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On the Differential Geometry of Coframe Bundle with Cheeger-Gromoll Metric

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ABSTRACT

In this paper we introduce the Cheeger-Gromoll type metric on the coframe bundle of a Riemannian manifold and investigate the Levi-Civita connection, curvature tensor, sectional curvature and geodesics of coframe bundle with this metric.

Keywords: *Coframe bundle, adapted frame, Cheeger-Gromoll metric, Levi-Civita connection, curvature tensor, geodesics.*

AMS Subject Classification (2020): *53C07, 53C22, 53C25*

1. Introduction

The special Riemannian metric on the tangent bundle, later called the Cheeger-Gromoll metric, was first introduced by J. Cheeger and D.Gromoll in [3] (see also [6], [10]). The curvatures of the Cheeger-Gromoll metric of the tangent bundle were studied by M. Sekizawa [14]. The geodesics of the mentioned metric were investigated in [13] by A. Salimov and S. Kazimova (see also [12]). The general Cheeger-Gromoll metrics on the tangent bundle of a Riemannian manifold introduced and investigated by M.Munteanu [9] and Z.Hou and L.Sun [7]. The Cheeger-Gromoll metric of the cotangent bundle was introduced by A. Salimov and F. Agca and studied in [1]. In [2], a new class of g-natural metrics was introduced on the cotangent bundle, to which the Cheeger-Gromoll metric belongs. A similar approach was implemented by K. Niedzialomski [11], applied to the bundle of linear frames.

In this paper, we shall define and study the Cheeger-Gromoll metric on the bundle of linear coframes of a Riemannian manifold. In 2 we briefly describe the definitions and results that are needed later, after which the adapted frame on coframe bundle introduced in 3. The Cheeger-Gromoll metric ${}^{CG}g$ on coframe bundle is determined in 4. In 5 we investigate the properties of Levi-Civita connection ${}^{CG}\nabla$ of metric ${}^{CG}g$. Christoffel symbols (components) ${}^{CG}\Gamma$ of connection ${}^{CG}\nabla$ are calculated in 6. In sections 7 and 8 we investigate the curvature tensor field, sectional curvature and geodesics on coframe bundle with Cheeger-Gromoll metric, respectively.

2. Preliminaries

In this section we shall summarize briefly the main definitions and results which be used later. Let (M,g) be an $n-$dimensional Riemannian manifold. The linear coframe bundle $F^*(M)$ over M consists of all pairs (x,u^*), where x is a point of M and u^* is a basis (coframe) for the cotangent space T^*_xM of M at x [5]. We denote by π the natural projection of $F^*(M)$ to M defined by $\pi(x,u^*)=x$. If $(U;x^1,x^2,...,x^n)$ is a system of local coordinates in M, then a coframe $u^*=(X^\alpha)=(X^1,X^2,...,X^n)$ for T^*_xM can be expressed uniquely in the form $X^\alpha=X^\alpha_i(dx^i)_x$. From mentioned above it follows that

$$\left(\pi^{-1}(U);x^1,x^2,...,x^n,X^1_1,X^1_2,...,X^n_n\right)$$

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is a system of local coordinates in $F^*(M)$ (see, [5]), that is $F^*(M)$ is a $C^\infty-$ manifold of dimension $n+n^2$. We note that indices $i,j,k,...,\alpha,\beta,\gamma,...$ have range in $\{1,2,...,n\}$, while indices $A,B,C,...$ have range in $\{1,...,n,n+1,...,n+n^2\}$. We put $i_\alpha=\alpha\cdot n+i$. Obviously that indices $i_\alpha,j_\beta,k_\gamma,...$ have range in $\{n+1,n+2,...,n+n^2\}$. Summation over repeated indices is always implied. Let ∇ be a symmetric linear connection on M with components Γ_{ij}^k. Then the tangent space $T_{(x,u^*)}(F^*(M))$ of $F^*(M)$ at $(x,u^*)\in F^*(M)$ splits into the horizontal and vertical subspaces with respect to ∇ :

$$T_{(x,u^*)}(F^*(M))=H_{(x,u^*)}(F^*(M))\oplus V_{(x,u^*)}(F^*(M)). \tag{2.1}$$

We denote by $\Im_s^r(M)$ the set of all differentiable tensor fields of type (r,s) on M. From (2.1) it follows that for every $X\in\Im_0^1(F^*(M))$ is obtained unique decomposing $X=hX+vX$, where $hX\in H(F^*(M))$, $vX\in V(F^*(M))$. $H(F^*(M))$ and $V(F^*(M))$ the horizontal and vertical distributions for $F^*(M)$, respectively. Now we define naturally n different vertical lifts of $1-$form $\omega\in\Im_1^0(M)$. If Y be a vector field on M, i.e. $Y\in\Im_0^1(M)$, then $i^\mu Y$ are functions on $F^*(M)$ defined by $(i^\mu Y)(x,u^*)=X^\mu(Y)$ for all $(x,u^*)=(x,X^1,X^2,...,X^n)\in F^*(M)$, where $\mu=1,2,...,n$. The vertical lifts ${}^{V_\lambda}\omega$ of ω to $F^*(M)$ are the n vector fields such that

$${}^{V_\lambda}\omega(i^\mu Y)=\omega(Y)\delta_\mu^\lambda \tag{2.2}$$

hold for all vector fields Y on M, where $\lambda,\mu=1,2,...,n$ and δ_μ^λ denote the Kronecker's delta. The n vertical lifts ${}^{V_\lambda}\omega$ are always uniquely determined and they are linearly independent if $\omega\neq 0$. If ${}^{V_\lambda}\omega={}^{V_\lambda}\omega^k\partial_k+{}^{V_\lambda}\omega^{k_\sigma}\partial_{k_\sigma}$, then from (2.2), we obtain:

$${}^{V_\lambda}\omega^k X_j^\mu\partial_k Y^j+{}^{V_\lambda}\omega^{k_\sigma}Y^k=\omega_l Y^l\delta_\mu^\lambda.$$

Since Y^k and $\partial_k Y^j$ can take any preassigned values at each point, we have from the above equality:

$${}^{V_\lambda}\omega^k\partial_k Y^j=0,\qquad {}^{V_\lambda}\omega^{k_\mu}=\omega_k\delta_\mu^\lambda.$$

So, we have ${}^{V_\lambda}\omega^k=0$ at all points of $F^*(M)$. Concequently, the vertical lifts ${}^{V_\lambda}\omega$ of ω to $F^*(M)$ have the components

$${}^{V_\lambda}\omega=\begin{pmatrix}{}^{V_\lambda}\omega^k\\ {}^{V_\lambda}\omega^{k_\mu}\end{pmatrix}=\begin{pmatrix}0\\ \omega_k\delta_\mu^\lambda\end{pmatrix} \tag{2.3}$$

with respect to the induced coordinates (x^i,X_i^α) in $F^*(M)$ (see, [5]).

Let $V\in\Im_0^1(M)$. The complete lift ${}^CV\in\Im_0^1(F^*(M))$ of V to the linear coframe bundle $F^*(M)$ is defined by

$${}^CV(i^\mu Y)=i^\mu(L_VY)=X_m^\mu(L_VY)^m$$

for all vector fields $Y\in\Im_0^1(M)$, where L_V be the Lie derivation with respect to V. The complete lift ${}^CV$has the components

$${}^CV=\begin{pmatrix}{}^CV^k\\ {}^CV^{k_\mu}\end{pmatrix}=\begin{pmatrix}V^k\\ -X_m^\mu\partial_kV^m\end{pmatrix} \tag{2.4}$$

with respect to the induced coordinates (x^i,X_i^α) in $F^*(M)$.

The horizontal lift ${}^HV\in\Im_0^1(F^*(M))$ of V to the linear coframe bundle $F^*(M)$ is defined by

$${}^HV(i^\mu Y)=i^\mu(\nabla_VY)=X_m^\mu(\nabla_VY)^m$$

for all vector fields $Y\in\Im_0^1(M)$, where ∇_V be the covariant derivative with respect to V. The horizontal lift ${}^HV$has the components

$${}^HV=\begin{pmatrix}{}^HV^k\\ {}^HV^{k_\mu}\end{pmatrix}=\begin{pmatrix}V^k\\ X_m^\mu\Gamma_{lk}^mV^l\end{pmatrix} \tag{2.5}$$

with respect to the induced coordinates (x^i,X_i^α) in $F^*(M)$, where Γ_{ij}^k are the components of Levi-Civita connection on M.

The bracket operation of vertical and horizontal vector fields is given by the formulas

$$\begin{aligned}&[{}^{V_\beta}\omega,{}^{V_\gamma}\theta]=0,\\ &[{}^HX,{}^{V_\gamma}\theta]={}^{V_\gamma}(\nabla_X\theta),\\ &[{}^HX,{}^HY]={}^H[X,Y]+\textstyle\sum_{\sigma=1}^n{}^{V_\sigma}(X^\sigma\circ R(X,Y))\end{aligned} \tag{2.6}$$

for all $X,Y\in\Im_0^1(M)$ and $\omega,\theta\in\Im_1^0(M)$, where R is the Riemannian curvature of g. If f is a differentiable function on M, ${}^Vf=f\circ\pi$ denotes its canonical vertical lift to the $F^*(M)$.

3. Adapted frames on $F^*(M)$

Suppose (U, x^i) be a local coordinate system in M.In $U \subset M$, we put

$$X_{(i)} = \partial/(\partial x^i), \quad \theta^{(i)} = dx^i,\ i = 1, 2, ..., n.$$

Taking into account of (2.3) and (2.5), we see that

$$^{H}X_{(i)} = D_i = \begin{pmatrix} \delta_i^j \\ X_m^{\beta}\Gamma_{ij}^m \end{pmatrix}, \tag{3.1}$$

$$^{V_\alpha}\theta^{(i)} = D_{i_\alpha} = \begin{pmatrix} 0 \\ \delta_\beta^\alpha \delta_j^i \end{pmatrix} \tag{3.2}$$

with respect to the natural frame $\{\partial_j, \partial_{j_\beta}\}$. It follows that this $n+n^2$ vector fields are linearly independent and generate, respectively the horizontal distribution of linear connection ∇ and the vertical distribution of linear coframe bundle $F^*(M)$. The set $\{D_I\} = \{D_i, D_{i_\alpha}\}$ is called the frame adapted to linear connection ∇ on $\pi^{-1}(U) \subset F^*(M)$. From (2.3), (2.5), (3.1) and (3.2), we deduce that the horizontal lift ^{H}V of $V \in \Im_0^1(M)$ and vertical lift $^{V_\alpha}\omega$ of $\omega \in \Im_1^0(M)$ for each $\alpha = 1, 2, ..., n$, have respectively, components:

$$^{H}V = V^j D_j = \begin{pmatrix} V^j \\ 0 \end{pmatrix}, \tag{3.3}$$

$$^{V_\alpha}\omega = \sum_h \omega_j \delta_\beta^\alpha D_{j_\beta} = \begin{pmatrix} 0 \\ \delta_\beta^\alpha \omega_j \end{pmatrix} \tag{3.4}$$

with respect to the adapted frame $\{D_J\}$. The non-holonomic objects $\Omega_{IL}{}^K$ of the adapted frame $\{D_J\}$are defined by

$$[D_I, D_L] = \Omega_{IL}{}^K D_K$$

and have the following non-zero components:

$$\left(\begin{array}{c} \Omega_{il_\beta}{}^{k_\gamma} = -\Omega_{l_\beta i}{}^{k_\gamma} = -\delta_\beta^\gamma \Gamma_{ik}^l, \\ \Omega_{il}{}^{k_\gamma} = X_m^\gamma R_{ilk}{}^m, \end{array} \right. \tag{3.5}$$

where $R_{ilk}{}^m$ local components of the Riemannian curvature R.

4. The Cheeger-Gromoll metric on the linear coframe bundle $F^*(M)$

Definition 4.1. Let (M, g) be an $n-$dimensional Riemannian manifold. A Riemannian metric $\tilde{g}$ on the linear coframe bundle $F^*(M)$ is said to be natural with respect to g on M if

$$\tilde{g}(^{H}X, {}^{H}Y) = g(X, Y),$$

$$\tilde{g}(^{H}X, {}^{V_\alpha}\omega) = 0$$

for all $X, Y \in \Im_0^1(M)$ and $\omega \in \Im_1^0(M)$.

