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Research Article

Differential subordination and harmonic means for the Koebe type n-fold symmetric functions

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Abstract

In this research, we develop some differential subordination results involving harmonic means of $f_b(z), f_b(z)+zf_b'(z)$ and $f_b(z)+\frac{zf_b'(z)}{f_b(z)}$, where $f_b(z)=\frac{z}{(1-z^n)^b}$, $b\geq 0; n\in\mathbb{N}=1,2,3...$ is an n-fold symmetric Koebe type functions defined in the unit disk with $f_b(0)=0, f_b'(z)\neq 0$. By using the admissibility conditions, we also study several applications in the geometric function theory.

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1. Introduction and definitions

Harmonic mean is a particular case of the power mean and it is used where the average rate is desired. It is Pythagorean mean, along with arithmetic and geometric mean. This is always less than either of them. The harmonic mean denoted by H of positive reals $x_1, x_2, ..., x_n$ is defined as that the reciprocals of the arithmetic average or mean of the reciprocals of $x_1, x_2, ..., x_n$, that is

$$H=\left(\frac{1}{k}\sum_{n=1}^{k}\frac{1}{x_n}\right)^{-1}$$

For a special choice, we write $H=\frac{2x_1x_2}{x_1+x_2}$. It is related to arithmetic mean A and geometric mean G by $G=\sqrt{AH}$. For more information, we refer [1–7, 9–12]. To obtain our major results, we are introducing some preliminary notations. Let $\mathbb{E}$ be the unit open disk in $\mathbb{C}$ defined by $|z|<1$ and let $\mathcal{H}(\mathbb{E})$ represent the class of functions holomorphic or analytic in $\mathbb{E}$. Let $\mathcal{A}_n$ be the subcollection of $\mathcal{H}(\mathbb{E})$ of functions $f: f(0)=f'(0)-1=0$. A function $f\in\mathcal{A}_n$ is represented by

$$f(z)=z+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+...,z\in\mathbb{E}. \tag{1.1}$$

A function $f\in\mathcal{A}_n$ is univalent, if it is one-to-one in $\mathbb{E}$. We denote this set by $\mathcal{S}$. For $f,g\in\mathcal{H}(\mathbb{E})$, we say that $f\prec g$, if there exists a well-known Schwarz function $w: f(z)=g(w(z))$,

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for $z \in \mathbb{E}$. If $g \in \mathcal{S}$, then this $\prec$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$. In the following, we present some definitions. Let $\mathcal{P}$ be the collection of functions $p : p \in \mathcal{H}(\mathbb{E})$, $p(0) = 1$, $\operatorname{Re} p(z) > 0$ and

$$p(z) = 1 + c_2 z^2 + c_3 z^3 + ..., z \in \mathbb{E}.$$

Definition 1.1. For $\Omega \subset \mathbb{C}$, $n \in \mathbb{N}$ and $h \in \mathcal{Q}$, the set of admissible functions $\Psi_n [\Omega, h]$ consists of $\Psi : \mathbb{C}^3 \times \mathbb{E} \to \mathbb{C}$ that meet the admissibility conditions:

$$\Psi(r, s, t, z) \notin \Omega, \text{ for } \; r = h(\xi), \; s = m\xi h'(\xi)$$

and

$$\operatorname{Re} \frac{t}{s} + 1 \geq m \operatorname{Re} \left(\frac{\xi h''(\xi)}{h'(\xi)} + 1 \right), \quad \xi \in \overline{\overline{\mathbb{E}}} \backslash \mathbb{E}(h), m \geq n, z \in \mathbb{E},$$

where the set $\mathcal{Q}$ represents analytic or holomorphic and injective functions on $\partial\overline{\mathbb{E}} \backslash \mathbb{E}(h)$: $\mathbb{E}(h) = \{\xi \in \partial\mathbb{E} : \lim_{z \to \xi} h(z) = \infty\}$, such that $h'(\xi) \neq 0$ for $\xi \in \partial\overline{\mathbb{E}} \backslash \mathbb{E}(h)$. If $h \in \mathcal{Q}$, then $h(\mathbb{E})$ is simply connected.

An $n-$fold symmetric Koebe type function f_b is defined by

$$f_b(z) = \frac{z}{(1 - z^n)^b}, b \geq 0; n = 1, 2, 3....$$

The function f_b becomes the Koebe function for $n = 1$ and $b = 2$. For the above definitions, we refer [2] and [8] respectively.

