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ÖKLİT UZAYI $\mathbb{E}^4$ DE KARIŞIK ÇARPIM

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ÖZET

Bu çalışmada 4-boyutlu Öklit uzayı $\mathbb{E}^4$ de karışık çarpım olarak nitelendirilen yeni bir çarpım yüzeyi tanımlanmıştır. Bu tip yüzeylerin Gauss, ortalama ve normal eğrilikleri hesaplanmış ve bazı sonuçlar elde edilmiştir. Sonuç olarak, özel örnekler verilmiştir.

Anahtar Kelimeler:. *Spherical Product, Gaussian curvature, Gaussian torsion, Mean curvature*

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MIXED PRODUCT SURFACES IN $\mathbb{E}^4$

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ABSTRACT

In the present study we define a new kind of product surfaces namely mixed products which are the product of two space curves in 4-dimensional Euclidean space $\mathbb{E}^4$. We investigate the Gaussian curvature, Gaussian torsion and mean curvature of these kind of surfaces. Further, we obtain some original results of mixed product surfaces in $\mathbb{E}^4$. Finally, we give some examples of these kind of surfaces.

Keywords: *Spherical Product, Gaussian curvature, Gaussian torsion, Mean curvature.*

1. INTRODUCTION

In classical differential geometry the first and second fundamental form provides an important role to describe the shape of the surfaces [3]. Gaussian curvature is an intrinsic surface invariant of a local surface. Consequently, both Gaussian and mean curvatures are important to recover the shape of the objects [6].

The rotational embeddings in Euclidean spaces are special products which are introduced first by N.H. Kuiper in 1970 [11]. It is known that the spherical products of 2D curves are the special type of rotational surfaces in $\mathbb{E}^3$ [1]. Quadrics are the simple type of these surfaces. So, superquadrics can be also considered as the spherical products of two 2D curves. In fact, superquadrics are the solid models of the smooth shapes [12, 15]. Superquadrics are the special type of supershapes, defined by Gielis and et.al. [9]. In[5], the present authors defined the spherical product of a $3D$ curve with a $2D$ curve in Euclidean 4-space $\mathbb{E}^4$. For more details see also [8] and [13].

In the present study we define a new kind of product surfaces that are product of a two space curves in $\mathbb{E}^4$ is called mixed product surface. Mixed products can be considered as the generalization of spherical products. The rest of the paper is organized as follows: In Section 2 we give necessary definitions and theorems as basic concepts. Section 3 gives the original results of mixed product surface patches in $\mathbb{E}^3$, which is recently, studied the present authors [4]. In Section 4 the authors calculate the Gaussian curvature, Gaussian torsion and mean curvature of these kind surfaces and give some examples.

2. BASIC CONCEPTS

Let M be a smooth surface in $\mathbb{E}^4$ given with the local patch $X(u,v):(u,v)\in D\subset\mathbb{E}^2$. The coefficients of the first fundamental form of M are given by

$$E=<X_u,X_u>,\ F=<X_u,X_v>,\ G=<X_v,X_v> \tag{1}$$

where $<,>$ is the inner product in $\mathbb{E}^4$, and X_u,X_v are the tangent vectors of M. We assume that $W^2=EG-F^2\neq 0$, i.e. the local patch $X(u,v)$ is regular. Further, given any local vector fields X_i,X_j tangent to M one can define the second fundamental form of M by

$$h(X_i,X_j)=\widetilde{\nabla}_{X_i}X_j-\nabla_{X_i}X_j,\quad 1\leq i,j\leq 2. \tag{2}$$

where $\widetilde{\nabla},\nabla$ are the Riemannian connection and induced connection of M respectively. This map is well-defined symmetric and bilinear.

For any arbitrary orthonormal normal frame field $\{N_1,N_2\}$ of M, recall the shape operator

$$A_{N_i}X_i=-\left(\widetilde{\nabla}_{X_i}N_i\right)^T,\quad X_i\in\chi(M). \tag{3}$$

This operator is bilinear, self-adjoint and satisfies the following equation:

$$<A_{N_k}X_i,X_j>=<h(X_i,X_j),N_k>=c_{ij}^k,\ 1\leq i,j,k\leq 2. \tag{4}$$

The equation (2) is called Gaussian formula, and

$$h(X_i,X_j)=\sum_{k=1}^{2}c_{ij}^kN_k,\quad 1\leq i,j,k\leq 2 \tag{5}$$

where c_{ij}^k are the coefficients of the second fundamental form.

