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Intuitionistic Smooth Fuzzy θ-Closure Operator

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Abstract

In this paper, the concepts of intuitionistic r-fuzzy θ-open (θ-closed) sets and intuitionistic r-fuzzy θ-closure operator are introduced and discussed in intuitionistic smooth fuzzy topological spaces. As applications of these concepts, certain functions are characterized in terms of intuitionistic smooth fuzzy θ-closure operator.

Keywords: *Intuitionistic Smooth Fuzzy topology; intuitionistic fuzzy r-open set; intuitionistic fuzzy r-θ-open set; intuitionistic r-fuzzy δ-open set.*
2010 Mathematics Subject Classification: *54A40.*

1. Introduction

Since the advent of the notion of fuzzy set by Zadeh [11], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [2] is one among them. Using the notion of intuitionistic fuzzy sets, Coker [4] introduced the notion of intuitionistic smooth fuzzy topological spaces. Samanta and Mondal [8, 9] introduced the definitions of the intuitionistic smooth fuzzy topological space in Sǒstak sense. The purpose of this paper is to present and investigate the concepts of intuitionistic fuzzy θ-open (θ-closed) sets and intuitionistic fuzzy θ-closure operator in the context of intuitionistic smooth fuzzy topological spaces. As applications of these concepts, certain functions are characterized in terms of intuitionistic smooth fuzzy θ-closure operator.

2. Preliminaris

Definition 2.1. *Let X be a nonempty fixed set and I the closed interval $[0,1]$. An intuitionistic fuzzy set A is an object of the following form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ where the mapping $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) for each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x)+\gamma_A(x) \leq 1$ for each $x \in X$. Obviously, every fuzzy set A on a nonempty set X is an intuitionistic fuzzy set of the following form $A = \{\langle x, \mu_A(x), \overline{1}-\mu_A(x)\rangle : x \in X\}$.*

Definition 2.2. *Let A and B be intuitionistic fuzzy sets of the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \gamma_B(x)\rangle : x \in X\}$. Then*

1. *$A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$;*
2. *$\bar{A} = \{\langle x, \gamma_A(x), \mu_A(x)\rangle : x \in X\}$*
3. *$A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x)\rangle : x \in X\}$.*
4. *$A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x)\rangle : x \in X\}$.*

We will use the notation $A = \langle x, \mu_A, \gamma_A\rangle$ instead of $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$. A constant fuzzy set α taking value $\alpha \in [0,1]$ will be denoted by $\underline{\alpha}$. The intuitionistic fuzzy sets $\overline{0}$ and $\overline{1}$ are defined by $\overline{0} = \{\langle x, 0, 1\rangle : x \in X\}$ and $\overline{1} = \{\langle x, 1, 0\rangle : x \in X\}$. Let f be a mapping from an ordinary set X into an ordinary set Y. If $B = \{\langle y, \mu_B(y), \gamma_B(y)\rangle : y \in Y\}$ is an intuitionistic fuzzy set in Y, then the inverse image of B under f is an intuitionistic fuzzy set defined by $f^{-1}(B) = \{\langle x, f^{-1}(\mu_B)(x), f^{-1}(\gamma_B)(x)\rangle : x \in X\}$. The image of an intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle : x \in X\}$ under f is an intuitionistic fuzzy set defined by $f(A) = \{\langle y, f(\mu_A)(y), f(\gamma_A)(y)\rangle : y \in Y\}$, where

$$f(\mu_A)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu_A(x), & f^{-1}(y) \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

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and
$$f(\mu_A)(y)=\begin{cases}\inf_{x\in f^{-1}(y)}\gamma_A(x), & f^{-1}(y)\neq 0,\\ 1 & \text{otherwise}\end{cases}$$
for each $y\in Y$.