For any $x \in M$ the scalar product on the cotangent space T_x^*M is defined by

$$g^{-1}(\omega, \theta) = g^{ij}\omega_i\theta_j$$

for all $\omega, \theta \in \Im_1^0(M)$.

The Cheeger-Gromoll metric ^{CG}g is a positive definite metric on linear coframe bundle $F^*(M)$ which is described in terms of lifted vector fields as follows.

Definition 4.2. Let g be a Riemannian metric on a manifold M. Then the Cheeger-Gromoll metric is a Riemannian metric ${}^{CG}g$ on the linear coframe bundle $F^*(M)$ such that

$$\begin{aligned}
&{}^{CG}g({}^{H}X, {}^{H}Y) = g(X,Y)),\\
&{}^{CG}g({}^{V_\alpha}\omega, {}^{H}Y) = 0,\\
&{}^{CG}g({}^{V_\alpha}\omega, {}^{V_\beta}\theta) = 0, \quad \alpha \neq \beta,\\
&{}^{CG}g({}^{V_\alpha}\omega, {}^{V_\alpha}\theta) = \tfrac{1}{1+r_\alpha^2}(g^{-1}(\omega,\theta) + g^{-1}(\omega, X^\alpha)g^{-1}(\theta, X^\alpha))
\end{aligned} \tag{4.1}$$

for all $X, Y \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$, where $r_\alpha^2 = |X^\alpha|^2 = g^{-1}(X^\alpha, X^\alpha)$.

From (4.1) we determine that metric ${}^{CG}g$ has components

$$\begin{aligned}
&{}^{CG}g_{ij} = {}^{CG}g(D_i, D_i) = {}^{V}(g(\partial_i, \partial_j)) = g_{ij},\\
&{}^{CG}g_{i_\alpha j} = {}^{CG}g(D_{i_\alpha}, D_j) = 0,\\
&{}^{CG}g_{i_\alpha j_\beta} = {}^{CG}g(D_{i_\alpha}, D_{j_\beta}) = 0, \quad \alpha \neq \beta,\\
&{}^{CG}g_{i_\alpha j_\alpha} = {}^{CG}g(D_{i_\alpha}, D_{j_\alpha}) = \tfrac{1}{1+r_\alpha^2}(g^{-1}(dx^i, dx^j)\\
&\qquad + g^{-1}(dx^i, X_r^\alpha)g^{-1}(dx^j, X_s^\alpha) = \tfrac{1}{1+r_\alpha^2}(g^{ij} + g^{ir}g^{js}X_r^\alpha X_s^\alpha)
\end{aligned}$$

with respect to the adapted frame $\{D_I\}$ of linear coframe bundle $F^*(M)$.

From (2.4) and (2.5), it follows that the complete lift ${}^{C}X$ of $X \in \Im_0^1(M)$ is expressed by

$$\begin{aligned}
{}^{C}X - {}^{H}X &= -X_m^\alpha \sum_i (\partial_i X^m - \Gamma_{ik}^m X^k)\partial_{i_\alpha}\\
&= -X_m^\alpha \sum_i \nabla_i X^m \partial_{i_\alpha} = -\delta_\alpha^\beta X_m^\alpha \nabla_i X^m \partial_{i_\beta} = -\sum_{\alpha=1}^{n} {}^{V_\alpha}(X_m^\alpha \nabla_i X^m),
\end{aligned}$$

i.e.,

$$ {}^{C}X = {}^{H}X - \sum_{\alpha=1}^{n} {}^{V_\alpha}(X^\alpha \circ \nabla X), \tag{4.2}$$

where

$$X^\alpha \circ \nabla X = X_m^\alpha \nabla_i X^m dx^i.$$

Using (4.1) and (4.2), we have

$$\begin{aligned}
&{}^{CG}g({}^{C}X, {}^{C}Y) = {}^{CG}g({}^{H}X - \sum_{\alpha=1}^{n} {}^{V_\alpha}(X^\alpha \circ \nabla X), {}^{H}Y - \sum_{\alpha=1}^{n} {}^{V_\alpha}(X^\alpha \circ \nabla Y))\\
&= {}^{V}(g(X,Y)) + \sum_{\alpha=1}^{n} \tfrac{1}{1+r_\alpha^2}(g^{-1}(X^\alpha \circ \nabla X, X^\alpha \circ \nabla Y)\\
&+ g^{-1}(X^\alpha \circ \nabla X, X^\alpha)g^{-1}(X^\alpha \circ \nabla Y, X^\alpha)),
\end{aligned} \tag{4.3}$$

where

$$g^{-1}(X^\alpha \circ \nabla X, X^\alpha \circ \nabla Y) = g^{ij}(X_m^\alpha \nabla_i X^m)(X_s^\alpha \nabla_j Y^s)$$

and

$$g^{-1}(X^\alpha \circ \nabla X, X^\alpha) = g^{ir}(X^\alpha \circ \nabla X)_i X_r^\alpha.$$

Since the tensor field ${}^{CG}g \in \Im_2^0(F^*(M))$ is completely determined also by its action on vector fields ${}^{C}X$ and ${}^{C}Y$, we have an alternative characterization of ${}^{CG}g$ on $F^*(M)$: ${}^{CG}g$ is completely determined by the condition (4.3).

5. The Levi-Civita connection of ${}^{CG}g$

Before we calculate the Levi-Civita connection ${}^{CG}\nabla$ of $F^*(M)$ with Cheeger-Gromoll metric ${}^{CG}g$, we will need some formulas concerning this metric.
Lemma 5.1. *The following equalities hold:*

$$ {}^{H}X\left(\frac{1}{1+r_\alpha^2}\right)=0, \tag{5.1}$$

$$ {}^{V_\beta}\theta\left(\frac{1}{1+r_\alpha^2}\right)=-\frac{2}{(1+r_\alpha^2)^2}\delta_\alpha^\beta g^{-1}(\theta,X^\alpha), \tag{5.2}$$

$$ {}^{H}X({}^{CG}g({}^{V_\beta}\theta,{}^{V_\beta}\xi))={}^{CG}g({}^{V_\beta}(\nabla_X\theta),{}^{V_\beta}\xi)+{}^{CG}g({}^{V_\beta}\theta,{}^{V_\beta}(\nabla_X\xi)), \tag{5.3}$$

$$ {}^{CG}g({}^{V_\beta}\theta,\gamma\delta)=g^{-1}(\theta,X^\beta) \tag{5.4}$$

for all $X\in\Im_0^1(M),\ \theta,\xi\in\Im_1^0(M)$.

Proof. i) Direct calculations using (3.3) give

$$ {}^{H}X\left(\frac{1}{1+r_\alpha^2}\right)=(X^iD_i)\left(\frac{1}{1+r_\alpha^2}\right)=X^i(\partial_i+X_r^\sigma\Gamma_{ip}^r\partial_{p_\sigma})\left(\frac{1}{1+g^{-1}(X^\alpha,X^\alpha)}\right)$$

$$ =X^i\partial_i\left(\frac{1}{1+g^{-1}(X^\alpha,X^\alpha)}\right)+\Gamma_{ip}^rX^iX_r^\sigma\partial_{p_\sigma}\left(\frac{1}{1+g^{-1}(X^\alpha,X^\alpha)}\right)$$

$$ =\frac{X^i(-\partial_ig^{-1})(X^\alpha,X^\alpha)}{(1+g^{-1}(X^\alpha,X^\alpha))^2}+\Gamma_{ip}^rX^iX_r^\sigma\frac{(-\partial_{p_\sigma}(g^{-1}(X^\alpha,X^\alpha)))}{(1+g^{-1}(X^\alpha,X^\alpha))^2}$$

$$ =\frac{X^i(-\partial_ig^{lm}X_l^\alpha X_m^\alpha)}{(1+g^{-1}(X^\alpha,X^\alpha))^2}+\Gamma_{ip}^rX^iX_r^\sigma\frac{(-\partial_{p_\sigma}(g^{lm}X_l^\alpha X_m^\alpha))}{(1+g^{-1}(X^\alpha,X^\alpha))^2}$$

$$ =\frac{X^i(\Gamma_{is}^lg^{sm}+\Gamma_{is}^mg^{ls})X_l^\alpha X_m^\alpha}{(1+g^{-1}(X^\alpha,X^\alpha))^2}+\Gamma_{ip}^rX^iX_r^\sigma\frac{(-g^{lm}\delta_\sigma^\alpha\delta_l^pX_m^\alpha-g^{lm}\delta_\sigma^\alpha\delta_m^pX_l^\alpha)}{(1+g^{-1}(X^\alpha,X^\alpha))^2}$$

$$ =\frac{X^iX_l^\alpha X_m^\alpha\Gamma_{is}^lg^{sm}+X^i\Gamma_{is}^mg^{ls}X_l^\alpha X_m^\alpha}{(1+g^{-1}(X^\alpha,X^\alpha))^2}$$

$$ -\frac{\Gamma_{il}^rX^iX_r^\alpha X_m^\alpha g^{lm}+\Gamma_{im}^rX^iX_r^\alpha X_l^\alpha g^{lm}}{(1+g^{-1}(X^\alpha,X^\alpha))^2}=0.$$

ii) Calculations like above using (3.4) give

$$ {}^{V_\beta}\theta\left(\tfrac{1}{1+r_\alpha^2}\right)=\sum_i\theta_i\delta_\sigma^\beta D_{i_\sigma}\left(\tfrac{1}{1+g^{rs}X_r^\alpha X_s^\alpha}\right)=\sum_i\theta_i\delta_\sigma^\beta\partial_{i_\sigma}\left(\tfrac{1}{1+g^{rs}X_r^\alpha X_s^\alpha}\right)$$

$$ =\delta_\sigma^\beta\theta_i\tfrac{1}{(1+g^{rs}X_r^\alpha X_s^\alpha)^2}(-g^{rs}(\delta_\alpha^\sigma\delta_r^iX_s^\alpha+\delta_\alpha^\sigma\delta_s^iX_r^\alpha))$$

$$ =\delta_\alpha^\beta\theta_i\tfrac{1}{(1+r_\alpha^2)^2}(-g^{is}X_s^\alpha-g^{ri}X_r^\alpha)=-\delta_\alpha^\beta\tfrac{2}{(1+r_\alpha^2)^2}g^{is}\theta_iX_s^\alpha$$

$$ =-\delta_\alpha^\beta\tfrac{2}{(1+r_\alpha^2)^2}g^{-1}(\theta,X^\alpha).$$

iii) Using (3.3), (4.1) and (5.1), we obtain

$$ {}^{H}X({}^{CG}g({}^{V_\beta}\theta,{}^{V_\beta}\xi))={}^{H}X\left(\tfrac{1}{1+r_\beta^2}\left[g^{-1}(\theta,\xi)+g^{-1}(\theta,X^\beta)g^{-1}(\xi,X^\beta)\right]\right)$$

$$ =\tfrac{1}{1+r_\beta^2}\left(X^iD_i(g^{rs}\theta_r\xi_s)+X^iD_i\left[(g^{rs}\theta_rX_s^\beta)(g^{lm}\xi_lX_m^\beta)\right]\right)$$

$$ =\tfrac{1}{1+r_\beta^2}\left(g^{-1}(\nabla_X\,\theta,\xi)+g^{-1}(\theta,\nabla_X\xi)+g^{-1}(\nabla_X\theta,X^\beta)g^{-1}(\xi,X^\beta)\right.$$

$$ \left.+g^{-1}(\theta,X^\beta)g^{-1}(\nabla_X\xi,X^\beta)\right)={}^{CG}g\left({}^{V_\beta}(\nabla_X\theta),{}^{V_\beta}\xi\right)+{}^{CG}g\left({}^{V_\beta}\theta,{}^{V_\beta}(\nabla_X\xi)\right).$$

iv) Calculations using (2.6) and (4.1) give

$$\begin{aligned}&{}^{CG}g({}^{V_\beta}\theta,\gamma\delta)={}^{CG}g({}^{V_\beta}\theta,\sum_{\sigma=1}^{n}{}^{V_\sigma}(X^\sigma\circ\delta))=\sum_{\sigma=1}^{n}{}^{CG}g({}^{V_\beta}\theta,{}^{V_\sigma}X^\sigma)\\&={}^{CG}g({}^{V_\beta}\theta,{}^{V_\beta}X^\beta)=\tfrac{1}{1+r_\beta^2}(g^{-1}(\theta,X^\beta)+g^{-1}(\theta,X^\beta)g^{-1}(X^\beta,X^\beta))\\&=\tfrac{1}{1+r_\beta^2}(g^{-1}(\theta,X^\beta)(1+g^{-1}(X^\beta,X^\beta))=g^{-1}(\theta,X^\beta).\end{aligned}$$