Further, the *Gaussian curvature* and *Gaussian torsion* of a regular patch $X(u,v)$ are given by

$$K=\frac{1}{W^2}\sum_{k=1}^{2}\left(c_{11}^kc_{22}^k-\left(c_{12}^k\right)^2\right), \tag{6}$$

and

$$K_N = \frac{1}{W^2}\left(E(c_{12}^1 c_{22}^2 - c_{12}^2 c_{22}^1) - F(c_{11}^1 c_{22}^2 - c_{11}^2 c_{22}^1) + G(c_{11}^1 c_{12}^2 - c_{11}^2 c_{12}^1)\right), \tag{7}$$

respectively.

Further, tha mean curvature vector of a regular patch $X(u, v)$ is defined by

$$\vec{H} = \frac{1}{2W^2}\sum_{k=1}^{2}\left(c_{11}^k G + c_{22}^k E - 2c_{12}^k F\right)N_k. \tag{8}$$

Recall that a surface M is said to be *minimal* if its mean curvature vector vanishes identically [7].

3. MIXED PRODUCT SURFACES IN $\mathbb{E}^3$

Definition 1. *Let $\alpha: I \subset \mathbb{R} \to \mathbb{E}^2$ be Euclidean plane curve and $\beta: J \subset \mathbb{R} \to \mathbb{E}^3$ a space curve respectively. Put $\alpha(u) = (f_1(u), f_2(u))$ and $\beta(v) = (g_1(v), g_2(v), g_3(v))$. Then we define their mixed product patch by*

$$X = \alpha \boxtimes \beta: \mathbb{E}^2 \to \mathbb{E}^3; X(u, v) = (f_1(u)g_1(v), f_1(u)g_2(v), f_2(u)g_3(v)); \tag{9}$$

where $u \in I = (u_0, u_1)$ and $v \in J = (v_0, v_1)$ [4].

If $\alpha(u)$ and $\beta(v)$ are not straight lines passing through the origin then the surface patch $X(u, v)$ is regular.

In [4] the present authors gave the following examples and results;

Example 1. *The mixed product $\alpha(u) = (f_1(u), f_2(u))$ with $\beta(v) = (g_1(v), g_2(v), 1)$ forms the surface patch*

$$X(u, v) = \left(f_1(u)g_1(v), f_1(u)g_2(v), f_2(u)\right),$$

which is a spherical product patch [12]. For $\beta(v) = (\cos v, \sin v, 1)$ the surface patch

$$X(u, v) = \left(f_1(u)\cos v, f_1(u)\sin v, f_2(u)\right),$$

becomes a surface of revolution [14].

Example 2. *The mixed product* $\alpha(u) = (u, 1)$ *with* $\beta(v) = (\cos v, \sin v, bv)$ *forms the surface patch*

$X(u, v) = (u \cos v, u \sin v, bv),$

becomes a helicoid which is a minimal surface in $\mathbb{E}^3$ *[14].*

Example 3. *The mixed product* $\alpha(u) = (\lambda, u)$ *with* $\beta(v) = (g_1(v), g_2(v), g_3(v))$ *forms the surface patch*

$X(u, v) = \lambda(g_1(v), g_2(v), 0) + u(0,0, g_3(v))$

which is a ruled surface. Further, for the given vector $\gamma = (0,0, g_3(v))$ *the cross product* $\gamma \times \gamma'$ *vanishes identically. So the ruled surface is cylindrical.*

Definition 2. *Let* $\beta: J \subset \mathbb{R} \to \mathbb{E}^3$ *be a regular curve in* $\mathbb{E}^3$*. If* $< \beta, \vec{B} >= 0$ *then* $\beta(v)$ *is called osculating curve in* $\mathbb{E}^3$*.*

Proposition 1. [4] *The mixed product of a straight line* $\alpha(u): y(u) = x(u)$ *with the space curve* $\beta(v) = (g_1(v), g_2(v), g_3(v))$ *forms the surface patch*

$X(u, v) = x(u)\beta(v)$

is a flat conical surface.

