Araştırma Makalesi / Research Article

Fractional Solutions of the Associated Legendre Equation

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Abstract
Fractional calculus and its generalizations are used for the solutions of some classes of linear ordinary and partial differential equations of the second and higher orders and fractional differential equations. In this paper, our aim is that obtaining fractional solutions of the associated Legendre equation via N-fractional calculus operator N^{μ} method.

Keywords: Fractional Calculus, N-Fractional Calculus Operator N^{μ} Method, Legendre Equation, Ordinary Differential Equation, Generalized Leibniz Rule, Index Law.

1. Introduction

The fractional calculus theory enables a set of axioms and methods to generalize the coordinate and corresponding derivative notions from integer k to arbitrary order μ, $\{x^k, \partial^k/\partial x^k\} \to \{x^{\mu}, \partial^{\mu}/\partial x^{\mu}\}$ in a good light. Fractional calculus that is an important subject of applied mathematics can be used in many fields such as robot technology, PID control systems, Schrödinger equation, heat transfer, relativity theory, economy, filtration, controller design, mechanics, optics, modelling and so on. Thus, this popular subject has contributed to science for 300 years [1-3]. Bas and Metin [4] defined a fractional singular Sturm-Liouville operator having Coulomb potential of type A/x. Theory of spectral properties for eigenvalues and eigenfunctions of Bessel type of fractional singular Sturm-Liouville problem is presented [5]. Explicit solutions of Bessel equation by means of fractional calculus techniques are obtained [6]. Yilmazer and Bas [7] introduced fractional solutions of a confluent hypergeometric equation by using N-fractional calculus operator. This method presents successful results for some singular differential equations [8,9]. And, we also apply this operator to the associated Legendre equation in this paper.

Some of most obvious formulations based on the fundamental definitions of Riemann-Liouville fractional integration and fractional differentiation are, respectively,

$$ {}_aD_t^{-\mu} f(t) = [f(t)]_{-\mu} = \frac{1}{\Gamma(\mu)} \int_a^t f(\xi)(t-\xi)^{\mu-1}\, d\xi \quad (t > a, \mu > 0), $$

$$ {}_aD_t^{\mu} f(t) = [f(t)]_{\mu} = \frac{1}{\Gamma(k-\mu)} \left(\frac{d}{dt}\right)^k \int_a^t f(\xi)(t-\xi)^{k-\mu-1}\, d\xi \quad (k-1 \le \mu < k, k \in \mathbb{N}), \tag{1} $$

where Γ stands for Euler's function gamma [10].

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2. Preliminaries

Definition 2.1 If the function $f(z)$ is analytic *(regular)* inside and on C, where $C = \{C^-, C^+\}$, C^- is a contour along the cut joining the points z and $-\infty + i\text{Im}(z)$, which starts from the point at $-\infty$, encircles the point z once counter-clockwise, and returns to the point at $-\infty$, and C^+ is a contour along the cut joining the points z and $\infty + i\text{Im}(z)$, which starts from the point at ∞, encircles the point z once counter-clockwise, and returns to the point at ∞,

$$f_\mu(z) = \big(f(z)\big)_\mu = \frac{\Gamma(\mu+1)}{2\pi i}\int_C \frac{f(t)dt}{(t-z)^{\mu+1}} \quad (\mu \neq -1, -2, \dots),$$
$$f_{-k}(z) = \lim_{\mu \to -k} f_\mu(z) \quad (k \in \mathbb{Z}^+), \tag{2}$$

where $t \neq z$,

$$-\pi \leq \arg(t-z) \leq \pi \quad \text{for } C^-,$$
$$0 \leq \arg(t-z) \leq 2\pi \quad \text{for } C^+. \tag{3}$$

In that case, $f_\mu(z)$ $(\mu > 0)$ is the fractional derivative of $f(z)$ of order μ and$f_\mu(z)$ $(\mu < 0)$ is the fractional integral of $f(z)$ of order $-\mu$, confirmed *(in each case)* that

$$\left|f_\mu(z)\right| < \infty \quad (\mu \in \mathbb{R}). \tag{4}$$

[10].

Lemma 2.1 ***(Linearity)*** Let $f(z)$ and $g(z)$ be single-valued and analytic functions. If f_μ and g_μ exist, then

$$(Kf + Lg)_\mu = Kf_\mu + Lg_\mu, \tag{5}$$

hold, where K and L are constants and $\mu \in \mathbb{R}, z \in \mathbb{C}$ [10].