□

Theorem 5.1. *Connection ${}^{CG}\nabla$ satisfies the following relations*

i) ${}^{CG}\nabla_{{}^HX}{}^HY={}^H(\nabla_XY)+\frac{1}{2}\sum_{\sigma=1}^{n}{}^{V_\sigma}(X^\sigma\circ R(X,Y))$,

ii) ${}^{CG}\nabla_{{}^HX}{}^{V_\beta}\theta={}^{V_\beta}(\nabla_X\theta)+\frac{1}{2h_\beta}{}^H(X^\beta(g^{-1}\circ R(\ ,X)\tilde\theta))$,

iii)

$${}^{CG}\nabla_{{}^{V_\alpha}\omega}{}^HY=\frac{1}{2h_\alpha}{}^H(X^\alpha(g^{-1}\circ R(\ ,Y)\overset{\leftrightarrow}{\omega})),\tag{5.5}$$

iv) ${}^{CG}\nabla_{{}^{V_\alpha}\omega}{}^{V_\beta}\theta=0$ *for* $\alpha\neq\beta$,

$${}^{CG}\nabla_{{}^{V_\alpha}\omega}{}^{V_\alpha}\theta=-\frac{1}{h_\alpha}({}^{CG}g({}^{V_\alpha}\omega,\gamma\delta){}^{V_\alpha}\theta+{}^{CG}g({}^{V_\alpha}\theta,\gamma\delta){}^{V_\alpha}\omega)$$

$$+\frac{1+h_\alpha}{h_\alpha}{}^{CG}g({}^{V_\alpha}\omega,{}^{V_\alpha}\theta)\gamma\delta-\frac{1}{h_\alpha}{}^{CG}g({}^{V_\alpha}\theta,\gamma\delta){}^{CG}g({}^{V_\alpha}\omega,\gamma\delta)\gamma\delta$$

for all $X,Y\in\Im_0^1(M)$, $\omega,\theta\in\Im_1^0(M)$, *where* $\tilde\omega=g^{-1}\circ\omega, R(\ ,X)\tilde\omega\in\Im_1^1(M)$, $h_\alpha=1+r_\alpha^2$, R *and* $\gamma\delta$ *denotes respectively the Riemannii an curvature of* g *and the canonical vertical vector field on* $F^*(M)$ *with local expression* $\gamma\delta=X_i^\sigma D_{i_\sigma}$.

Proof. The Levi-Civita connection ${}^{CG}\nabla$ of $F^*(M)$ with Cheeger-Gromoll metric ${}^{CG}g$ is characterized by the Koszul formula

$$\begin{aligned}2{}^{CG}g({}^{CG}\nabla_{\bar X}\bar Y,\bar Z)&=\bar X({}^{CG}g(\bar Y,\bar Z))+\bar Y({}^{CG}g(\bar Z,\bar X))-\bar Z({}^{CG}g(\bar X,\bar Y))\\&-{}^{CG}g(\bar X,[\bar Y,\bar Z])+{}^{CG}g(\bar Y,[\bar Z,\bar X])+{}^{CG}g(\bar Z,[\bar X,\bar Y])\end{aligned}\tag{5.6}$$

for any $\bar X,\bar Y,\bar Z\in\Im_0^1(F^*(M))$.

Let $X,Y,Z\in\Im_0^1(M)$, $\omega,\theta,\xi\in\Im_1^0(M)$. We calculate ${}^{CG}\nabla$ using the Koszul formulas for g and ${}^{CG}g$.

i) Direct calculations using (2.6), (4.1) and (5.6) give

$$\begin{aligned}&2{}^{CG}g({}^{CG}\nabla_{{}^HX}{}^HY,{}^HZ)={}^HX(g(Y,Z))+{}^HY(g(Z,X))-{}^HZ(g(X,Y))\\&-{}^{CG}g({}^HX,{}^H[X,Y]+\gamma R(Y,Z))+{}^{CG}g({}^HY,{}^H[Z,X]+\gamma R(Z,X))\\&+{}^{CG}g({}^HZ,{}^H[X,Y]+\gamma R(X,Y))=X(g(Y,Z))+Y(g(Z,X))\\&-Z(g(X,Y))-g(X,[Y,Z])+g(Y,[Z,X])+g(Z,[X,Y])\\&=2g(\nabla_XY,Z)\end{aligned}$$

and

$$\begin{aligned}&2{}^{CG}g({}^{CG}\nabla_{{}^HX}{}^HY,{}^{V_\gamma}\xi)={}^HX(g(Y,Z))-{}^{CG}g({}^HX,{}^{V_\gamma}(\nabla_Y\xi))\\&+{}^{CG}g({}^HY,-{}^{V_\gamma}(\nabla_X\xi))+{}^{CG}g({}^{V_\gamma}\xi,{}^H[X,Y]+\gamma R(X,Y))\\&={}^{CG}g({}^{V_\gamma}\xi,\gamma R(X,Y))={}^{CG}g({}^{V_\gamma}\xi,\textstyle\sum_{\sigma=1}^{n}{}^{V_\sigma}(X^\sigma\circ R(X,Y))\end{aligned}$$

from which it follows that

$${}^{CG}\nabla_{{}^HX}{}^HY={}^H(\nabla_XY)+\frac{1}{2}\sum_{\sigma=1}^{n}{}^{V_\sigma}(X^\sigma\circ R(X,Y)).$$

ii) Calculations similar to those in *i)* give

$$2\,{}^{CG}g({}^{CG}\nabla_{{}^HX}{}^{V_\beta}\theta, {}^HZ) = {}^{V_\beta}\theta(g(Z,X)) - {}^{CG}g({}^HX, -{}^{V_\beta}(\nabla_Z\theta))$$

$$+{}^{CG}g({}^{V_\beta}\theta, {}^H[Z,X] + \gamma R(Z,X)) + {}^{CG}g({}^HZ, {}^{V_\beta}(\nabla_X\theta)) = {}^{CG}g({}^{V_\beta}\theta, {}^H[Z,X])$$

$$+{}^{CG}g({}^{V_\beta}\theta, \sum_{\sigma=1}^{n}{}^{V_\sigma}(X^\sigma \circ R(Z,X))) = {}^{CG}g({}^{V_\beta}\theta, {}^{V_\beta}(X^\beta \circ R(Z,X)))$$

$$= \tfrac{1}{h_\beta}(g^{-1}(\theta, X^\beta \circ R(Z,X)) + g^{-1}(\theta, X^\beta)g^{-1}(X^\beta \circ R(Z,X), X^\beta)),$$

where $h_\beta = 1 + r_\beta^2$.
It is easily sean that

$$g^{-1}(\theta, X^\beta \circ R(Z,X)) = g^{kl}\theta_l(X^\beta \circ R(Z,X))_k = (g^{kl}\theta_l X_s^\beta R_{ijl}{}^s Z^i X^j)$$

$$= (g_{mi}X_s^\beta R^m_{\cdot jk}{}^s Z^i X^j \tilde{\theta}^k) = g(X^\beta(g^{-1} \circ R(\ \ , X)\tilde{\theta}, Z)$$

$$= {}^{CG}g({}^H(X^\beta(g^{-1} \circ R(\ \ , X)\tilde{\theta}), {}^HZ)$$

and

$$g^{-1}(X^\beta \circ R(Z,X), X^\beta) = (g^{ij}X_s^\beta R_{abi}{}^s Z^a X^b X_j^\beta)$$

$$= (X_s^\beta g^{ls} R_{abil} Z^a X^b \tilde{X}^{\beta i}) = (R_{abil} Z^a X^b \tilde{X}^{\beta l}\tilde{X}^{\beta i}) = (R_{ilab} Z^a X^b \tilde{X}^{\beta l}\tilde{X}^{\beta i})$$

$$= (-g_{ta}R_{ilb}{}^t Z^a X^b \tilde{X}^{\beta l}\tilde{X}^{\beta i}) = g(-R(\tilde{X}^\beta, \tilde{X}^\beta)X, Z) = 0,$$

where

$$\tilde{\theta}^k = g^{kl}\theta_l, \tilde{X}^{\beta i} = g^{is}X_s^\beta.$$

Thus, we have

$$2\,{}^{CG}g({}^{CG}\nabla_{{}^HX}{}^\beta\theta, {}^HZ) = \frac{1}{h_\beta}{}^{CG}g({}^H(X^\beta(g^{-1} \circ R(\ \ , X)\tilde{\theta}))). \tag{5.7}$$

Also using (4.1), (5.3) and (5.6), we have

$$\begin{aligned} &2\,{}^{CG}g({}^{CG}\nabla_{{}^HX}{}^\beta\theta, {}^{V_\beta}\xi) = {}^HX({}^{CG}g({}^{V_\beta}\theta, {}^{V_\beta}\xi)) - {}^{CG}g({}^{V_\beta}\theta, {}^{V_\beta}(\nabla_X\xi)) \\ &+{}^{CG}g({}^{V_\beta}\xi, {}^{V_\beta}(\nabla_X\theta)) = {}^{CG}g({}^{V_\beta}(\nabla_X\theta), {}^{V_\beta}\xi)) + {}^{CG}g({}^{V_\beta}\theta, {}^{V_\beta}(\nabla_X\xi)) \\ &-{}^{CG}g({}^{V_\beta}\theta, {}^{V_\beta}(\nabla_X\xi)) + {}^{CG}g({}^{V_\beta}\xi, {}^{V_\beta}(\nabla_X\theta)) = 2\,{}^{CG}g({}^{V_\beta}(\nabla_X\theta), {}^{V_\beta}\xi)). \end{aligned} \tag{5.8}$$

From (5.7) and (5.8) it follows that

$${}^{CG}\nabla_{{}^HX}{}^{V_\beta}\theta = {}^{V_\beta}(\nabla_X\theta) + \frac{1}{2h_\beta}{}^H(X^\beta(g^{-1} \circ R(\ \ , X)\tilde{\theta})).$$

iii) Calculations using (2.6), (4.1), (5.3) and (5.6) give

$$2\,{}^{CG}g({}^{CG}\nabla_{V_\alpha\omega}{}^HY, {}^HZ) = -{}^{CG}g({}^{V_\alpha}\omega, \sum_{\sigma=1}^{n}{}^{V_\sigma}(X^\sigma \circ R(Y,Z)))$$

$$= {}^{CG}g({}^{V_\alpha}\omega, \sum_{\sigma=1}^{n}{}^{V_\sigma}(X^\sigma \circ R(Z,Y))) = {}^{CG}g({}^{V_\alpha}\omega, {}^{V_\alpha}(X^\alpha \circ R(Z,Y)))$$

$$= \tfrac{1}{h_\alpha}{}^{CG}g({}^H(X^\alpha(g^{-1} \circ R(\ \ , Y)\tilde{\omega})), {}^HZ)$$

and

$${}^{CG}g({}^{CG}\nabla_{V_\alpha\omega}{}^HY, {}^{V_\gamma}\xi) = {}^HY({}^{CG}g({}^{V_\gamma}\xi, {}^{V_\alpha}\omega) - {}^{CG}g({}^{V_\alpha}\omega, [{}^HY, {}^{V_\gamma}\xi])$$

$$+{}^{CG}g({}^{V_\gamma}\xi, [{}^{V_\alpha}\omega, {}^HY]) = {}^{CG}g({}^{V_\gamma}\xi, {}^{V\alpha}(\nabla_Y\omega)) + {}^{CG}g({}^{V_\gamma}(\nabla_Y\xi), {}^{V_\alpha}\omega)$$