2. Preliminaries

To discuss our prime theorems, we need the following useful lemma.

Lemma 2.1 ([8])**.** *Let $h \in \mathcal{Q} : h(0) = a$, and $p(z) = a + a_n z^n + ...$ with $p(z) \neq a$ and $n \geq 1$. If $p \not\prec h$, then for $z_0 = r_0 e^{i\theta} \in \mathbb{E}$, $\xi_0 \in \partial\overline{\mathbb{E}} \backslash \mathbb{E}(h)$ and $m \geq n \geq 1$, $p(\mathbb{E}_{r_0}) \subset h(\mathbb{E})$, $p(z_0) = h(\xi_0)$, $\;h(\xi_0) = m\xi_0 h'(\xi_0)$ and $\;\operatorname{Re} \frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m \operatorname{Re} \left(\frac{\xi_0 h''(\xi_0)}{h'(\xi_0)} + 1 \right)$.*

3. Main results

Theorem 3.1. *Let $f_b(z) = \frac{z}{(1-z^n)^b}$ be holomorphic or analytic in $\mathbb{E}$ with $f_b(z) \neq 1$. If*

$$\operatorname{Re} \left\{ \frac{2 f_b(z) \left[f_b(z) + z f_b'(z) \right]}{2 f_b(z) + z f_b'(z)} \right\} \in \mathcal{P}, \tag{3.1}$$

then $f_b \in \mathcal{P}$.

Proof. Let us suppose that

$$h(z) = \frac{1 - (nb - 1) z}{1 - z} \tag{3.2}$$

with $h(\mathbb{E}) = \{w : \operatorname{Re} w > 0\}$ and $b > 1$ such that $h(0) = 1, \mathbb{E}(h) = \{1\}$, $h \in \mathcal{Q}$. Then, from (3.1), we want to prove that $f_b \prec h$. Assume that $f_b \not\prec h$. The Lemma2.1 shows that there is a point $z_0 \in \mathbb{E}$ and $\xi_0 \in \partial\overline{\mathbb{E}} \backslash \{1\}$ such that $f_b(z_0) = h(\xi_0)$ and $\operatorname{Re} f_b(z) > 0$, $z \in \mathbb{E}_{|z_0| = r}$. Thus $\operatorname{Re} f_b(z_0) = 0$. Therefore, we choose $f_b(z_0) = ix$, where x is real. Because of symmetry, we consider that $x > 0$. Take $h^{-1}(z) = \frac{z-1}{z-(nb-1)}$. Then

$$\xi_0 = h^{-1}(f_b(z_0)) = \frac{1 - f_b(z_0)}{nb - 1 - f_b(z_0)}$$

or

$$\begin{aligned}
\xi_0 h'(\xi_0) &= -\frac{(1-f_b(z_0))\,(f_b(z_0)-(nb-1))}{2-nb} = -\frac{(1-f_b(z_0))\,(f_b(z_0)-(2\operatorname{Re} f_b(z)-1))}{2-2\operatorname{Re} f_b(z)} \\
&= -\frac{(f_b(z_0)-1)\,(1-\operatorname{Re} f_b(z))}{2-2\operatorname{Re} f_b(z)} = \frac{(1-f_b(z_0))\,(\operatorname{Re} f_b(z)-1)}{2\,(1-\operatorname{Re} f_b(z))} \\
&\approx -\frac{(1-f_b(z_0))\left(\overline{1-f_b(z)}\right)}{2\,(1-\operatorname{Re} f_b(z))} = -\frac{|f_b(z_0)-1|^2}{2\,(1-\operatorname{Re} f_b(z))} = -\frac{|f_b(z_0)-1|^2}{2\operatorname{Re}(1-f_b(z))}.
\end{aligned}$$

Since for $\operatorname{Re} f_b(z) \geq \frac{nb}{2}, n \geq 2$ and $b > 1$, if $z_0 \in \mathbb{E}, \xi_0 \in \partial\overline{\mathbb{E}}$ and

$$\operatorname{Re} f_b(z_0) = \min\{\operatorname{Re} f_b(z) : |z| \leq |z_0|\},$$

then

$$\frac{z_0 f_b'(z_0)}{f_b(z_0)} \leq \frac{n}{2}\frac{|f_b(z_0)-1|^2}{\operatorname{Re}(1-f_b(z_0))}$$

or we can write

$$\operatorname{Re}\frac{z_0 f_b''(z_0)}{f_b'(z_0)} + 1 \geq 0.$$

Therefore, we see that

$$z_0 f_b'(z_0) = m\xi_0 h'(\xi_0) = \frac{-m}{2}\frac{|f_b(z_0)-1|^2}{\operatorname{Re}(1-f_b(z))}.$$