Proposition 2. [4] *Let* M *be a mixed product of the straight line* $\alpha(u): y(u) = x(u)$ *with unit speed curve* $\beta(u) = (g_1(v), g_2(v), g_3(v))$*. If* $\beta(v)$ *is an osculating space curve then* M *is a minimal surface.*

4. MIXED PRODUCT SURFACES IN $\mathbb{E}^4$

Definition 3. *Let* $\alpha: I \subset \mathbb{R} \to \mathbb{E}^3$ *and* $\beta: J \subset \mathbb{R} \to \mathbb{E}^3$ *be Euclidean space curve. Put* $\alpha(u) = (f_1(u), f_2(u), f_3(u))$ *and* $\beta(v) = (g_1(v), g_2(v), g_3(v))$*. Then we define their mixed product patch by*

$X = \alpha \boxtimes \beta: \mathbb{E}^2 \to \mathbb{E}^4;$

$$X(u, v) = (f_1(u)g_1(v), f_1(u)g_2(v), f_2(u)g_3(v), f_3(u)g_3(v)); \qquad (10)$$

$u \in I = (u_0, u_1)$, $v \in J = (v_0, v_1)$. We call the local surface given with the patch (10) a mixed product surface.

We assume that surface patch $X(u,v)$ is a regular. So $\alpha(u)$ and $\beta(v)$ can not be considered as straight lines passing through the origin.

A spherical product surface in $\mathbb{E}^4$ has the parametrization

$X(u,v) = (g_1(v), g_2(v), g_3(v)\cos u, g_3(v)\sin u)$

which are studied with many geometers ([2,5,8,10,11]). In fact, these surfaces can be considered as the mixed products of the curve $\alpha(u) = (1, \cos u, \sin u)$ with $\beta(v) = (g_1(v), g_2(v), g_3(v))$. Furthermore, if we take $\alpha(u) = (f(u), \cos u, \sin u)$ and $\beta(v) = (\cos v, \sin v, g(v))$ the mixed product patch becomes

$$X(u,v) = \alpha(u) \boxtimes \beta(v) = \\ (f(u)\cos v, f(u)\sin v, g(v)\cos u, g(v)\sin u), \tag{11}$$

where f and g are some smooth functions.

Then we proved the following result.

Theorem 3. *Let M be the mixed product surface given with the patch (11). Then the Gaussian curvature K and Gaussian torsion K_N of M become*

$$K = -\frac{1}{W^4}\begin{Bmatrix} (f(u)^2 g(v) g''(v) + f'(u)^2 g'(v)^2)(g(v)^2 + f'(u)^2) \\ +(g(v)^2 f(u) f''(u) + f'(u)^2 g'(v)^2)(f(u)^2 + g'(v)^2) \end{Bmatrix}, \tag{12}$$

and

$$K_N = \frac{1}{W^4}\begin{Bmatrix} f(u) f'(u) g'(v)(g(v)^2 + f'(u)^2)(g(v) - g''(v)) \\ +g(v)(f(u)^2 + g'(v)^2)(f(u)^2 g(v) + f'(u) g'(v) f''(u)) \end{Bmatrix}, \tag{13}$$

respectively.

Proof. The tangent space of M is spanned by the vector fields

$$\frac{\partial X}{\partial u} = (f'(u)\cos v, f'(u)\sin v, -g(v)\sin u, g(v)\cos u), \tag{14}$$

$$\frac{\partial X}{\partial v} = (-f(u)\sin v, f(u)\cos v, g'(v)\cos u, g'(v)\sin u).$$

Hence the coefficients of the first fundamental forms of the surface are

$$E =< X_u, X_u >= f'(u)^2 + g(v)^2,$$
$$F =< X_u, X_v >= 0, \quad (15)$$
$$G =< X_v, X_v >= f(u)^2 + g'(v)^2,$$

where $<,>$ is the standard scalar product in $\mathbb{E}^4$.

The second partial derivatives of $X(u,v)$ are expressed as follows

$$X_{uu} = (f''(u)\cos v, f''(u)\sin v, -g(v)\cos u, -g(v)\sin u),$$
$$X_{uv} = (-f'(u)\sin v, f'(u)\cos v, -g'(v)\sin u, g'(v)\cos u), \quad (16)$$
$$X_{vv} = (-f(u)\cos v, -f(u)\sin v, g''(v)\cos u, g''(v)\sin u).$$

Further, the normal space of M is spanned by the vector fields

$$N_1 = \frac{1}{\sqrt{f(u)^2+g\prime(v)^2}}(-g'(v)\sin v, g'(v)\cos v, f(u)\cos u, f(u)\sin u), \quad (17)$$
$$N_2 = \frac{1}{\sqrt{f\prime(u)^2+g(v)^2}}(g(v)\cos v, g(v)\sin v, f'(u)\sin u, -f'(u)\cos u).$$