$$-{}^{CG}g({}^{V_\alpha}\omega, {}^{V_\gamma}(\nabla_Y\xi)) - {}^{CG}g({}^{V_\gamma}\xi, {}^{V\alpha}(\nabla_Y\omega)) = 0,$$

which implies that

$$^{CG}\nabla_{V_\alpha \omega}{}^{H}Y = \frac{1}{2h_\alpha}{}^{H}(X^\alpha(g^{-1}\circ R(\ \ ,Y)\tilde{\omega})).$$

iv) If $\alpha \neq \beta$. Using (2.6), (4.1) and (5.6), we get

$$2^{CG}g(^{CG}\nabla_{V_\alpha \omega}{}^{V_\beta}\theta, {}^{H}Z) = -^{CG}g(^{V_\alpha}\omega, [^{V_\beta}\theta, {}^{H}Z]) + {}^{CG}g(^{V_\beta}\theta, [^{H}Z, {}^{V_\alpha}\omega])$$

$$= -^{CG}g(^{V_\alpha}\omega, {}^{V_\beta}(\nabla_Z\theta)) + {}^{CG}g(^{V_\beta}\theta, {}^{V_\alpha}(\nabla_Z\omega)) = 0$$

and

$$2^{CG}g(^{CG}\nabla_{V_\alpha \omega}{}^{V_\beta}\theta, {}^{V_\gamma}\xi) = {}^{V_\alpha}\omega(^{CG}g(^{V_\beta}\theta, {}^{V_\gamma}\xi)) + {}^{V_\beta}\theta(^{CG}g(^{V_\gamma}\xi, {}^{V_\alpha}\omega))$$

$$-{}^{V_\gamma}\xi(^{CG}g(^{V_\alpha}\omega, {}^{V_\beta}\theta)).$$

Let $\gamma = \alpha \neq \beta$. Using (5.2) we have

$$^{CG}g(^{CG}\nabla_{V_\alpha \omega}{}^{V_\beta}\theta, {}^{V_\alpha}\xi) = {}^{V_\beta}\theta(^{CG}g(^{V_\alpha}\omega, {}^{V_\alpha}\xi)) = {}^{V_\beta}\theta\left(\tfrac{1}{h_\alpha}\ (g^{-1}(\xi,\omega)\right.$$

$$+g^{-1}(\xi, X^\alpha)g^{-1}\ (\omega, X^\alpha)) = -\delta^\beta_\alpha \tfrac{2}{h^2_\alpha} g^{-1}(\theta, X^\alpha)g^{-1}(\xi,\omega)$$

$$-\delta^\beta_\alpha \tfrac{2}{h^2_\alpha} g^{-1}(\theta, X^\alpha)g^{-1}(\xi, X^\alpha)g^{-1}(\omega, X^\alpha)$$

$$+\delta^\beta_\alpha \theta_i \partial_{i_\alpha}((g^{rs}\xi_r X^\alpha_s)(g^{pq}\omega_p X^\alpha_q)) = 0.$$

From above calculations it follows that
$^{CG}\nabla_{V_\alpha \omega}{}^{V_\beta}\theta = 0$ for $\alpha \neq \beta$.
Now suppose that $\beta = \alpha$. Calculations using (2.6), (5.3) and (5.6) give

$$2^{CG}g(^{CG}\nabla_{V_\alpha \omega}{}^{V_\alpha}\theta, {}^{H}Z) = -{}^{H}Z(^{CG}g(^{V_\alpha}\omega, {}^{V_\alpha}\theta) - {}^{CG}g(^{V_\alpha}\omega, [^{V_\alpha}\theta, {}^{H}Z])$$

$$+{}^{CG}g(^{V_\alpha}\theta, [^{H}Z, {}^{V_\alpha}\omega]) = -{}^{CG}g(^{V_\alpha}\omega, {}^{V_\alpha}(\nabla_Z\theta)) - {}^{CG}g(^{V_\alpha}\theta, {}^{V_\alpha}(\nabla_Z\omega))$$

$$+{}^{CG}g(^{V_\alpha}\omega, {}^{V_\alpha}(\nabla_Z\theta)) + {}^{CG}g(^{V_\alpha}\theta, {}^{V_\alpha}(\nabla_Z\omega)) = 0$$

and

$$2^{CG}g(^{CG}\nabla_{V_\alpha \omega}{}^{V_\alpha}\theta, {}^{V_\gamma}\xi) = {}^{V_\alpha}\omega(^{CG}g(^{V_\alpha}\theta, {}^{V_\gamma}\xi)) + {}^{V_\alpha}\theta(^{CG}g(^{V_\gamma}\xi, {}^{V_\alpha}\omega))$$

$$-{}^{V_\gamma}\xi(^{CG}g(^{V_\alpha}\omega, {}^{V_\alpha}\theta)).$$

If we put $\gamma \neq \alpha$. Then by using (4.1) and (5.2) we get

$$2^{CG}g(^{CG}\nabla_{V_\alpha \omega}{}^{V_\alpha}\theta, {}^{V_\gamma}\xi) = -{}^{V_\gamma}\xi(^{CG}g(^{V_\alpha}\omega, {}^{V_\alpha}\theta)) = -{}^{V_\gamma}\xi\left(\tfrac{1}{h_\alpha}\ (g^{-1}(\omega,\theta)\right.$$

$$+g^{-1}(\omega, X^\alpha)g^{-1}\ (\theta, X^\alpha))) = \tfrac{2}{h^2_\alpha}\delta^\gamma_\alpha g^{-1}(\xi, X^\alpha)g^{-1}(\omega,\theta)$$

$$+\tfrac{2}{h^2_\alpha}\delta^\gamma_\alpha g^{-1}(\xi, X^\alpha)g^{-1}(\omega, X^\alpha)g^{-1}(\theta, X^\alpha) - \tfrac{1}{h^2_\alpha}\delta^\gamma_\alpha(g^{-1}(\omega,\xi)g^{-1}(\theta, X^\alpha)$$

$$+g^{-1}(\omega, X^\alpha)g^{-1}(\theta,\xi)) = 0.$$

If we put $\gamma = \alpha$. Calculations like above give

$$\begin{aligned}
&2^{CG}g(^{CG}\nabla_{V_\alpha \omega}{}^{V_\alpha}\theta, {}^{V_\alpha}\xi) = {}^{V_\alpha}\omega(^{CG}g(^{V_\alpha}\theta, {}^{V_\alpha}\varsigma)) + {}^{V_\alpha}\theta(^{CG}g(^{V_\alpha}\xi, {}^{V_\alpha}\omega)) \\
&-{}^{V_\alpha}\xi(^{CG}g(^{V_\alpha}\omega, {}^{V_\alpha}\theta)) = {}^{V_\alpha}\omega(\tfrac{1}{h_\alpha}(g^{-1}(\theta,\xi) + g^{-1}(\theta, X^\alpha)g^{-1}(\xi, X^\alpha))) \\
&+{}^{V_\alpha}\theta(\tfrac{1}{h_\alpha}(g^{-1}(\xi,\omega) + g^{-1}(\xi, X^\alpha)g^{-1}(\omega, X^\alpha))) - {}^{V_\alpha}\xi(\tfrac{1}{h_\alpha}(g^{-1}(\omega,\theta) \\
&+g^{-1}(\omega, X^\alpha)g^{-1}(\theta, X^\alpha))) = -\tfrac{2}{h^2_\alpha}g^{-1}(\omega, X^\alpha)g^{-1}(\theta,\xi) \\
&-\tfrac{2}{h^2_\alpha}g^{-1}(\theta, X^\alpha)g^{-1}(\xi,\omega) + \tfrac{2}{h^2_\alpha}g^{-1}(\xi, X^\alpha)g^{-1}(\omega,\theta) \\
&-\tfrac{2}{h^2_\alpha}g^{-1}(\theta, X^\alpha)g^{-1}(\xi, X^\alpha)g^{-1}(\omega, X^\alpha) + \tfrac{2}{h_\alpha}g^{-1}(\theta,\omega)g^{-1}(\xi, X^\alpha).
\end{aligned} \tag{5.9}$$

Taking into account (4.1) and (5.4) in (5.9), we get

$$h_\alpha^2 {}^{CG}g({}^{CG}\nabla_{{}^{V_\alpha}\omega} {}^{V_\alpha}\theta, {}^{V_\alpha}\xi) = -g^{-1}(\omega, X^\alpha)g^{-1}(\theta,\xi) - g^{-1}(\theta, X^\alpha)g^{-1}(\xi,\omega)$$
$$+g^{-1}(\xi, X^\alpha)g^{-1}(\omega,\theta) - g^{-1}(\theta, X^\alpha)g^{-1}(\xi, X^\alpha)g^{-1}(\omega, X^\alpha)$$
$$+h_\alpha g^{-1}(\theta,\omega)g^{-1}(\xi, X^\alpha) = -h_\alpha g^{-1}(\omega, X^\alpha){}^{CG}g({}^{V_\alpha}\theta, {}^{V_\alpha}\xi)$$
$$-h_\alpha g^{-1}(\theta, X^\alpha){}^{CG}g({}^{V_\alpha}\xi, {}^{V_\alpha}\theta) + h_\alpha g^{-1}(\xi, X^\alpha){}^{CG}g({}^{V_\alpha}\theta, {}^{V_\alpha}\omega)$$
$$+h_\alpha^2 g^{-1}(\xi, X^\alpha){}^{CG}g({}^{V_\alpha}\theta, {}^{V_\alpha}\omega) - h_\alpha g^{-1}(\xi, X^\alpha)g^{-1}(\theta, X^\alpha)g^{-1}(\omega, X^\alpha)$$
$$= -h_\alpha {}^{CG}g({}^{V_\alpha}\omega, \gamma\delta){}^{CG}g({}^{V_\alpha}\theta, {}^{V_\alpha}\xi) - h_\alpha {}^{CG}g({}^{V_\alpha}\theta, \gamma\delta){}^{CG}g({}^{V_\alpha}\xi, {}^{V_\alpha}\omega) \tag{5.10}$$
$$+h_\alpha(1+h_\alpha){}^{CG}g({}^{V_\alpha}\xi, \gamma\delta){}^{CG}g({}^{V_\alpha}\omega, {}^{V_\alpha}\theta)$$
$$-h_\alpha {}^{CG}g({}^{V_\alpha}\xi, \gamma\delta){}^{CG}g({}^{V_\alpha}\theta, \gamma\delta){}^{CG}g({}^{V_\alpha}\omega, \gamma\delta)$$
$$= {}^{CG}g(-h_\alpha {}^{CG}g({}^{V_\alpha}\omega, \gamma\delta){}^{V_\alpha}\theta - h_\alpha {}^{CG}g({}^{V_\alpha}\theta, \gamma\delta){}^{V_\alpha}\omega$$
$$+h_\alpha(1+h_\alpha){}^{CG}g({}^{V_\alpha}\omega, {}^{V_\alpha}\theta)\gamma\delta$$
$$-h_\alpha {}^{CG}g({}^{V_\alpha}\theta, \gamma\delta){}^{CG}g({}^{V_\alpha}\omega, \gamma\delta)\gamma\delta, {}^{V_\alpha}\xi),$$

From (5.9) and (5.10) implies that

$${}^{CG}\nabla_{{}^{V_\alpha}\omega} {}^{V_\alpha}\theta = -\tfrac{1}{h_\alpha}({}^{CG}g({}^{V_\alpha}\omega, \gamma\delta){}^{V_\alpha}\theta + {}^{CG}g({}^{V_\alpha}\theta, \gamma\delta){}^{V_\alpha}\omega$$
$$+\tfrac{1+h_\alpha}{h_\alpha}{}^{CG}g({}^{V_\alpha}\omega, {}^{V_\alpha}\theta)\gamma\delta - \tfrac{1}{h_\alpha}{}^{CG}g({}^{V_\alpha}\theta, \gamma\delta){}^{CG}g({}^{V_\alpha}\omega, \gamma\delta)\gamma\delta.$$

Hence theorem is proved.