For $f_b(z_0) = ix$, the above equation takes the form

$$z_0 f_b'(z_0) = m\xi_0 h'(\xi_0) = -\frac{m}{2}\frac{|ix-1|^2}{\operatorname{Re}(1-ix)} = -\frac{m}{2}\frac{x^2+1}{1} = -\frac{m\,(x^2+1)}{2} = y. \tag{3.3}$$

By using (3.3), we thus prove that

$$\operatorname{Re}\left\{\frac{2f_b(z_0)\,[f_b(z_0)+z_0 f_b'(z_0)]}{2f_b(z_0)+z_0 f_b'(z_0)}\right\} = \operatorname{Re}\left[\frac{2ix\,(ix+y)}{2ix+y}\right] = \frac{2x^2 y}{4x^2+y^2} < 0.$$

This contradicts our hypothesis. Therefore $f_b \prec h$ and this leads to the desired proof. □

From the above result, we have the following corollary as a particular case:

Corollary 3.2. *If* $\phi(z) = \frac{f_b(z)}{z}$ *such that* $\frac{2\phi(z)\phi'(z)}{\phi(z)+z\phi'(z)} \in \mathcal{P}$, *then* $\phi \in \mathcal{P}$.

Theorem 3.3. *Let* f_b *be analytic in* $\mathbb{E}$ *with* $f_b(z) \neq 1$. *Then* $\frac{2f_b(z)+2zf_b'(z)}{1+f_b^2(z)+zf_b(z)f_b'(z)} \in \mathcal{P}$ *implies that* $f_b \in \mathcal{P}$.

Proof. Following the procedure adopted by Theorem 3.1 and setting $f_b(z_0) = ix$, $x > 0$, $z_0 f_b'(z_0) = y$, $y < 0$, we obtain

$$\operatorname{Re}\left\{\frac{2f_b(z_0)+2z_0 f_b'(z_0)}{1+f_b^2(z_0)+z_0 f_b(z_0)f_b'(z_0)}\right\} = \operatorname{Re}\left[\frac{2ix+y}{-x^2+ixy}\right] = \frac{2y}{(1-x^2)^2+x^2y^2} < 0.$$

This follows the required proof. □

Corollary 3.4. *Let* $\phi(z) = \frac{f_b(z)}{z}$. *Then*

$$\frac{2\phi^{-1}(z)\,(\phi(z)+z\phi'(z))}{\phi^{-1}(z)+z\,(\phi(z)+z\phi'(z))} = \frac{2\frac{z}{f_b(z)}f_b'(z)}{\frac{z}{f_b(z)}+zf_b'(z)} \in \mathcal{P} \quad \textit{implies that } \frac{f_b(z)}{z} \in \mathcal{P}.$$

Theorem 3.5. *Let* f_b *be analytic in* $\mathbb{E}$ *and with* $f_b \neq 1$. *Then*

$$\frac{f_b(z)+\frac{zf_b'(z)}{f_b(z)}}{2+\frac{zf_b'(z)}{f_b^2(z)}} \in \mathcal{P} \quad \textit{implies that } f_b \in \mathcal{P}.$$

Proof. We just show that $f_b \prec h$, where h is defined by (3.2). As done in the Theorem 3.1, we have a point $z_0 \in \mathbb{E} : f_b(z_0) = ix, x > 0$ and $z_0 f_b'(z_0) = y$, $y < 0$, so that we can write

$$\operatorname{Re}\left(\frac{f_b(z_0) + \frac{z_0 f_b'(z_0)}{f_b(z_0)}}{2 + \frac{z_0 f_b'(z_0)}{f_b^2(z_0)}}\right) = \operatorname{Re}\left(\frac{ix + \frac{y}{ix}}{2 - \frac{y}{x}}\right) = 0.$$

Hence the result follows. □

Corollary 3.6. *Let f_b be holomorphic or analytic in $\mathbb{E}$ and $\frac{zf_b'(z)}{f_b(z)} = \phi(z)$ with $1 + \frac{zf_b''(z)}{f_b'(z)} = \frac{z\phi'(z)}{\phi(z)} - \phi(z)$. Then*

$$\frac{\phi^2(z)\left(\frac{z\phi'(z)}{\phi(z)} - \phi(z)\right)}{z\phi'(z)} \in \mathcal{P} \quad \textit{implies that } \phi \in \mathcal{P}.$$