Lemma 2.2 ***(Index law)*** Let $f(z)$ be single-valued and analytic function. If $(f_\nu)_\mu$ and $\left(f_\mu\right)_\nu$ exist, then

$$(f_\nu)_\mu = f_{\nu+\mu} = \left(f_\mu\right)_\nu, \tag{6}$$

where $\nu, \mu \in \mathbb{R}, z \in \mathbb{C}$ and $\left|\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+1)\Gamma(\mu+1)}\right| < \infty$ [10].

Lemma 1.3 ***(N^μ method)*** Let $f(z)$ and $g(z)$ be single-valued and analytic functions. If f_μ and g_μ exist, then, generalized Leibniz rule is defined by

$$N^\mu(f.g) = (f.g)_\mu = \sum_{k=0}^{\infty} \frac{\Gamma(\mu+1)}{\Gamma(\mu-k+1)\Gamma(k+1)} f_{\mu-k}.g_k, \tag{7}$$

where $\mu \in \mathbb{R}, z \in \mathbb{C}$ and $\left|\frac{\Gamma(\mu+1)}{\Gamma(\mu-k+1)\Gamma(k+1)}\right| < \infty$ [10].

Property 2.1

$$(e^{\omega z})_\mu = \omega^\mu e^{\omega z} \quad (\omega \neq 0, \mu \in \mathbb{R}, z \in \mathbb{C}), \tag{8}$$

$$(e^{-\omega z})_\mu = e^{-i\pi\mu}\omega^\mu e^{-\omega z} \quad (\omega \neq 0, \mu \in \mathbb{R}, z \in \mathbb{C}), \tag{9}$$

$$(z^\omega)_\mu = e^{-i\pi\mu}\frac{\Gamma(\mu-\omega)}{\Gamma(-\omega)} z^{\omega-\mu} \quad \left(\mu \in \mathbb{R}, z \in \mathbb{C}, \left|\frac{\Gamma(\mu-\omega)}{\Gamma(-\omega)}\right| < \infty\right), \tag{10}$$

where ω is a constant [10].

Remark 2.1 The familiar Bessel differential equation of general order l:

$$z^2\frac{d^2f}{dz^2} + z\frac{df}{dz} + (z^2 - l^2)f = 0, \tag{11}$$

which is named after F. Wilheim Bessel. More precisely, just as in the earlier works [11,12], we aim here at demonstrating how the underlying simple fractional-calculus approach to the solutions of the classical differential equation (11), which were considered in the earlier works [11,12], would lead us analogously to several interesting consequences including *(for example)* an alternative investigation of solutions of the following family of differential equations *(cf.* [13, vol. I, p. 121, Eq. 3.2(1)]*; see also* [14, Chapter 15]*)*:

$$(1 - z^2)f_2 - 2zf_1 + \left[l(l+1) - \frac{m^2}{1-z^2}\right]f = 0, \tag{12}$$

known as Legendre's differential equation where $f_k = d^kf/dz^k \quad (k = 0,1,2,\dots),\ f_0 = f = f(z),\ z \in \mathbb{C}$.

3. Fractional Solutions of the Associated Legendre Equation

Theorem 3.1 Let $f \in \{f: 0 \neq |\ f_\mu| < \infty; \mu \in \mathbb{R}\}$. Eq. (12) has particular solutions as follows

$$f^{(\mathrm{I})} = A(z^2-1)^{m/2}\left[(z^2-1)^{-\sigma/2}\right]_{(m-2(\lambda+1))/2}, \tag{13}$$

$$f^{(\mathrm{II})} = B(z^2-1)^{m/2}\left[(z^2-1)^{-\tau/2}\right]_{(m+2(\lambda-1))/2}, \tag{14}$$

where A and B are arbitrary constants, σ, τ and λ will define in the proof.

Proof. Let $\varphi = \varphi(z)$. Set

$$f = (z^2-1)^{m/2}\varphi \quad (z \in \mathbb{C}\backslash\{-1,1\}). \tag{15}$$

Hence

$$f_1 = (z^2-1)^{m/2}[mz(z^2-1)^{-1}\varphi + \varphi_1], \tag{16}$$

and

$$f_2 = (z^2-1)^{m/2}\{m(z^2-1)^{-1}[1+(m-2)z^2]\varphi + 2mz(z^2-1)^{-1}\varphi_1 + \varphi_2\}. \tag{17}$$

By substituting (15), (16) and (17) into (12), we have

$$(1-z^2)\varphi_2 - 2(m+1)z\varphi_1 - (m-l)(m+l+1)\varphi = 0 \quad (m, l \in \mathbb{R}), \tag{18}$$

the homogeneous as well as nonhomogeneous versions of which *(together with their numerous interesting special cases for different choices of the parameters m and l, and the nonhomogeneous terms)* were considered recently by Lin and Nishimoto *(see, for details,* [15,16]*)*.
By applying the operator N^{μ} to Eq. (18), we have

$$(1-z^2)\varphi_{2+\mu} - 2(\mu+m+1)z\varphi_{1+\mu} - 2\left[\mu^2 + m\mu + \frac{\kappa}{2}\right]\varphi_\mu = 0, \tag{19}$$

where $\kappa = (m-l)(m+l+1)$.
Now, we choose μ such that $\mu^2 + m\mu + \kappa/2 = 0$, that is,

$$\mu = (-m \pm 2\lambda)/2, \tag{20}$$

where $\lambda = \sqrt{m^2 - 2\kappa}/2$.