□

6. Components of connection ${}^{CG}\nabla$

We write

$${}^{CG}\nabla_{D_I} D_J = {}^{CG}\Gamma_{IJ}^K D_K$$

with respect to the adapted frame $\{D_K\}$ of linear coframe bundle $F^*(M)$, where ${}^{CG}\Gamma_{IJ}^K$ denote the components (Christoffel symbols) of Levi-Civita connection ${}^{CG}\nabla$. Then by using Theorem 5.2, we immediately get following

Theorem 6.1. *Let (M, g) be a Riemannian manifold and ${}^{CG}\nabla$ be the Levi-Civita connection of the linear coframe bundle $F^*(M)$ equipped with the metric ${}^{CG}g$. Then particular values of ${}^{CG}\Gamma_{IJ}^K$ for different indices by taking account of (5.5) are then found to be*

$${}^{CG}\Gamma_{ij}^k = \Gamma_{ij}^k, \; {}^{CG}\Gamma_{ij}^{k_\gamma} = \frac{1}{2}X_m^\gamma R_{ijk}{}^m,$$
$${}^{CG}\Gamma_{ij_\beta}^k = \frac{1}{2h_\beta}X_m^\beta R_{\cdot\, i\, \cdot}^{k\; jm}, \; {}^{CG}\Gamma_{ij_\beta}^{k_\gamma} = -\delta_\beta^\gamma \Gamma_{ik}^j,$$
$${}^{CG}\Gamma_{i_\alpha j}^k = \frac{1}{2h_\alpha}X_m^\alpha R_{\cdot\, j\, \cdot}^{k\; im}, \; {}^{CG}\Gamma_{i_\alpha j}^{k_\gamma} = {}^{CG}\Gamma_{i_\alpha j_\beta}^k = 0, \tag{6.1}$$
$${}^{CG}\Gamma_{i_\alpha j_\beta}^{k_\gamma} = 0 \text{ for } \alpha \neq \beta,$$
$${}^{CG}\Gamma_{i_\alpha j_\alpha}^{k_\gamma} = -\tfrac{1}{h_\alpha}(\tilde{X}^{\alpha i}\delta_\gamma^\alpha \delta_k^j + \tilde{X}^{\alpha j}\delta_\gamma^\alpha \delta_k^i) + \tfrac{1+h_\alpha}{h_\alpha^2} g^{ij} X_k^\gamma$$
$$+\tfrac{1}{h_\alpha^2}\tilde{X}^{\alpha i}\tilde{X}^{\alpha j}X_k^\gamma,$$

where $\tilde{X}^{\alpha i} = g^{is}X_s^\alpha$.

Proof. Let $X, Y \in \Im^1_0(M),\ \ \omega,\theta \in \Im^0_1(M)$. Using formulas (3.3) and (5.5), we obtain

$$\begin{gathered}{}^{CG}\nabla_{{}^HX}{}^HY = {}^{CG}\nabla_{X^iD_i}(Y^jD_j) = X^{i\,CG}\nabla_{D_i}(Y^jD_j) = X^i(Y^{j\,CG}\nabla_{D_i}D_j \\ +D_iY^jD_j) = X^iY^{j\,CG}\Gamma^k_{ij}D_k + X^iY^{j\,CG}\Gamma^{k_\gamma}_{ij}D_{k_\gamma} + X^i\partial_iY^jD_j\end{gathered} \tag{6.2}$$

and

$$\begin{gathered}{}^H(\nabla_XY) + \tfrac{1}{2}\sum_{\sigma=1}^{n}{}^{V_\sigma}(X^\sigma \circ R(X,Y)) = (\nabla_XY)^iD_i \\ +\tfrac{1}{2}\sum_{\sigma=1}^{n}\delta^\gamma_\sigma X^\sigma_m R_{pqh}{}^mX^pY^qD_{h_\gamma} = X^j\partial_jY^iD_i + X^jY^s\Gamma^i_{js}D_i \\ +\tfrac{1}{2}\sum_{\sigma=1}^{n}\delta^\gamma_\sigma X^\sigma_m R^m_{pqh}X^pY^qD_{h_\gamma}.\end{gathered} \tag{6.3}$$

Equating the right-hand sides of equalities (6.2) and (6.3), we will have

$${}^{CG}\Gamma^k_{ij} = \Gamma^k_{ij},\ {}^{CG}\Gamma^{k_\gamma}_{ij} = \frac{1}{2}X^\gamma_m R_{ijk}{}^m.$$

Similarly, calculations using (3.3), (3.4) and (5.5) give

$$\begin{gathered}{}^{CG}\nabla_{{}^HX}{}^{V_\beta}\theta = {}^{CG}\nabla_{X^iD_i}(\delta^\beta_\sigma\theta_jD_{j_\sigma}) = X^{i\,CG}\nabla_{D_i}(\delta^\beta_\sigma\theta_jD_{j_\sigma}) \\ = \delta^\beta_\omega X^i(D_i\theta_jD_{j_\sigma} + \theta_j{}^{CG}\nabla_{D_i}D_{j_\sigma}) = \delta^\beta_\sigma X^i\partial_i\theta_jD_{j_\sigma} + \delta^\beta_\sigma X^i\theta_j{}^{CG}\Gamma^k_{ij_\sigma}D_k \\ +\delta^\beta_\sigma X^i\theta_j{}^{CG}\Gamma^{k_\gamma}_{ij_\sigma}D_{k_\gamma}\end{gathered} \tag{6.4}$$

and

$$\begin{gathered}{}^{V_\beta}(\nabla_X\theta) + \tfrac{1}{2h_\beta}{}^H(X^\beta(g^{-1}\circ R(\ \ ,X)\tilde{\theta})) = \delta^\beta_\sigma(X^i(\partial_i\theta_j - \Gamma^m_{ij}\theta_m)D_{j_\sigma} \\ +\tfrac{1}{2h_\beta}(X^\beta_m R^l{}_{\cdot ik}{}^m X^ig^{ks}\theta_s)D_l = \delta^\beta_\sigma X^i\partial_i\theta_jD_{j_\sigma} - \delta^\beta_\sigma X^i\Gamma^m_{ij}\theta_mD_{j_\sigma} \\ +\tfrac{1}{2h_\beta}X^\beta_m R^l{}_{\cdot i\,\cdot}{}^{sm}X^i\theta_s)D_l.\end{gathered} \tag{6.5}$$

Comparing the right-hand sides of equalities (6.4) and (6.5), we arrive at the following

$${}^{CG}\Gamma^k_{ij_\beta} = \frac{1}{2h_\beta}X^\beta_m R^k{}_{\cdot\, i\,\cdot}{}^{jm},\ {}^{CG}\Gamma^{k_\gamma}_{ij_\beta} = -\delta^\beta_\gamma\Gamma^j_{ik}.$$

By calculations similar to those above we yield

$$\begin{gathered}{}^{CG}\nabla_{{}^{V_\alpha}\omega}{}^HY = {}^{CG}\nabla_{\delta^\alpha_\sigma\omega D_{i_\sigma}}(Y^jD_j) = \delta^\alpha_\sigma\omega_i{}^{CG}\nabla_{D_{i_\sigma}}(Y^jD_j) \\ = \delta^\alpha_\sigma\omega_iY^{j\,CG}\nabla_{D_{i_\sigma}}D_j = \delta^\alpha_\sigma\omega_iY^{j\,CG}\Gamma^K_{i_\sigma j}D_K = \delta^\alpha_\sigma\omega_iY^{j\,CG}\Gamma^k_{i_\sigma j}D_k \\ +\delta^\alpha_\sigma\omega_iY^{j\,CG}\Gamma^{k_\gamma}_{i_\sigma j}D_{k_\gamma}\end{gathered} \tag{6.6}$$

and

$$\begin{gathered}\tfrac{1}{2h_\alpha}{}^H(X^\alpha(g^{-1}\circ R(\ \ ,Y)\tilde{\omega})) = \tfrac{1}{2h_\alpha}(X^\alpha_m R^l{}_{\cdot jk}{}^mY^jg^{ks}\omega_s)D_l \\ = \tfrac{1}{2h_\alpha}X^\alpha_m R^l{}_{\cdot j\,\cdot}{}^{sm}Y^j\omega_sD_l.\end{gathered} \tag{6.7}$$

From (6.6) and (6.7), we get

$${}^{CG}\Gamma^k_{i_\alpha j} = \frac{1}{2h_\alpha}X^\alpha_m R^k{}_{\cdot\, j\,\cdot}{}^{im},\ {}^{CG}\Gamma^{k_\gamma}_{i_\alpha j} = 0.$$

Now we assume that $\alpha \neq \beta$. Then by using of (3.3), (3.4) and (5.5), we have

$$\begin{gathered}{}^{CG}\nabla_{{}^{V_\alpha}\omega}{}^{V_\beta}\theta = {}^{CG}\nabla_{\delta^\alpha_\sigma\omega D_{i_\sigma}}(\delta^\beta_\tau\theta_jD_{j_\tau}) = \delta^\alpha_\sigma\omega_i{}^{CG}\nabla_{D_{i_\sigma}}(\delta^\beta_\tau\theta_jD_{j_\tau}) \\ = \delta^\alpha_\sigma\omega_i\delta^\beta_\tau({}^{CG}\nabla_{D_{i_\sigma}}\theta_j)D_{j_\tau} + \delta^\alpha_\sigma\omega_i\delta^\beta_\tau\theta_j{}^{CG}\nabla_{D_{i_\sigma}}D_{j_\tau} \\ = \delta^\alpha_\sigma\omega_i\delta^\beta_\tau\theta_j{}^{CG}\Gamma^k_{i_\sigma j_\tau}D_k + \delta^\alpha_\sigma\omega_i\delta^\beta_\tau\theta_j{}^{CG}\Gamma^{k_\gamma}_{i_\sigma j_\tau}D_{k_\gamma} = 0.\end{gathered}$$

The last relation shows that

$$^{CG}\Gamma^{k}_{i_\alpha j_\beta}=0, {}^{CG}\Gamma^{k_\gamma}_{i_\alpha j_\beta}=0 \text{ for } \alpha\neq\beta.$$

If $\alpha=\beta$. Calculations like above give

$$\begin{aligned}&^{CG}\nabla_{V_\alpha\omega}{}^{V_\alpha}\theta=\delta^\alpha_\sigma\omega_i\delta^\alpha_\tau\theta_j{}^{CG}\Gamma^k_{i_\sigma j_\tau}D_k+\delta^\alpha_\sigma\omega_i\delta^\alpha_\tau\theta_j{}^{CG}\Gamma^{k_\gamma}_{i_\sigma j_\tau}D_{k_\gamma}\\&=\omega_i\theta_j{}^{CG}\Gamma^k_{i_\alpha j_\alpha}D_k+\omega_i\theta_j{}^{CG}\Gamma^{k_\gamma}_{i_\alpha j_\alpha}D_{k_\gamma}=-\tfrac{1}{h_\alpha}({}^{CG}g({}^{V\alpha}\omega,\gamma\delta)^{V_\alpha}\theta\\&+{}^{CG}g({}^{V\alpha}\theta,\gamma\delta)^{V_\alpha}\omega)+\tfrac{1+h_\alpha}{h_\alpha}{}^{CG}g({}^{V\alpha}\omega,{}^{V_\alpha}\theta)\gamma\delta\\&-\tfrac{1}{h_\alpha}{}^{CG}g({}^{V\alpha}\theta,\gamma\delta){}^{CG}g({}^{V\alpha}\omega,\gamma\delta)\gamma\delta=-\tfrac{1}{h_\alpha}(g^{is}\omega_iX^\alpha_s\delta^\alpha_\tau\theta_jD_{j_\tau}\\&+g^{js}\theta_jX^\alpha_s\delta^\alpha_\sigma\omega_iD_{i_\sigma})+\tfrac{1+h_\alpha}{h^2_\alpha}(g^{ij}\omega_i\theta_j+\end{aligned}$$

$$+g^{is}\omega_iX^\alpha_sg^{jm}\theta_jX^\alpha_m)\sum_{\mu=1}^n\delta^\mu_\gamma X^\mu_kD_{k_\gamma}-\frac{1}{h_\alpha}g^{jm}\theta_jX^\alpha_mg^{is}\omega_iX^\alpha_s)\sum_{\mu=1}^n\delta^\mu_\gamma X^\mu_kD_{k_\gamma}$$

$$=\left[-\frac{1}{h_\alpha}\,(g^{is}X^\alpha_s\delta^\alpha_\gamma\delta^j_k+g^{js}X^\alpha_s\delta^\alpha_\gamma\delta^i_k)+\frac{1+h_\alpha}{h^2_\alpha}(g^{ij}+g^{is}X^\alpha_sg^{jm}X^\alpha_m)X^\gamma_k-\frac{1}{h_\alpha}g^{is}X^\alpha_sg^{jm}X^\alpha_mX^\gamma_k\right]\omega_i\theta_jD_{k_\gamma},$$

from which it follows that

$$\begin{aligned}&^{CG}\Gamma^k_{i_\alpha j_\alpha}=0,\\&^{CG}\Gamma^{k_\gamma}_{i_\alpha j_\alpha}=-\tfrac{1}{h_\alpha}(g^{is}X^\alpha_s\delta^\alpha_\gamma\delta^j_k+g^{js}X^\alpha_s\delta^\alpha_\gamma\delta^i_k)+\tfrac{1+h_\alpha}{h^2_\alpha}(g^{ij}+\\&+g^{is}X^\alpha_sg^{jm}X^\alpha_m)X^\gamma_k-\tfrac{1}{h_\alpha}g^{is}X^\alpha_sg^{jm}X^\alpha_mX^\gamma_k=-\tfrac{1}{h_\alpha}(\tilde X^{\alpha i}\delta^\alpha_\gamma\delta^j_k\\&+\tilde X^{\alpha j}\delta^\alpha_\gamma\delta^i_k)+\tfrac{1+h_\alpha}{h^2_\alpha}g^{ij}X^\gamma_k+\tfrac{1}{h^2_\alpha}\tilde X^{\alpha i}\tilde X^{\alpha j}X^\gamma_k,\end{aligned}$$

where $\tilde X^{\alpha i}=g^{is}X^\alpha_s$. This completes the proof.