Theorem 3.7. *Let f_b be holomorphic or analytic in $\mathbb{E}$ with $f_b(z) \neq 1$. Then*

$$\frac{2f_b(z) + \frac{zf_b'(z)}{f_b(z)}}{1 + f_b^2(z) + zf_b'(z)} \in \mathcal{P} \quad \textit{implies that } \phi \in \mathcal{P}.$$

Proof. Following the procedure as in Theorem 3.1 and setting $f_b(z_0) = ix$, $x > 0$, $z_0 f_b'(z_0) = y$, $y < 0$, we obtain

$$\operatorname{Re}\left(\frac{2f_b(z_0) + \frac{z_0 f_b'(z_0)}{f_b(z_0)}}{1 + f_b^2(z_0) + z_0 f_b'(z_0)}\right) = \operatorname{Re}\left(\frac{2ix + \frac{y}{ix}}{1 + (ix)^2 + ixy}\right) = 0.$$

Hence the result follows. □

Corollary 3.8. *Let $\phi(z) = \frac{zf_b'(z)}{f_b(z)}$. Then*

$$\frac{2\phi^2(z)\left(\frac{z\phi'(z)}{\phi(z)} - \phi(z)\right)}{z\phi'(z)} \in \mathcal{P} \quad \textit{implies } \phi \in \mathcal{P}.$$

Theorem 3.9. *Let f_b be holomorphic or analytic in $\mathbb{E}$ with $f_b(z) \neq 1$ and $M \in \mathbb{R}^+$. Then*

$$\left|\frac{2f_b(z)\left[f_b(z) + zf_b'(z)\right]}{2f_b(z) + zf_b'(z)} - 1\right| < M \quad \textit{implies that } |1 - f_b(z)| < M.$$

Proof. We suppose that

$$h(z) = 1 + Mz, \quad 0 < M < 1 \tag{3.4}$$

with $h(\mathbb{E}) = \{w : |1 - w| < M\}$, $h(0) = 1$ and $h \in \mathcal{Q}$. Then (3.3) takes the form

$$\left|\frac{2f_b(z)\left[f_b(z) + zf_b'(z)\right]}{2f_b(z) + zf_b'(z)} - 1\right| < M \quad \textit{which implies that} \;\; f_b \prec h.$$

Suppose that $f_b \not\prec h$. Then Lemma 2.1 shows that there exist $z_0 \in \mathbb{E}, \xi_0 \in \partial\mathbb{E}$ and $m \geq 1$, such that

$$f_b(z_0) = h(\xi_0),$$
$$|f_b(z_0) - 1| < M, \quad z \in \mathbb{E}_{|z_0|}$$

and

$$|f_b(z_0) - 1| = |h(\xi_0) - 1| = M.$$

We can select $f_b(z_0)$ of the form $f_b(z_0) = 1 + Me^{i\theta}$ where θ is real. Using (3.4), we have

$$\xi_0 = h^{-1}(f_b(z_0)) = \frac{f_b(z_0) - 1}{M} \quad \text{with} \;\; z_0 f_b'(z_0) = m\xi_0 h'(\xi_0) = mMe^{i\theta}, \quad m \geq 1.$$

We can also write

$$\left|\frac{2f_b(z_0)\left[f_b(z_0)+z_0f'(z_0)\right]}{2f_b(z_0)+z_0'f_b'(z_0)}-1\right|$$
$$=\left|\frac{2f_b^2(z_0)+2z_0f_b(z_0)f_b'(z_0)-2f_b(z_0)-z_0f_b'(z_0)}{2f_b(z_0)+z_0f_b'(z_0)}\right|=|M|\left|1+\frac{m\left(Me^{i\theta}+1\right)}{2+(2+m)\,Me^{i\theta}}\right|.$$

For contradiction, we show that the last expression is greater than M which is equivalent to

$$\left|1+\frac{\left(1+Me^{i\theta}\right)m}{2+(2+m)\,Me^{i\theta}}\right|\geq 1$$

or we observe that

$$\left|2+(m+2)\,Me^{i\theta}+m\left(Me^{i\theta}+1\right)\right|^2\geq\left|2+M\,(2+m)\,e^{i\theta}\right|^2,$$

which further implies that

$$-\,(3m+4)\left(\frac{m+4}{3m+4}-M\right)(M-1)\geq 0.$$

For $M>0$ and $m\geq 1$, $(3m+4)\left(\frac{m+4}{3m+4}-M\right)(1-M)>0$ which contradicts the assumptions of the theorem. Therefore $f_b\prec h$ and thus the proof follows. □

