = m$ and $x_1^{(l)} = x_2^{(l)} = \ldots = x_n^{(l)} = x^{(l)}$, (6) is obtained.

Result 2.

$$\prod_{j=1}^{3}\sum_{i=1}^{3} x_j^{(i)} = \left(x_1^{(1)}+x_1^{(2)}+x_1^{(3)}\right)\left(x_2^{(1)}+x_2^{(2)}+x_2^{(3)}\right)\left(x_3^{(1)}+x_3^{(2)}+x_3^{(3)}\right)$$

$$=\sum_{k_1=0}^{3}\sum_{k_2=0}^{3-k_1}\frac{1}{k_1!k_2!(3-k_1-k_2)!}\ per[\underset{k_1}{X_1}\quad \underset{k_2}{X_2}\quad \underset{3-k_1-k_2}{X_3}\ ]$$

$$=\sum_{k_1=0}^{3}\sum_{k_2=0}^{3-k_1}\frac{1}{k_1!k_2!(3-k_1-k_2)!}\sum_{n_{s_1},\,n_{s_2}} per[\underset{k_1}{X_1}][s_1/.)\,per[\underset{k_2}{X_2}][s_2/.)\,per[\underset{3-k_1-k_2}{X_3}][s_3/.)$$

$$=\sum_{k_1=0}^{3}\sum_{k_2=0}^{3-k_1}\frac{1}{k_1!k_2!(3-k_1-k_2)!}\sum_{P}\left(\prod_{\ell_1=1}^{k_1}x_{i_{\ell_1}}^{(1)}\right)\left(\prod_{\ell_2=k_1+1}^{k_1+k_2}x_{i_{\ell_2}}^{(2)}\right)\prod_{\ell_3=k_1+k_2+1}^{3}x_{i_{\ell_3}}^{(3)}$$

$$=\sum_{k_1=0}^{3}\sum_{k_2=0}^{3-k_1}\sum_{n_{s_1},\,n_{s_2}}\left(\prod_{i_1=1}^{k_1}x_{s_1^{(i_1)}}^{(1)}\right)\left(\prod_{i_2=1}^{k_2}x_{s_2^{(i_2)}}^{(2)}\right)\prod_{i_3=1}^{3-k_1-k_2}x_{s_3^{(i_3)}}^{(3)} \qquad (7)$$

where $X_1=(x_1^{(1)},x_2^{(1)},x_3^{(1)})'$, $X_2=(x_1^{(2)},x_2^{(2)},x_3^{(2)})'$, $X_3=(x_1^{(3)},x_2^{(3)},x_3^{(3)})'$.

Proof. In (1) - (5), if $n=3$ and $m_1=m_2=m_3=m=3$, (7) is obtained.

Result 4.

$$\prod_{j=1}^{2}\sum_{i=1}^{2}x_j^{(i)}=\left(x_1^{(1)}+x_1^{(2)}\right)\left(x_2^{(1)}+x_2^{(2)}\right)$$

$$=\sum_{k_1=0}^{2}\frac{1}{k_1!(2-k_1)!}\,per[\underset{k_1}{X_1}\quad\underset{2-k_1}{X_2}]$$

$$=\sum_{k_1=0}^{2}\frac{1}{k_1!(2-k_1)!}\sum_{n_{s_1}}per[\underset{k_1}{X_1}][s_1/.)\,per[\underset{2-k_1}{X_2}][s_2/.)$$

$$=\sum_{k_1=0}^{2}\frac{1}{k_1!(2-k_1)!}\sum_{P}\left(\prod_{\ell_1=1}^{k_1}x_{i_{\ell_1}}^{(1)}\right)\prod_{\ell_2=k_1+1}^{2}x_{i_{\ell_2}}^{(2)}\quad = \sum_{k_1=0}^{2}\sum_{n_{s_1}}\left(\prod_{i_1=1}^{k_1}x_{s_1^{(i_1)}}^{(1)}\right)\prod_{i_2=1}^{2-k_1}x_{s_2^{(i_2)}}^{(2)} \tag{9}$$

where $X_1=(x_1^{(1)},x_2^{(1)},x_3^{(1)})'$, $X_2=(x_1^{(2)},x_2^{(2)},x_3^{(2)})'$.

Proof. In (1) - (5), if $n=2$ and $m_1=m_2=m=2$, (9) is obtained.

Result 5.

$$(x^{(1)}+x^{(2)})^n=\sum_{k_1=0}^{n}\frac{n!}{k_1!(n-k_1)!}\,(x^{(1)})^{k_1}\,(x^{(2)})^{n-k_1} \tag{10}$$

3 Proof. In (6), if $m=2$, (10) is obtained. The above result is binomial expansion.

Kaynaklar

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