7. The Riemannian curvature tensor of $F^*(M)$ with ^{CG}g

Let ^{CG}R be a curvature tensor field of ^{CG}g. The curvature tensor field ^{CG}R has components

$$\begin{aligned}&^{CG}R_{IJK}{}^L=D_I{}^{CG}\Gamma^L_{JK}-D_J{}^{CG}\Gamma^L_{IK}+{}^{CG}\Gamma^L_{IS}{}^{CG}\Gamma^S_{JK}-\\&^{CG}\Gamma^L_{JS}{}^{CG}\Gamma^S_{IK}-\Omega_{IJ}{}^{S\,CG}\Gamma^L_{SK},\end{aligned}\tag{7.1}$$

with respect to the adapted frame $\{D_I\}$, where $\Omega_{IJ}{}^K$ be a non-holonomic object.

Taking account (3.5), (5.5), (6.1) and (7.1), we find the components of curvature tensor field ^{CG}R.

$$\begin{aligned}&^{CG}R_{ijk}{}^l=R_{ijk}{}^l+\sum_{\sigma=1}^n\tfrac{1}{4h_\sigma}X^\sigma_mX^\sigma_r(R^l{}_{.\,i\,.}{}^{sm}R_{jks}{}^r-R^l{}_{.\,j\,.}{}^{sm}R_{iks}{}^r)\\&-\sum_{\sigma=1}^n\tfrac{1}{2h_\sigma}X^\sigma_mX^\sigma_rR_{ijs}{}^mR^l{}_{.\,k\,.}{}^{sr},\end{aligned}$$

$$^{CG}R_{i_\alpha jk}{}^l=-\frac{1}{2h_\alpha}X^\alpha_s\nabla_jR^l{}_{.\,k\,.}{}^{is},$$

$$^{CG}R_{ijk_\gamma}{}^l=\frac{1}{2h_\gamma}X^\gamma_m(\nabla_iR^l{}_{.\,j\,.}{}^{ks}-\nabla_jR^l{}_{.\,i\,.}{}^{ks}),$$

$$
\begin{aligned}
{}^{CG}R_{ijk}{}^{l_\tau} &= \frac{1}{2}X^\tau_m(\nabla_i R_{jkl}{}^m - \nabla_j R_{ikl}{}^m),\\
{}^{CG}R_{ijk_\gamma}{}^{l_\tau} &= \delta^\gamma_\tau R_{jil}{}^k + \tfrac{1}{4h_\gamma}X^\tau_m X^\gamma_r(R_{isl}{}^m R^s{}_{.\,j\,.}{}^{kr} - R_{jsl}{}^m R^s{}_{.\,i\,.}{}^{kr})\\
&-\tfrac{1+h_\gamma}{h_\gamma^2}X^\gamma_m X^\tau_l R_{ij\,.}{}^{km} + \tfrac{1}{h_\gamma}X^\gamma_m\delta^\gamma_\tau(R_{ijs}{}^m\tilde{X}^{\gamma s}\delta^k_l + R_{ijl}{}^m\tilde{X}^{\gamma k}) \qquad (7.2)\\
&-\tfrac{1}{h_\gamma^2}\tilde{X}^{\gamma s}\tilde{X}^{\gamma k}X^\tau_l X^\gamma_m R_{ijs}{}^m,\\
{}^{CG}R_{i_\alpha jk}{}^{l_\tau} &= \tfrac{1}{2}\delta^\alpha_\tau R_{jkl}{}^i + \tfrac{1}{2h_\alpha}\delta^\alpha_\tau X^\alpha_m R_{jks}{}^m(\tilde{X}^{\alpha i}\delta^s_l + \tilde{X}^{\alpha s}\delta^i_l)\\
&+\tfrac{1+h_\alpha}{2h_\alpha^2}X^\alpha_m R_{jk\,.}{}^{im}X^\tau_l + \tfrac{1}{2h_\alpha^2}X^\alpha_m R_{jks}{}^m\tilde{X}^{\alpha i}\tilde{X}^{\alpha s}X^\tau_l +\\
&-\frac{1}{4h_\alpha}X^\tau_m X^\alpha_r R_{jsl}{}^m R^s{}_{.\,k\,.}{}^{ir},
\end{aligned}
$$

$$
{}^{CG}R_{i_\alpha j_\beta k}{}^l = \frac{1}{4h_\alpha h_\beta}X^\alpha_m X^\beta_r(R^l{}_{.s\,.}{}^{im}R^s{}_{.k\,.}{}^{jr} - R^l{}_{.s\,.}{}^{jr}R^s{}_{.k\,.}{}^{im}) \text{ for } \alpha\neq\beta,
$$

$$
\begin{aligned}
{}^{CG}R_{i_\alpha j_\alpha k}{}^l &= \tfrac{1}{2h_\alpha}(R^l{}_{.\,k\,.}{}^{ji} - R^l{}_{.\,k\,.}{}^{ij})\\
&+\tfrac{1}{4h_\alpha^2}X^\alpha_m X^\alpha_r(R^l{}_{.s\,.}{}^{im}R^s{}_{.k\,.}{}^{jr} - R^l{}_{.s\,.}{}^{jr}R^s{}_{.k\,.}{}^{im})\\
&+\tfrac{1}{h_\alpha^2}X^\alpha_m(\tilde{X}^{\alpha j}R^l{}_{.k\,.}{}^{im} - \tilde{X}^{\alpha i}R^l{}_{.k\,.}{}^{jm}),
\end{aligned}
$$

$$
{}^{CG}R_{i_\alpha jk_\gamma}{}^l = \frac{1}{h_\alpha h_\gamma}X^\alpha_m X^\gamma_r R^l{}_{.s\,.}{}^{im}R^s{}_{.j\,.}{}^{kr} \text{ for } \alpha\neq\gamma,
$$

$$
\begin{aligned}
{}^{CG}R_{i_\alpha jk_\alpha}{}^l &= \tfrac{1}{2h_\alpha}R^l{}_{.j\,.}{}^{ki} + \tfrac{1}{2h_\alpha^2}(\tilde{X}^{\alpha k}R^l{}_{.j\,.}{}^{im} - \tilde{X}^{\alpha i}R^l{}_{.j\,.}{}^{km})X^\alpha_m\\
&+\tfrac{1}{4h_\alpha}X^\alpha_m X^\alpha_r R^l{}_{.s\,.}{}^{im}R^s{}_{.j\,.}{}^{kr} - \sum_{\sigma=1}^{n}\tfrac{1}{2h_\sigma h_\alpha}X^\sigma_m X^\alpha_s R^l{}_{.j\,.}{}^{sm}g^{ik}\\
&-\sum_{\sigma=1}^{n}\tfrac{1}{2h_\sigma h_\alpha^2}X^\sigma_m X^\alpha_s R^l{}_{.j\,.}{}^{sm}(g^{ik} + \tilde{X}^{\alpha i}\tilde{X}^{\alpha k}),
\end{aligned}
$$

$$
{}^{CG}R_{i_\alpha jk_\alpha}{}^{l_\tau} = \frac{1}{h_\alpha}A + \frac{1+h_\alpha}{h_\alpha^2}B + \frac{1}{h_\alpha^2}C,
$$

where

$$
\begin{aligned}
A &= \delta^\alpha_\tau(\Gamma^k_{js}(\tilde{X}^{\alpha i}\delta^s_l + \tilde{X}^{\alpha s}\delta^i_l) + \Gamma^s_{jl}(\tilde{X}^{\alpha i}\delta^k_s + \tilde{X}^{\alpha k}\delta^i_s)\\
&+\Gamma^i_{js}(\tilde{X}^{\alpha s}\delta^k_l + \tilde{X}^{\alpha k}\delta^s_l),\\
B &= -\Gamma^k_{js}g^{is}X^\tau_l - \Gamma^s_{jl}g^{ik}X^\tau_s - \Gamma^i_{js}g^{sk}X^\tau_l,\\
C &= -\Gamma^k_{js}\tilde{X}^{\alpha i}\tilde{X}^{\alpha s}X^\tau_l - \Gamma^s_{jl}\tilde{X}^{\alpha i}\tilde{X}^{\alpha k}X^\tau_s - \Gamma^i_{js}\tilde{X}^{\alpha s}\tilde{X}^{\alpha k}X^\tau_l,\\
{}^{CG}R_{i_\alpha j_\beta k_\alpha}{}^{l_\tau} &= \tfrac{1}{h_\beta h_\alpha^2}\Big[\delta^\beta_\tau\delta^j_l\ (2h_\beta - 1)(g^{ik}(1+h_\alpha) + \tilde{X}^{\alpha i}\tilde{X}^{\alpha k})\\
&+3\tilde{X}^{\beta j}g^{ik}X^\tau_l(h_\alpha + 1) + 3\ \tilde{X}^{\alpha i}\tilde{X}^{\alpha k}\tilde{X}^{\beta j}X^\tau_l\Big]
\end{aligned}
$$

for $\alpha\neq\beta$,

$$
\begin{aligned}
{}^{CG}R_{i_\alpha j_\alpha k_\alpha}{}^{l_\tau} &= \tfrac{h_\alpha-1}{h_\alpha^3}\tilde{X}^{\alpha k}\delta^\alpha_\tau(\tilde{X}^{\alpha i}\delta^j_l - \tilde{X}^{\alpha j}\delta^i_l)\\
&+\tfrac{h_\alpha+2}{h_\alpha^3}X^\tau_l(\tilde{X}^{\alpha j}g^{ki} - \tilde{X}^{\alpha i}g^{kj}) + \tfrac{h_\alpha^2+h_\alpha+1}{h_\alpha^3}\delta^\alpha_\tau(g^{jk}\delta^l_i - g^{ik}\delta^j_l),\\
{}^{CG}R_{i_\alpha j_\beta k_\gamma}{}^l &= {}^{CG}R_{i_\alpha j_\beta k}{}^{l_\tau} = 0,
\end{aligned}
$$

$$ {}^{CG}R_{i_\alpha j k_\gamma}{}^{l_\tau} = {}^{CG}R_{i_\alpha j_\alpha k_\gamma}{}^{l_\tau} = 0 \text{ for } \alpha \neq \gamma. $$

