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On Parafree Leibniz Algebras

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Abstract

The parafree Leibniz algebras are a special class of Leibniz algebras which have many properties with a free Leibniz algebra. In this note, we introduce the structure of parafree Leibniz algebras. We survey the important results in parafree Leibniz algebras which are analogs of corresponding results in parafree Lie algebras. We first investigate some properties of subalgebras and quotient algebras of parafree Leibniz algebras. Then, we describe the direct sum of parafree Leibniz algebras. We show that the direct sum of two parafree Leibniz algebras is a Leibniz algebra. Furthermore, we prove that the direct sum of two parafree Leibniz algebras is again parafree.

Keywords: Parafree Leibniz algebra, subalgebras, quotient algebras, direct sum

1. Introduction

Let F be a field with characteristic zero. A Leibniz algebra L over a field F is a non-associative algebra with multiplication $[\ ,\ ]: L \times L \to L$ satisfying the Leibniz identity. Firstly, Leibniz algebras were introduced by Loday in the early 90's as a generalization of Lie algebras. The free Leibniz algebra was described by Loday [9,10] and Pirashvili [10]. H. Baur [4] defined a non-free parafree Lie algebra. Further, N. Ekici and Z. Velioğlu have worked on some important results of parafree Lie algebras [6,7] and Z. Velioğlu has investigated the subalgebras and quotient algebras of parafree Lie algebras [11]. In this paper following [2] we focus on parafree Leibniz algebras and state all important known results for such algebras. We turn our attention to the structure of subalgebras of parafree Leibniz algebras. Our aim is to investigate some properties of subalgebras and quotient algebras of parafree Leibniz algebras.

Moreover, we prove that the direct sum of two parafree Leibniz algebras is a Leibniz algebra and also this direct sum is again parafree.

2. Notations and Definitions

In this section we recall some basic definitions and some basic results which we need for our aims (see [1,3,5]). We use standard notation.

Throughout this note, F denotes a field with characteristic zero. The algebra L is called a (left) Leibniz algebra if it satisfies the Leibniz identity

$$[[x,y],z] = [x,[y,z]] - [y,[x,z]] \qquad \textbf{(2.1)}$$

for all $x,y,z \in L$. Throughout of this paper, we prefer to work with left Leibniz algebra. Define the series of ideals

$$L^1 \supseteq L^2 \supseteq \ldots \supseteq L^k \supseteq \ldots$$

where $L^1 = L,\ L^2 = [L,L], \ldots, L^{k+1} = [L^k, L]$ for $k \geq 1$. This series is called the lower central sequence of L. The Leibniz algebra L is called a nilpotent Leibniz algebra if there exists a positive integer $k \geq 1$ such that $L^k = \{0\}$. We say that two Leibniz algebras L_1 and L_2 have the same lower central sequence if $L_1/L_1^n \cong L_2/L_2^n$.

Let X be a set and $A(X)$ be the free non-associative algebra on X over F, I be the two-sided ideal of $A(X)$ generated by the elements

$$[[x,y],z] - [x,[y,z]] + [y,[x,z]]$$

for all $x,y,z \in A(X)$. Then the algebra $L(X) = A(X)/I$ is a free Leibniz algebra.

Now we give some definitions on Leibniz algebras analogs to those of Lie algebras.

Definition 2.1. A Leibniz algebra is called Hopfian if it satisfies the following equivalent conditions
(i) it is isomorphic to any of its proper quotients,
(ii) every surjective endomorphism of it is an automorphism.

Definition 2.2. Any Leibniz algebra L is said to be residually nilpotent if $\bigcap_{n=1}^{\infty} L^n = \{0\}$.

Definition 2.3. Let $L(X)$ be the free Leibniz algebra freely generated by X. A Leibniz algebra P is said to be parafree over a set X if

(i) P is residually nilpotent,

(ii) for every $n \geq 1$, $P/P^n = L(X)/L(X)^n$, i.e. P has the same lower central sequence as $L(X)$.

The cardinality of X is called the rank of P.

3. Subalgebras and Quotient algebras of Parafree Leibniz algebras

In this section we give the proofs of our main results on subalgebras and quotient algebras of parafree Leibniz algebras. In [11], the main theorem states that a subalgebra of a parafree Lie algebra is parafree. Unfortunately, this case for parafree Leibniz algebras is not true. Since every subalgebra of free Leibniz algebra is not free, a subalgebra of a parafree Leibniz algebra need not to be parafree. By the following theorem, we show that every free subalgebra of a parafree Leibniz algebra is parafree.