(i) Let $\mu = (-m+2\lambda)/2$. By substituting μ into (19), we have

$$(1-z^2)\left[\varphi_{(-m+2(\lambda+1))/2}\right]_1 - \sigma z\varphi_{(-m+2(\lambda+1))/2} = 0, \tag{21}$$

where $\sigma = m + 2(\lambda+1)$.
Set

$$\varphi_{(-m+2(\lambda+1))/2} = u = u(z) \quad \left(\varphi(z) = u_{(m-2(\lambda+1))/2}\right), \tag{22}$$

then, we have

$$u_1 + \frac{\sigma z}{(z^2-1)}u = 0, \tag{23}$$

from (21). A particular solution of ordinary differential equation (23) is given by

$$u = A(z^2-1)^{-\sigma/2}. \tag{24}$$

Inversely, (24) satisfies (23), then

$$\varphi(z) = A\left[(z^2-1)^{-\sigma/2}\right]_{(m-2(\lambda+1))/2}, \tag{25}$$

satisfies (21). By substituting (25) into (15), we have

$$f(z) = A(z^2-1)^{m/2}\left[(z^2-1)^{-\sigma/2}\right]_{(m-2(\lambda+1))/2}. \tag{26}$$

(ii) Let $\mu = -(m+2\lambda)/2$. By substituting μ into (19), we have

$$(1-z^2)\left[\varphi_{(-m-2(\lambda-1))/2}\right]_1 - \tau z \varphi_{(-m-2(\lambda-1))/2} = 0, \tag{27}$$

where $\tau = m - 2(\lambda-1)$. Set

$$\varphi_{(-m-2(\lambda-1))/2} = \vartheta = \vartheta(z) \quad \left(\varphi(z) = \vartheta_{(m+2(\lambda-1))/2}\right), \tag{28}$$

then, we have

$$\vartheta_1 + \frac{\tau z}{(z^2-1)}\vartheta = 0, \tag{29}$$

from (28). A particular solution of ordinary differential equation (29) is given by

$$\vartheta = B(z^2-1)^{-\tau/2}. \tag{30}$$

Inversely, (30) satisfies (29), then

$$\varphi(z) = B\left[(z^2-1)^{-\tau/2}\right]_{(m+2(\lambda-1))/2}, \tag{31}$$

satisfies (27). By substituting (31) into (15), we have

$$f(z) = B(z^2-1)^{m/2}\left[(z^2-1)^{-\tau/2}\right]_{(m+2(\lambda-1))/2}. \tag{32}$$

4. Hypergeometric Forms of the Fractional Solutions

Theorem 4.1 We have [17]
(i)

$$\left\{\left[(z-b)^\beta - c\right]^\alpha\right\}_\gamma = e^{-i\pi\gamma}(z-b)^{\alpha\beta-\gamma}\sum_{k=0}^{\infty}\frac{[-\alpha]_k\Gamma(\beta k-\alpha\beta+\gamma)}{\Gamma(\beta k-\alpha\beta)\Gamma(k+1)}\left(\frac{c}{(z-b)^\beta}\right)^k, \tag{33}$$
$$\left(\left|\frac{\Gamma(\beta k-\alpha\beta+\gamma)}{\Gamma(\beta k-\alpha\beta)}\right| < \infty\right),$$

(ii)

$$\left\{\left[(z-b)^\beta - c\right]^\alpha\right\}_n = (-1)^n(z-b)^{\alpha\beta-n}\sum_{k=0}^{\infty}\frac{[-\alpha]_k[\beta k-\alpha\beta]_n}{\Gamma(k+1)}\left(\frac{c}{(z-b)^\beta}\right)^k, \tag{34}$$
$$\left(n \in \mathbb{Z}_0^+, \left|\frac{c}{(z-b)^\beta}\right| < 1\right),$$

where symbol of Pochhammer is defined by $[\alpha]_k = \alpha(\alpha+1)\dots(\alpha+k-1) = \Gamma(\alpha+k)/\Gamma(\alpha)$ and $[\alpha]_0 = 1$.