It is known that the sectional curvature (see [8, p. 200]) on $(F^*(M), {}^{CG}g)$ for $P(U,V)$ is given by

$$ {}^{CG}K(P) = -\frac{{}^{CG}R_{kijs}U^kV^iU^jV^s}{({}^{CG}g_{kj}{}^{CG}g_{is} - {}^{CG}g_{ks}{}^{CG}g_{ij})U^kV^iU^jV^s}, \tag{7.3} $$

where $P(U,V)$ denotes the plane spanned by (U,V). Let $\{X_i\}$ and $\{\omega^i\}$, $i = 1,...,n$, be a local orthonormal frame and coframe on M, respectively. Then from (4.1) we see that $\{{}^HX_1,...,{}^HX_n{}^{V_1}\omega^1,...,{}^{V_n}\omega^1,...,{}^{V_1}\omega^n,...,{}^{V_1}\omega^n,...,{}^{V_n}\omega^n\}$ is a local orthonormal frame on $F^*(M)$. Let ${}^{CG}K({}^HX,{}^HY), {}^{CG}K({}^HX,{}^{V_\beta}\theta), {}^{CG}K({}^{V_\alpha}\omega,{}^{V_\alpha}\theta)$ and ${}^{CG}K({}^{V_\alpha}\omega,{}^{V_\beta}\theta)$ denote the sectional curvature of the plane spanned by $({}^HX,{}^HY), ({}^HX,{}^{V_\beta}\theta), ({}^{V_\alpha}\omega,{}^{V_\alpha}\theta)$ and $({}^{V_\alpha}\omega,{}^{V_\beta}\theta)$ on $F^*(M)$, respectively. Then direct calculations using (3.3), (3.4), (4.2) and (7.3) give

i)

$$ {}^{CG}K({}^HX,{}^HY) = -\frac{{}^{CG}R_{kijs}{}^HX^k{}^HY^i{}^HX^j{}^HY^s}{({}^{CG}g_{kj}{}^{CG}g_{is} - {}^{CG}g_{ks}{}^{CG}g_{ij}){}^HX^k{}^HY^i{}^HX^j{}^HY^s} $$

$$ = \frac{{}^{CG}R_{kij}{}^l{}^{CG}g_{sl}{}^HX^k{}^HY^i{}^HX^j{}^HY^s + {}^{CG}R_{kij}{}^{l_\tau}{}^{CG}g_{sl_\tau}{}^HX^k{}^HY^i{}^HX^j{}^HY^s}{({}^{CG}g_{kj}{}^{CG}g_{is} - {}^{CG}g_{ks}{}^{CG}g_{ij}){}^HX^k{}^HY^i{}^HX^j{}^HY^s} $$

$$ = \left(\frac{-R^l_{kij} + \sum_{\sigma=1}^n \frac{1}{2h_\sigma} X^\sigma_m X^\sigma_r R_{kit}{}^r R^l{}_{\cdot\, j\, \cdot}{}^{tm} - \sum_{\sigma=1}^n \frac{1}{4h_\sigma} X^\sigma_m X^\sigma_r (R^l{}_{\cdot\, k\, \cdot}{}^{tm} R_{ijt}{}^r}{(g_{kj}g_{is} - g_{ks}g_{ij})X^kY^iX^jY^s} \right. $$

$$ \left. - \frac{R^l{}_{\cdot\, i\, \cdot}{}^{tm} R_{kjt}{}^r)X^kY^iX^jY^s}{(g_{kj}g_{is} - g_{ks}g_{ij})X^kY^iX^jY^s} \right) = K(X,Y) $$

$$ - \frac{\sum_{\sigma=1}^n \frac{1}{2h_\sigma} g^{tf}(X^\sigma \circ R(X,Y))_t (X^\sigma \circ R(X,Y))_f}{g(X,X)g(Y,Y) - g(X,Y)g(Y,X)} $$

$$ + \frac{\sum_{\sigma=1}^n \frac{1}{4h_\sigma} g^{tf}(X^\sigma \circ R(Y,Y))_t (X^\sigma \circ R(X,X))_f}{g(X,X)g(Y,Y) - g(X,Y)g(Y,X)} $$

$$ = K(X,Y) - \sum_{\sigma=1}^n \frac{3}{4h_\sigma} \left|(X^\sigma \circ R(X,Y))\right|^2, $$

ii)

$$ {}^{CG}K({}^HX,{}^{V_\beta}\theta) = -\frac{{}^{CG}R_{ki_\beta j s_\beta}{}^HX^k{}^{V_\beta}\theta^{i_\beta}{}^HX^j{}^{V_\beta}\theta^{s_\beta}}{({}^{CG}g_{kj}{}^{CG}g_{i_\beta s_\beta} - {}^{CG}g_{ks_\beta}{}^{CG}g_{i_\beta j}){}^HX^k{}^{V_\beta}\theta^{i_\beta}{}^HX^j{}^{V_\beta}\theta^{s_\beta}} $$

$$ = -\frac{{}^{CG}R_{ki_\beta j}{}^l{}^{CG}g_{s_\beta l}X^k\theta_iX^j\theta_s + {}^{CG}R_{ki_\beta j}{}^{l_\beta}{}^{CG}g_{s_\beta l_\beta}X^k\theta_iX^j\theta_s}{g_{kj}\left(\frac{1}{h_\beta}(g^{is} + g^{ia}g^{sb}X^\alpha_aX^\alpha_b)\right)X^k\theta_iX^j\theta_s} $$

$$ = \left(\frac{\frac{1}{2}R_{kjl}{}^i - \frac{1}{2h_\beta}X^\beta_m R_{kjl}{}^m \tilde{X}^{\beta i} - \frac{1}{2h_\beta}X^\beta_m R_{kjt}{}^m \tilde{X}^{\beta t}\delta^i_l}{\left(\frac{1}{h_\beta}(g_{kj}g^{is} + g_{kj}g^{ia}g^{sb}X^\beta_aX^\beta_b)\right)X^k\theta_iX^j\theta_s} \right. $$

$$ + \frac{-\frac{1}{4h_\beta}X^\beta_mX^\beta_r R_{ktl}{}^m R^t{}_{\cdot\, j\, \cdot}{}^{ir} + \frac{1+h_\beta}{2h^2_\beta}X^\beta_m R_{kj\,\cdot}{}^{im}X^\beta_l}{\left(\frac{1}{h_\beta}(g_{kj}g^{is} + g_{kj}g^{ia}g^{sb}X^\beta_aX^\beta_b)\right)X^k\theta_iX^j\theta_s} $$

$$ \left. + \frac{\frac{1}{2h^2_\beta}X^\beta_m R_{kjr}{}^m \tilde{X}^{\beta i}\tilde{X}^{\beta r}X^\beta_l}{\left(\frac{1}{h_\beta}(g_{kj}g^{is} + g_{kj}g^{ia}g^{sb}X^\beta_aX^\beta_b)\right)X^k\theta_iX^j\theta_s} \right) \left(\frac{1}{h_\beta}(g^{sl} + +g^{sa}g^{lb}X^\beta_aX^\beta_b)X^k\theta_iX^j\theta_s \right) $$

$$= \frac{-\frac{1}{4h_\beta^2} X_m^\beta X_r^\beta R_{ktl}{}^m R^t{}_{\cdot\, j}{}^{ir}_{\cdot} g^{sl} X^k \theta_i X^j \theta_s}{\frac{1}{h_\beta}(g(X,X)g^{-1}(\theta,\theta) + g(X,X)gg^{-1}(\theta,X^\beta)g^{-1}(\theta,X^\beta))}$$

$$+ \frac{-\frac{1}{4h_\beta^2} X_m^\beta X_r^\beta R_{ktl}{}^m R^t{}_{\cdot\, j}{}^{ir}_{\cdot} g^{sa} g^{lb} X_a^\beta X_b^\beta X^k \theta_i X^j \theta_s}{\frac{1}{h_\beta}(g(X,X)g^{-1}(\theta,\theta) + g(X,X)g^{-1}(\theta,X^\beta)g^{-1}(\theta,X^\beta))}$$

$$= \frac{\frac{1}{4h_\beta^2} g^{tf} X_m^\beta X_r^\beta \tilde{\theta}^l R_{tkl}{}^m X^k R_{fjp}{}^r g^{pi} X^j \theta_i}{\frac{1}{h_\beta}(1 + (g^{-1}(X^\beta,\theta))^2)} = \frac{1}{4h_\beta} \frac{\left|X^\beta \circ R(\quad, X)\tilde{\theta}\right|^2}{(1 + (g^{-1}(X^\beta,\theta))^2)},$$

iii)

$$^{CG}K(^{V_\alpha}\omega, {}^{V_\alpha}\theta)$$

$$= -\frac{^{CG}R_{k_\alpha i_\alpha j_\alpha s_\alpha} {}^{V_\alpha}\omega^{k_\alpha} {}^{V_\alpha}\theta^{i_\alpha} {}^{V_\alpha}\omega^{j_\alpha} {}^{V_\alpha}\theta^{s_\alpha}}{(^{CG}g_{k_\alpha j_\alpha} {}^{CG}g_{i_\alpha s_\alpha} - {}^{CG}g_{k_\alpha s_\alpha} {}^{CG}g_{i_\alpha j_\alpha})^{V_\alpha}\omega^{k_\alpha} {}^{V_\alpha}\theta^{i_\alpha} {}^{V_\alpha}\omega^{j_\alpha} {}^{V_\alpha}\theta^{s_\alpha}}$$

$$= -\frac{^{CG}R_{k_\alpha i_\alpha j_\alpha}{}^{l_\alpha} \, {}^{CG}g_{s_\alpha l_\alpha} \omega_k \theta_i \omega_j \theta_s}{L}$$

$$= \left(\frac{-\frac{h_\alpha - 1}{h_\alpha^3} \tilde{X}^{\alpha j} (\overset{\leftrightarrow}{X}{}^{\alpha j} \delta_l^k - \tilde{X}^{\alpha k} \delta_l^i) - \frac{h_\alpha + 2}{h_\alpha^3} X_l^\alpha (\tilde{X}^{\alpha i} g^{jk} - \tilde{X}^{\alpha k} g^{ji})}{L} \right.$$

$$\left. - \frac{\frac{h_\alpha^2 + h_\alpha + 1}{h_\alpha^3}(g^{ij}\delta_l^k - g^{kj}\delta_l^i}{L} \right) \left(\frac{1}{h_\alpha}(g^{sl} + g^{sa} g^{lb} X_a^\alpha X_b^\alpha) \omega_k \theta_i \omega_j \theta_s \right)$$

$$= \frac{\frac{1-h_\alpha}{h_\alpha^4}(1 + (g^{-1}(X^\alpha,\omega))^2 + (g^{-1}(X^\alpha,\theta))^2) + \frac{h_\alpha + 2}{h_\alpha^3}}{\frac{1}{h_\alpha^2}(1 + (g^{-1}(X^\alpha,\omega))^2 + (g^{-1}(X^\alpha,\theta))^2)}$$

$$= \frac{1 - h_\alpha}{h_\alpha^2} + \frac{h_\alpha + 2}{h_\alpha} \frac{1}{(1 + (g^{-1}(X^\alpha,\omega))^2 + (g^{-1}(X^\alpha,\theta))^2)},$$

where

$$L = (^{CG}g_{k_\alpha j_\alpha} {}^{CG}g_{i_\alpha s_\alpha} - {}^{CG}g_{k_\alpha s_\alpha} {}^{CG}g_{i_\alpha j_\alpha}) \omega_k \theta_i \omega_j \theta_s$$

$$= \frac{1}{h_\alpha}(g^{kj} + g^{ka} g^{jb} X_a^\alpha X_b^\alpha) \frac{1}{h_\alpha}(g^{is} + g^{it} g^{sf} X_t^\alpha X_f^\alpha) \omega_k \theta_i \omega_j \theta_s$$

$$- \frac{1}{h_\alpha}(g^{ks} + g^{kc} g^{sd} X_c^\alpha X_d^\alpha) \frac{1}{h_\alpha}(g^{ij} + g^{iu} g^{sv} X_u^\alpha X_v^\alpha) \omega_k \theta_i \omega_j \theta_s$$

$$= \frac{1}{h_\alpha^2}(1 + (g^{-1}(X^\alpha,\omega))^2 + (g(X^\alpha,\theta))^2),$$

iv) Calculations similar to those in *iii)* show that

$$^{CG}K(^{V_\alpha}\omega, {}^{V_\beta}\theta) = 0, \text{ for } \alpha \neq \beta.$$

Therefore, the following theorem holds.