Corollary 3.10. *Let $f_b$be analytic in $\mathbb{E}$ and $p(z)=\frac{f_b(z)}{z}$ alongwith*

$$\frac{z\,(p\,(z)+zp'(z))}{zp\,(z)}=\frac{zf_b'(z)}{f_b(z)}.$$

For $0<M<1$, the inequality

$$2\left|\frac{zp'\,(z)+p(z)}{\frac{zp'(z)}{p(z)}+1}\right|<M \quad \textit{implies that} \quad |p(z)-1|<M.$$

Corollary 3.11. *Let $p(z)=f_b'(z)$ and $M\in(0,1)$. Then*

$$\left|\frac{2f_b'(z)+zf_b''(z)}{2+\frac{zf_b''(z)}{f_b'(z)}}-1\right|<M \quad \textit{implies that} \quad \left|f_b'(z)-1\right|<M.$$

Corollary 3.12. *Let $p(z)=\frac{zf_b'(z)}{f_b(z)}$ and $0<M<1$. Then*

$$\left|\frac{2\frac{zf_b'(z)}{f_b(z)}\left[2+\frac{zf_b''(z)}{f_b'(z)}-\frac{zf_b'(z)}{f_b(z)}\right]}{3+\frac{zf_b''(z)}{f_b'(z)}-\frac{zf_b'(z)}{f_b(z)}}-1\right|<M \quad \textit{implies that} \quad \left|\frac{zf_b'(z)}{f_b(z)}-1\right|<M.$$

Theorem 3.13. *Let f_b be holomorphic or analytic in $\mathbb{E}$ with $f_b\,(z)\neq 1$ and $0<\gamma\leq 1$. Then*

$$\left|\arg\frac{2f_b(z)\left[f_b(z)+zf_b'(z)\right]}{2f_b(z)+zf_b'(z)}-1\right|<\gamma\frac{\pi}{2} \quad \textit{implies that} \quad |\arg f_b(z)|<\gamma\frac{\pi}{2}.$$

Proof. Let us suppose that

$$h(z)=\left[\frac{(nb-1)\,z-1}{z-1}\right]^{\gamma} \quad or \quad h'(z)=\gamma\left[\frac{(nb-1)\,z-1}{z-1}\right]^{\gamma-1}\frac{2-nb}{(1-z)^2}. \tag{3.5}$$

with $h(\mathbb{E}) = \left\{w : |\arg w| < \gamma\frac{\pi}{2}\right\}$, $h(0) = a$, and $h \in \mathcal{Q}$. Then the condition in (3.5) is rewritten as

$$\left|\arg \frac{2f_b(z)\left[f_b(z) + zf_b'(z)\right]}{2f_b(z) + zf_b'(z)} - 1\right| < \gamma\frac{\pi}{2} \quad \textit{implies that} \quad p \prec h.$$

Suppose that $p \not\prec h$. Then, from Lemma 2.1, we have $z_0 \in \mathbb{E}, \xi_0 \in \partial\mathbb{E}\backslash\{1\}$ and $m \geq 1$, such that

$$f_b(z_0) = h(\xi_0) \;\text{ or } z_0 f_b'(z_0) = m\xi_0 h'(\xi_0).$$

This implies that

$$f_b(z_0) = h(\xi_0) = (ix)^\gamma = x^\gamma e^{\gamma i \frac{\pi}{2}}, \tag{3.6}$$

where $x \in \mathbb{R}$. Because of symmetry, we consider $x > 0$. We can write

$$\xi_0 = h^{-1}\left(f_b(z_0)\right) = \frac{\left[f_b(z_0)\right]^{\frac{1}{\gamma}} - 1}{\left[f_b(z_0)\right]^{\frac{1}{\gamma}} - (nb-1)},$$

$$h'(z) = \gamma \frac{\left[\left(1 - (nb-1)\,z\right)\right]^{\gamma - 1}}{(1-z)^{\gamma+1}}\left[2 - nb\right]$$