Property 4.1 We have [17]

$$\Gamma(2\rho) = \frac{2^{2\rho-1}}{\sqrt{\pi}}\Gamma(\rho)\Gamma\left(\rho+\frac{1}{2}\right), \tag{35}$$

$$\begin{aligned}\Gamma(2k+\rho) = \Gamma\left(2\left(k+\frac{\rho}{2}\right)\right) &= \frac{2^{2k+\rho-1}}{\sqrt{\pi}}\Gamma\left(k+\frac{\rho}{2}\right)\Gamma\left(k+\frac{\rho}{2}+\frac{1}{2}\right),\\ &= \frac{2^{2k+\rho-1}}{\sqrt{\pi}}\Gamma\left(\frac{\rho}{2}\right)\Gamma\left(\frac{\rho+1}{2}\right)\left[\frac{\rho}{2}\right]_k\left[\frac{\rho+1}{2}\right]_k,\end{aligned} \tag{36}$$

$$\frac{\sqrt{\pi}}{2^{\rho-1}}\Gamma(\rho) = \Gamma\left(\frac{\rho}{2}\right)\Gamma\left(\frac{\rho+1}{2}\right). \tag{38}$$

Theorem 4.2 Let $\left|\left[(z^2-1)^{-\sigma/2}\right]_{\nu-\sigma}\right| < \infty$ $(n \in \mathbb{Z}_0^+)$, $z \neq 0$ and $\left|\frac{1}{z^2}\right| < 1$. Eq. (13) can be written as follows

$$f^{(\mathrm{I})} = A(z^2-1)^{m/2}e^{-i\pi(\nu-\sigma)}z^{-\nu}\frac{\Gamma(\nu)}{\Gamma(\sigma)}\ {}_2F_1\left(\frac{\nu}{2},\frac{\nu+1}{2};\frac{\sigma+1}{2};\frac{1}{z^2}\right), \tag{39}$$

where ${}_2F_1$ is the Gauss hypergeometric function and $\nu = \frac{2\sigma+m-2(\lambda+1)}{2}$.

Proof. By applying (33) to (13), we have

$$\begin{aligned}&f^{(\mathrm{I})} = A(z^2-1)^{m/2}e^{-i\pi(\nu-\sigma)}z^{-\nu}\\ &\times\sum_{k=0}^{\infty}\frac{\left[\frac{\sigma}{2}\right]_k\Gamma(2k+\nu)}{\Gamma(2k+\sigma)\Gamma(k+1)}\left(\frac{1}{z^2}\right)^k \quad \left(\left|\frac{\Gamma(2k+\nu)}{\Gamma(2k+\sigma)}\right| < \infty\right).\end{aligned} \tag{40}$$

By using (35) and (36), we obtain

$$f^{(\mathrm{I})} = A(z^2-1)^{m/2}e^{-i\pi(\nu-\sigma)}z^{-\nu}2^{\nu-\sigma}\frac{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\sigma}{2}\right)\Gamma\left(\frac{\sigma+1}{2}\right)}\sum_{k=0}^{\infty}\frac{\left[\frac{\nu}{2}\right]_k\left[\frac{\nu+1}{2}\right]_k}{\left[\frac{\sigma+1}{2}\right]_k k!}\left(\frac{1}{z^2}\right)^k. \tag{41}$$

By applying (38) to (41), then we have

$$f^{(\mathrm{I})} = A(z^2-1)^{m/2}e^{-i\pi(\nu-\sigma)}z^{-\nu}\frac{\Gamma(\nu)}{\Gamma(\sigma)}\ {}_2F_1\left(\frac{\nu}{2},\frac{\nu+1}{2};\frac{\sigma+1}{2};\frac{1}{z^2}\right). \tag{42}$$

Theorem 4.3 Let $\left|\left[(z^2-1)^{-\tau/2}\right]_{\upsilon-\tau}\right| < \infty$ $(n \in \mathbb{Z}_0^+)$, $z \neq 0$ and $\left|\frac{1}{z^2}\right| < 1$. Eq. (14) can be written as follows

$$f^{(\mathrm{II})} = B(z^2-1)^{m/2}e^{-i\pi(\upsilon-\tau)}z^{-\upsilon}\frac{\Gamma(\upsilon)}{\Gamma(\tau)}\ {}_2F_1\left(\frac{\upsilon}{2},\frac{\upsilon+1}{2};\frac{\tau+1}{2};\frac{1}{z^2}\right), \tag{43}$$

where $_2F_1$ is the Gauss hypergeometric function and $\upsilon = \frac{2\tau+m+2(\lambda-1)}{2}$.

5. Conclusion

In this paper, we used N^{μ} method for the associated Legendre equation. We also obtained hypergeometric forms of the fractional solutions. The most important advantage of this method is applicable for singular equations.

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