Theorem 7.1. *Let (M, g) be a Riemannian manifold and $F^*(M)$ be its coframe bundle equipped with Cheeger-Gromoll metric ^{CG}g. Then the sectional curvature ^{CG}K of $(F^*(M), {}^{CG}g)$ satisfy the following:*

i)

$$^{CG}K(^H X, {}^H Y) = K(X,Y) - \sum_{\sigma=1}^{n} \frac{3}{4h_\sigma} \left|(X^\sigma \circ R(X,Y))\right|^2, \tag{7.4}$$

ii)

$$^{CG}K(^H X, {}^{V_\beta}\theta) = \frac{1}{4h_\beta} \frac{\left|X^\beta \circ R(\quad, X)\tilde{\theta}\right|^2}{(1 + (g^{-1}(X^\beta,\theta))^2)}, \tag{7.5}$$

iii)

$$ {}^{CG}K({}^{V_\alpha}\omega, {}^{V_\alpha}\theta) = \frac{1-h_\alpha}{h_\alpha^2} + \frac{h_\alpha+2}{h_\alpha}\frac{1}{(1+(g^{-1}(X^\alpha,\omega))^2+(g^{-1}(X^\alpha,\theta))^2)}, \tag{7.6} $$

iv)

$$ {}^{CG}K({}^{V_\alpha}\omega, {}^{V_\beta}\theta) = 0, \text{ for } \alpha \neq \beta, $$

where K is a sectional curvature of (M, g) and $\tilde{\theta} = g^{-1} \circ \theta = (g^{ij}\theta_{ij}) \in \Im_0^1(M)$, $R(\quad, X)\tilde{\theta} \in \Im_1^1(M)$.

Theorem 7.2. *Let (M, g) be a space of constant curvature κ and ${}^{CG}K$ the sectional curvature of the coframe bundle $F^*(M)$ with Cheeger-Gromoll metric ${}^{CG}g$. Then ${}^{CG}K({}^HX, {}^HY)$ is nonnegative if $0 \le \kappa \le 4/3$, ${}^{CG}K({}^HX, {}^{V_\beta}\theta)$ and ${}^{CG}K({}^{V_\alpha}\omega, {}^{V_\alpha}\theta)$ are nonnegative if $\kappa \ge 0$.*

Proof. Since M has constant sectional curvature κ,using (7.4), we have

$$ {}^{CG}K({}^HX, {}^HY) = \kappa - \sum_{\sigma=1}^{n} \frac{3}{4h_\sigma} g^{ij}(X^\sigma \circ R(X,Y))_i (X^\sigma \circ R(X,Y))_j $$

$$ = \kappa - \sum_{\sigma=1}^{n} \frac{3}{4h_\sigma} \kappa^2((g^{-1}(X^\sigma, \tilde{X}))^2 + (g^{-1}(X^\sigma, \tilde{Y}))^2). $$

Let $\{E^1, E^2, ..., E^n\}$ be an orthonormal basis for cotangent space T_x^*M such that $E^1 = \tilde{X}, \tilde{E}^2 = \tilde{Y}$. Then

$$ (g^{-1}(X^\sigma, \tilde{X}))^2 + (g^{-1}(X^\sigma, \tilde{Y}))^2 \le \sum_{i=1}^{n} (g^{-1}(X^\sigma, E^i))^2 = |X^\sigma|^2 $$

which together with $|X^\sigma|^2 = r_\sigma^2 < 1 + r_\sigma^2 = h_\sigma$ implies that ${}^{CG}K({}^HX, {}^HY)$ is nonnegative if $0 \le \kappa \le 4/3$.

The assertion for ${}^{CG}K({}^HX, {}^{V_\alpha}\theta)$ and ${}^{CG}K({}^{V_\alpha}\omega, {}^{V_\alpha}\theta)$ is clear by (7.5) and (7.6). □

8. Geodesics of $F^*(M)$ with metric ${}^{CG}g$

Various problems associated with geodesics in fiber bundles have been very well investigated (see, for example, [4, p. 70-71, 97-100], [15, p. 57-61, 114-117]). Geodesics of tangent bundle with Cheeger-Cromoll metric were considered by A. Salimov and S. Kazimova in [13], while the question of geodesics of the cotangent bundle with a similar metric was touched upon by A. Salimov and F. Agca in [1]. In this section we will investigate the geodesic curves of the linear coframe bundle $F^*(M)$ with the Cheeger-Gromoll metric ${}^{CG}g$.

Let $\tilde{C} = \tilde{C}(t)$ be a curve on the coframe bundle $F^*(M)$, locally defined by equations $x^h = x^h(t)$, $x^{h_\beta} = X_h^\beta(t)$ with respect to the natural frame $(x^i, x^{i_\alpha}) = (x^i, X_i^\alpha)$, where parameter t is the arc length of the curve $\tilde{C}$. Then curve $C = \pi \circ \tilde{C}$ on a manifold M is called the projection of curve $\tilde{C}$. Note that a curve C is locally defined by equations $x^h = x^h(t)$.

By definition, a curve $\tilde{C}$ is a geodesic of linear coframe bundle $F^*(M)$ with the Cheeger-Gromoll metric ${}^{CG}g$if and only if this curve satisfies differential equations

$$ \frac{d}{dt}\left(\frac{\tilde{\eta}^I}{dt}\right) + {}^{CG}\Gamma_{JK}^I \frac{\tilde{\eta}^J}{dt}\frac{\tilde{\eta}^K}{dt} = 0 \tag{8.1} $$

with respect to the adapted frame $\{D_I\}$, where $\{\tilde{\eta}^J\}$ is a conjugate coframe to the adapted frame $\{D_I\}$, and $\frac{\tilde{\eta}^h}{dt} = \frac{dx^h}{dt}$, $\frac{\tilde{\eta}^{h_\beta}}{dt} = \frac{\delta X_h^\beta}{dt}$ with respect to a curve $\tilde{C}$. Using (6.1), equations (8.1) are reduced to

$$ \frac{d}{dt}\left(\frac{\overset{\cdots}{\eta}^i}{dt}\right) + \Gamma_{jk}^i \frac{dx^j}{dt}\frac{dx^k}{dt} + \frac{1}{h_\alpha} X_m^\alpha R_{\cdot k \cdot}^{ijm} \frac{\delta X_j^\alpha}{dt}\frac{dx^k}{dt} = 0 \tag{8.2} $$

$$ \begin{aligned} &\frac{d}{dt}\left(\frac{\delta X_i^\alpha}{dt}\right) + \frac{1}{2} X_m^\alpha R_{jki}{}^m \frac{dx^j}{dt}\frac{dx^k}{dt} - \Gamma_{ji}^k \frac{dx^j}{dt}\frac{\delta X_k^\alpha}{dt} \\ &+ \left[-\frac{1}{h_\beta}\,(\tilde{X}^{\beta j}\delta_\beta^\alpha \delta_i^k + \tilde{X}^{\beta k}\delta_\beta^\alpha \delta_i^j) + \frac{1+h_\beta}{h_\beta^2} g^{jk} X_i^\alpha \right. \\ &\left. + \frac{1}{h_\beta^2}\tilde{X}^{\beta j}\tilde{X}^{\beta k} X_i^\alpha \right] \frac{\delta X_j^\beta}{dt}\frac{\delta X_k^\beta}{dr} = 0. \end{aligned} \tag{8.3} $$

We transform (8.2) as follows

$$\frac{\delta^2 x^i}{dt^2}+\frac{1}{h_\alpha}X^\alpha_m R^i_{\cdot k}{}^{jm}_{\cdot}\frac{\delta X^\alpha_j}{dt}\frac{dx^k}{dt}=0. \tag{8.4}$$

Now, using identity

$$\frac{1}{2}X^\alpha_m R_{jki}{}^m\frac{dx^j}{dt}\frac{dx^k}{dt}=0$$

which is a consequence of relation $R^m_{(jk)i}=0$, we transform (8.3):

$$\begin{aligned}&\tfrac{\delta^2 X^\alpha_i}{dt^2}+\Big[-\tfrac{1}{h_\beta}\,(\tilde{X}^{\beta j}\delta^\alpha_\beta\delta^k_i+\tilde{X}^{\beta k}\delta^\alpha_\beta\delta^j_i)+\tfrac{1+h_\beta}{h^2_\beta}g^{jk}X^\alpha_i\\ &+\tfrac{1}{h^2_\beta}\tilde{X}^{\beta j}\tilde{X}^{\beta k}X^\alpha_i\Big]\tfrac{\delta X^\beta_j}{dt}\tfrac{\delta X^\beta_k}{dr}=0.\end{aligned} \tag{8.5}$$

Thus we have the following theorem.

Theorem 8.1. *Let $\tilde{C}$ be a curve on $F^*(M)$ and locally expressed by equations $x^h=x^h(t),\ \ x^{h_\beta}=X^\beta_h(t)$ with respect to the induced coordinates $(x^i,x^{i_\alpha})\subset\pi^{-1}(U)\subset F^*(M)$. The curve $\tilde{C}$ is a geodesic in $F^*(M)$ with the Cheeger-Gromoll metric ${}^{CG}g$ if it satisfies equations (8.4) and (8.5).*

If the curve $\tilde{C}$ satisfies at all the points the relation

$$\frac{\delta X^\beta_h}{dt}=\frac{dX^\beta_h}{dt}-\Gamma^i_{jh}\frac{dx^j}{dt}X^\beta_i=0, \tag{8.6}$$

then the curve $\tilde{C}$ is said to be a horizontal lift of the curve C in M.

As a consequence of equations (8.4), (8.5) and (8.6), we obtain the following.

Theorem 8.2. *The horizontal lift of a geodesic in (M,g) is always geodesic in linear coframe bundle $F^*(M)$ with the Cheeger-Gromoll metric ${}^{CG}g$.*

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References

[1] Agca, F., Salimov, A.: *Some notes concerning Cheeger-Gromoll metrics.* Hacet. J. Math. Stat. **42**(5), 533-549 (2013).
[2] Agca, F.: *$g-$natural metrics on the cotangent bundle.* Int. Electron. J. Geom., **6** (1), 129-146 (2013).
[3] Cheeger, J., Gromoll, D.: *On the structure of complete manifolds of nonnegative curvature.* Ann. of Math. **96**, 413-443 (1972).
[4] Cordero, L., Dodson, C., Leon, M.: *Differential geometry of frame bundles.* Kluwer, Dordrecht, (1988).
[5] Fattayev, H.,: *Some notes on the differential geometry of linear coframe bundle of a Riemann manifold.* Adv. Studies: Euro-Tbilisi Math. J. **14**(4), 81-95 (2021).
[6] Gudmondson, S., Kappos, E.: *On the geometry of the tangent bundles.* Expo. Math. **20**(1), 1-41 (2002).
[7] Hou, Z., Sun, L.: *Geometry of tangent bundle with Cheeger-Gromoll type metric.* J. Math. Anal. Apll. **402**, 493-504 (2013).
[8] Kobayashi, S., Nomizu, K.: *Foundations of differential Geometry, Vol. I.* Interscience Publishers, New York-London, (1963).

[9] Munteanu, M.: *Cheeger-Gromoll type metrics on the tangent bundle.* Sci. Ann. Univ. Agric. Sci. Vet. Med. **49**(2), 257-268 (2006).
[10] Musso, E., Tricerri, F .: *Riemannian metrics on tangent bundles.* Ann. Math. Pura. Appl. **150**(4), 1-20, (1988).
[11] Niedzialomski, K.: *On the geometry of frame bundles.* Archivum Mathematicum (BRNO). **48**, 197-206 (2012).
[12] Salimov, A., Akbulut, K.: *A note on a paraholomorphic Cheeger-Gromoll metric.* Proc. Indian Acad. Sci. **119**(2), 187-195 (2009).
[13] Salimov, A., Kazimova, S.: *Geodesics of the Cheeger-Gromoll metric.* Turk. J. Math. **33**, 99-105 (2009).
[14] Sekizawa, M.: *Curvatures of tangent bundles with Cheeger-Gromoll metric.* Tokyo J. Math. **14**(2), 407-417 (1991).
[15] Yano, K., Ishihara, S.: *Tangent and cotangent bundles.* Marsel Dekker Inc., New York, (1973).

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