and

$$\begin{aligned} h'(\xi_0) &= \frac{\gamma\left(1 - (nb-1)\frac{\left[f_b(z_0)\right]^{\frac{1}{\gamma}} - 1}{\left[f_b(z_0)\right]^{\frac{1}{\gamma}} - (nb-1)}\right)^{\gamma-1}(2-nb)}{\left[1 - \frac{\left[f_b(z_0)\right]^{\frac{1}{\gamma}} - 1}{\left[f_b(z_0)\right]^{\frac{1}{\gamma}} - (nb-1)}\right]^{\gamma+1}} \\ &= \frac{\gamma\left(\left[f_b(z_0)\right]^{\frac{1}{\gamma}}\right)^{\gamma-1}\left(\left[f_b(z_0)\right]^{\frac{1}{\gamma}} - (nb-1)\right)^2}{(2-nb)}. \end{aligned}$$

By using (3.6), we see that

$$\begin{aligned} \xi_0 h'(\xi_0) &= \frac{\gamma\left[\left[f_b(z_0)\right]^{\frac{1}{\gamma}} - 1\right]\left(\left[f_b(z_0)\right]^{\frac{1}{\gamma}}\right)^{\gamma-1}\left(\left[f_b(z_0)\right]^{\frac{1}{\gamma}} - (nb-1)\right)}{(2-nb)} \\ &= \frac{\gamma\left[ix - 1\right](ix)^{\gamma-1}\left(ix - (nb-1)\right)}{(2-nb)} \end{aligned}$$

or

$$\begin{aligned} z_0 f_b'(z_0) &= m\xi_0 h'(\xi_0) \\ &= \frac{m\gamma}{2\left(1 - \frac{nb}{2}\right)} \frac{(ix-1)\,(ix)^{\gamma-1}}{\left[ix - (nb-1)\right]^{-1}} \\ &= \frac{m\gamma x^{\gamma-1} e^{i\frac{\pi}{2}(\gamma-1)}\left(ix - (nb-1)\right)(ix-1)}{2\left(1 - \frac{nb}{2}\right)} \\ &= \frac{m\gamma x^\gamma\left[(nb-1) - x^2 - nbxi\right] e^{i\frac{\pi}{2}(\gamma-1)}}{2x}. \end{aligned}$$

We also see that

$$\frac{z_0 f_b'(z_0)}{f_b(z_0)} = \frac{m\gamma}{2}\left[\frac{(nb-1) - x^2 - nbxi}{x}\right] e^{-i\frac{\pi}{2}} = \frac{im\gamma}{2}\left[\frac{x^2 - (nb-1) + nbxi}{x}\right] = iu.$$

Thus we can write

$$\left|\arg \frac{2f_b(z_0)\left[f_b(z_0)+z_0f_b'(z_0)\right]}{2f_b(z_0)+z_0f_b'(z_0)}-1\right| = \left|\arg \frac{2f_b(z_0)\left[1+\frac{z_0f_b'(z_0)}{f_b(z_0)}\right]}{2+\frac{z_0f_b'(z_0)}{f_b(z_0)}}\right|$$

$$= \left|\arg f_b(z_0) + \arg \frac{1+\frac{z_0f_b'(z_0)}{f_b(z_0)}}{2+\frac{z_0f_b'(z_0)}{f_b(z_0)}}\right| = \left|\gamma\frac{\pi}{2}+\arg\frac{1+iu}{2+iu}\right| \geq \gamma\frac{\pi}{2}.$$

This is a contradiction to our hypothesis. Therefore $p \prec h$ and we have the desired proof. □

Corollary 3.14. *Let f_b be analytic in $\mathbb{E}$ and $p(z)=\frac{f_b(z)}{z}$. Then*

$$\left|\arg\frac{2f_b(z)f_b'(z)}{f_b(z)+zf_b'(z)}\right| < \gamma\frac{\pi}{2} \quad \textit{implies that} \quad \left|\arg\frac{f_b(z)}{z}\right| < \gamma\frac{\pi}{2}.$$

4. Concluding remarks and observations

In this research, we developed differential subordination associated with the harmonic means of $f_b(z), f_b(z)+zf_b'(z)$ along with $f_b(z)+\frac{zf_b'(z)}{f_b(z)}$, where $f_b(z)=\frac{z}{(1-z^n)^b}$, $b\geq 0; n\in\mathbb{N}$ is an $n-$fold symmetric Koebe type function defined in $\mathbb{E} : f_b(0)=0, f_b'(z)\neq 0$. By using the admissibility conditions, we also studied some applications in geometric function theory.

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