Using (4), (16) and (17) we can calculate the coefficients of the second fundamental form as follows:

$$c_{11}^1 =< X_{uu}, N_1 >= \frac{-f(u)g(v)}{\sqrt{f(u)^2 + g'(v)^2}},$$
$$c_{12}^1 =< X_{uv}, N_1 >= \frac{-f'(u)g'(v)}{\sqrt{f(u)^2 + g'(v)^2}},$$
$$c_{22}^1 =< X_{vv}, N_1 >= \frac{f(u)g\prime\prime(v)}{\sqrt{f(u)^2+g\prime(v)^2}}, \quad (18)$$
$$c_{11}^2 =< X_{uu}, N_2 \geq \frac{f''(u)g(v)}{\sqrt{f'(u)^2 + g(v)^2}},$$
$$c_{12}^2 =< X_{uv}, N_2 \geq \frac{-f'(u)g'(v)}{\sqrt{f'(u)^2 + g(v)^2}},$$

$$c_{22}^2 = < X_{vv}, N_2 > = \frac{-f(u)g(v)}{\sqrt{f'(u)^2 + g(v)^2}}.$$

Further, substituting (15) and (18) into (6) and (7) we get (12) and (13). ■

As a consequence of Theorem 3 we can give the following examples.

Example 4. *The surfaces given with the following mixed product patches have vanishing Gaussian curvatures;*

i) $X(u,v) = (\lambda \cos v, \lambda \sin v, \mu \cos u, \mu \sin u)$, *i.e., a Clifford torus,*

ii) $X(u,v) = (\lambda \cos v, \lambda \sin v, (\mu v + a)\cos u, (\mu v + a)\sin u)$,

iii) $X(u,v) = ((\lambda u + b)\cos v, (\lambda u + b)\sin v, \mu \cos u, \mu \sin u)$,

where $a, b \in \mathbb{R}$, λ *and* μ *are nonzero real constants.*

Example 5. *The surfaces given with the following mixed product patches have vanishing Gaussian torsions;*

i) $X(u,v) = (e^u \cos v, e^u \sin v, e^{-v}\cos u, e^{-v}\sin u)$,

ii) $X(u,v) = (e^{-u}\cos v, e^{-u}\sin v, e^v \cos u, e^v \sin u)$.

By the use of (13) we obtain the following results.

Proposition 4. *Let* M *be the mixed product surface given with the patch (11). If the Gaussian torsion* K_N *of* M *is a real constant then*

$$\begin{aligned} 0 = (f^2 + g^2)\{g(f^2 g + f'g'f'') - c(g^2 + f'^2)^2(f^2 + g^2)\} \\ + \{ff'g'(g - g'')(g^2 + f'^2)\} \end{aligned}$$

holds, where $f = f(u)$, $g = g(v)$ *are smooth functions and* $K_N = c \in \mathbb{R}$.

As a consequence of Proposition 4 we can give the following example.

Example 6. *The surfaces given with the following mixed product patches have constant Gaussian torsions;*

i) $X(u,v) = \left(\lambda\cos v, \lambda\sin v, \frac{\lambda}{c}\cos u, \frac{\lambda}{c}\sin u\right)$, *i.e., a Clifford torus,*

ii) $X(u,v) = (\mu\cos v, \mu\sin v, (\delta u + a)\cos u, (\delta u + a)\sin u)$,

where a, λ, μ *and* δ *are nonzero real constants with* $K_N = c^2$ *and* $\delta = \sqrt{-\mu(\mu c \mp 1)}$

Theorem 5. *Let* M *be the mixed product surface given with the patch (11). Then the mean curvature vector* $\vec{H}$ *of* M *becomes*

$$\vec{H} = \frac{f(u)g''(v)\left(g(v)^2+f'(u)^2\right)-f(u)g(v)\left(f(u)^2+g'(v)^2\right)}{2W^2\sqrt{\left(f(u)^2+g'(v)^2\right)}}N_1 \tag{19}$$

$$+\frac{g(v)f''(u)\left(f(u)^2+g\prime(v)^2\right)-f(u)g(v)\left(g(v)^2+f'(u)^2\right)}{2W^2\sqrt{\left(g(v)^2+f'(u)^2\right)}}N_2.$$

Proof. Using the equations (8), (15) and (18) we get the result.

Corollary 6. *Let* M *be the mixed product surface given with the patch (11). If,* $f(u) = e^u \pm e^{-u}$ and $g(v) = e^v \pm e^{-v}$ then M has vanishing mean curvature.

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