Theorem 3.1. A free subalgebra of a parafree Leibniz algebra is parafree.

Proof. Let P be a parafree Leibniz algebra with the same lower central sequence as a free Leibniz algebra $L(X)$. The canonical mapping $\varphi: P \to L(X)$ induces the isomorphisms

$$\varphi_n: P/P^n \to L(X)/L(X)^n,$$

for $n \geq 2$. We take a free subalgebra of P, say H. We have $H \cap P^n = H^n$, then $\bigcap_{i=1}^{\infty} H^i \subset \bigcap_{i=1}^{\infty} P^i$. Since P is parafree, then P is residually nilpotent, that is,

$$\bigcap_{i=1}^{\infty} P^i = \{0\}.$$

Hence, we obtain $\bigcap_{i=1}^{\infty} H^i = \{0\}$. This means that H is residually nilpotent. Since H is free Leibniz subalgebra, H has the same lower central sequence as a free Leibniz algebra. Thus the free subalgebra H is parafree.

Now we have the following theorem for parafree Leibniz algebras with an easy proof using the corresponding parafree Lie algebra result which is given in [11].

Theorem 3.2. A quotient of a parafree Leibniz algebra is parafree.

Proof. Let P be a parafree Leibniz algebra and I be an ideal of P. Firstly we need to show that the quotient algebra P/I is residually nilpotent.

Suppose that $x \in \bigcap_{n=1}^{\infty} (P/I)^n$. Therefore, for all n,

$x \in (P/I)^n = (P^n + I)/I$. We have $x = y + I$, where $y \in P^n + I$. Clearly, $y \in \bigcap_{n=1}^{\infty} (P^n + I)$. By the residually nilpotence of P, P/I is residually nilpotent. It remains to prove that P / I has the same lower central sequence as a free Leibniz algebra.

We consider $(P / I)/(P / I)^n$. We have $(P^n + I)/I \cong P^n/I$. Therefore, we get

$$\left.(P/I)\middle/(P/I)^n\right. \cong (P/I)/((P^n + I)/I)$$

$$\cong (P/I)/(P^n/I) \cong P/P^n.$$

This demostrates that $(P / I) / (P/I)^n$ has the same lower central sequence as a free Leibniz algebra. Therefore,

$$(P/I)/(P/I)^n \cong L(X)/(L(X))^n.$$

Namely, P/I is parafree.

Lemma 3.3. Let P be a parafree Leibniz algebra with the finite rank and I be an ideal of P. If P and P/I have the same rank, then $I = \{0\}$.

Proof. We suppose that P and P/I have the same rank. For every positive integer n,

$$P/I \cong (P/I)/(P/I)^n \cong (P/I)/((P^n + I)/I)$$
$$\cong P/(P^n + I).$$

By the Theorem 3.2, P/I is residually nilpotent. Then by [8], P/I is Hopfian. Moreover,

$$P/(P^n + I) \subseteq P/P^n$$

and

$$P/P^n \cong P/(P^n + I).$$

Since P/I is Hopfian, a contradiction. Hence for all n, $I \subseteq P^n$, then $I = \{0\}$.

4. Direct sums of Leibniz algebras

Let $L_1, L_2, \ldots, L_n$ be Leibniz algebras. We define the direct sum $L = L_1 \oplus L_2 \oplus \ldots \oplus L_n$ as the vector space direct sum of the L_i with the Leibniz product $[\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i] = \sum_{i=1}^{n} [x_i, y_i]$, where $[x_i, y_i] \in L_i \cap L_j = \{0\}$ for $i \neq j, x_i \in L_i, x_j \in L_j$.

Lemma 4.1. Let L_1, L_2 be Leibniz algebras. The direct sum $L = L_1 \oplus L_2$ is Leibniz algebra with the product $[x_1 + x_2, y_1 + y_2] = [x_1, y_1] + [x_2, y_2]$ for $x_1, y_1 \in L_1$, $x_2, y_2 \in L_2$.

Proof. By (2.1), for $x = x_1 + x_2$, $y = y_1 + y_2$, $z = z_1 + z_2 \in L_1 \oplus L_2$ where $x_1, y_1, z_1 \in L_1$ and $x_2, y_2, z_2 \in L_2$, we have

$$[[x,y],z] = [[x_1 + x_2, y_1 + y_2], z_1 + z_2]$$
$$= [[x_1, y_1] + [x_2, y_2], z_1 + z_2]$$
$$= [[x_1, y_1], z_1 + z_2] + [[x_2, y_2], z_1 + z_2]$$
$$= [[x_1, y_1], z_1] + [[x_2, y_2], z_2]$$
$$[x,[y,z]] = [x_1 + x_2, [y_1 + y_2, z_1 + z_2]]$$
$$= [x_1 + x_2, [y_1 + z_1] + [y_2 + z_2]]$$
$$= [x_1, [y_1, z_1]] + [x_2, [y_2, z_2]]$$
$$[y,[x,z]] = [y_1 + y_2, [x_1 + x_2, z_1 + z_2]]$$
$$= [y_1 + y_2, [x_1, z_1] + [x_2, z_2]]$$
$$= [y_1, [x_1, z_1]] + [y_2, [x_2, z_2]].$$

It is clear to see that

$$[[x,y],z] = [x,[y,z]] - [y,[x,z]].$$

Therefore, $L = L_1 \oplus L_2$ is a Leibniz algebra.

Theorem 4.2. Let F_1 and F_2 be free Leibniz algebras. Then $F_1 \oplus F_2$ is again free.

Proof. By Lemma 4.1, $F_1 \oplus F_2$ is Leibniz algebra. Let X, Y be freely generating sets of F_1 and F_2 respectively. Then $X \cup Y$ freely generates $F_1 \oplus F_2$. Hence $F_1 \oplus F_2$ is a free Leibniz algebra freely generated by $X \cup Y$.

The following theorem for parafree Leibniz algebras is proved by using the corresponding parafree Lie algebra result which is given in [12].

Theorem 4.3. Let P_1 and P_2 be parafree Leibniz algebras and $P = P_1 \oplus P_2$. Then P is parafree.

Proof. First, we need to prove that P is residually nilpotent and P has the same lower central sequence as a free Leibniz algebra. Now, we need to show that

$$P^n = (P_1 \oplus P_2)^n = P_1^n \oplus P_2^n.$$

By induction on n, for $n = 1$ we have

$$P^1 = (P_1 \oplus P_2)^1 = P_1^1 \oplus P_2^1.$$

We suppose that for all $k < n$, $P^k = P_1^k \oplus P_2^k$. Then, we have

$$P^n = [P^{n-1}, P] = [P_1^{n-1} \oplus P_2^{n-1}, P_1 \oplus P_2].$$

Therefore,

$$P^n = [P_1^{n-1} + P_2^{n-1}, P_1 + P_2]$$
$$= [P_1^{n-1}, P_1] + [P_2^{n-1}, P_2]$$
$$= P_1^n + P_2^n$$

and

$$P_1^n \cap P_2^n = \{0\}.$$

Thus, we obtain $P^n = P_1^n \oplus P_2^n$. In order to prove residually nilpotency of P, we compute

$$\cap_{n=1}^{\infty} P^n = \cap_{n=1}^{\infty} (P_1^n \oplus P_2^n).$$

By using the definition of direct sum, we have

$$\bigcap_{n=1}^{\infty} (P_1^n \oplus P_2^n) = \bigcap_{n=1}^{\infty} P_1^n \oplus \bigcap_{n=1}^{\infty} P_2^n.$$

Since P_1 and P_2 are parafree, then we have

$$\cap_{n=1}^{\infty} P_1^n = \{0\} \text{ and } \cap_{n=1}^{\infty} P_2^n = \{0\}.$$

Therefore, $\cap_{n=1}^{\infty} P^n = \{0\}$, namely P is residually nilpotent. Now we show that P has the same lower central sequence as a free Leibniz algebra. Since P_1 and P_2 are parafree Leibniz algebras, then there exist free Leibniz algebra F_1 and F_2 such that

$$P_i / P_i^n \cong F_i / F_i^n,$$

$i = 1,2;\ n \geq 1$. Hence

$$P/P^n = (P_1 \oplus P_2)/(P_1 \oplus P_2)^n \cong P_1/P_1^n \oplus P_2/P_2^n$$
$$\cong {F_1}/{F_1^n} \oplus {F_2}/{F_2^n}$$
$$\cong {(F_1 \oplus F_2)}/{(F_1 \oplus F_2)^n}.$$

Since $F_1 \oplus F_2$ is free, P is parafree.

Author's Contributions

Nil Mansuroğlu: Drafted and wrote the manuscript, performed the experiment and result analysis.

Ethics

There are no ethical issues after the publication of this manuscript.

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