Definition 2.3. *[8, 9] An intuitionistic gradation of openness on X is an ordered pair $(\tau,\tau^\star)$ of functions from I^X to I such that*

1. $\tau(\lambda)+\tau^\star(\lambda)\leq 1$ *for all* $\lambda\in I^X$,
2. $\tau(\overline{0})=\tau(\overline{1})=1$, $\tau^\star(\overline{0})=\tau^\star(\overline{1})=0$
3. $\tau(\lambda_1\wedge\lambda_2)\geq\tau(\lambda_1)\wedge\tau(\lambda_2)$ *and* $\tau^\star(\lambda_1\wedge\lambda_2)\leq\tau^\star(\lambda_1)\vee\tau^\star(\lambda_2)$ *for each* $\lambda_1,\lambda_2\in I^X$.
4. $\tau(\bigvee_{i\in\Gamma}\lambda_i)\geq\bigwedge_{i\in\Gamma}\tau(\lambda_i)$ *and* $\tau^\star(\bigvee_{i\in\Gamma}\lambda_i)\leq\bigvee_{i\in\Gamma}\tau^\star(\lambda_i)$ *for each* $\lambda_i\in I^X, i\in\Gamma$.

The triplet $(X,\tau,\tau^\star)$ is called an intuitionistic smooth fuzzy topological space.

Theorem 2.4. *[8, 9] Let $(X,\tau,\tau^\star)$ be an intuitionistic smooth fuzzy topological space. Then for each $r\in I_0$, $\lambda\in I^X$ we define the operators $C_{\tau,\tau^\star}, I_{\tau,\tau^\star}: I^X\times I_0\to I^X$ as follows*
$$C_{\tau,\tau^\star}(\lambda,r)=\bigwedge\{\mu\in I^X\mid \lambda\leq\mu,\tau(\overline{1}-\mu)\geq r,\tau^\star(\overline{1}-\mu)\leq 1-r\},$$
$$I_{\tau,\tau^\star}(\lambda,r)=\bigvee\{\mu\in I^X\mid \mu\leq\lambda,\tau(\mu)\geq r,\tau^\star(\mu)\leq 1-r\}.$$

Definition 2.5. *[5] Let X be a nonempty set and $c\in X$ a fixed element in X. If $a\in(0,1]$ and $b\in[0,1)$ are two fixed real numbers such that $a+b\leq 1$, then the intuitionistic fuzzy set $c(a,b)=\langle x,c_a,1-c_{1-b}\rangle$ is called an intuitionistic fuzzy point in X, where a denotes the degree of membership of $c(a,b)$, b the degree of nonmembership of $c(a,b)$, and $c\in X$ the support of $c(a,b)$.*

Definition 2.6. *[5] Let $c(a,b)$ be an intuitionistic fuzzy point in X and $A=\langle x,\mu_A,\gamma_A\rangle$ be an intuitionistic fuzzy set in X. Suppose further that $a,b\in(0,1)$. $c(a,b)$ is said to be properly contained in A ($c(a,b)\in U$, for short) if and only if $a<\mu_A(c)$ and $b>\gamma_A(c)$.*

Definition 2.7. *[5]*

1. *An intuitionistic fuzzy point $c(a,b)$ in X is said to be quasi-coincident with the intuitionistic fuzzy set $A=\langle x,\mu_A,\gamma_A\rangle$ denoted by $c(a,b)\ q\ A$, if and only if $a<\mu_A(c)$ or $b>\gamma_A(c)$.*
2. *Let $A=\langle x,\mu_A,\gamma_A\rangle$ and $B=\langle x,\mu_B,\gamma_B\rangle$ are two intuitionistic fuzzy sets in X. Then A and B are said to be quasi-coincident, denoted by $A\ q\ B$ if and only if there exists an element $x\in X$ such that $\mu_A(x)>\gamma_B(x)$ or $\gamma_A(x)<\mu_B(x)$.*

The expression not quasi-coincident will be abbreviated as $\overline{q}$.

Proposition 2.8. *[5] Let U and V be two intuitionistic fuzzy sets and $c(a,b)$ an intuitionistic fuzzy point in X. Then*

1. $U\ \overline{q}\ \overline{1}-V$ *if and only if* $U\leq V$,
2. $U\ q\ V$ *if and only if* $U\not\leq\overline{1}-V$,
3. $c(a,b)\leq U$ *if and only if* $c(a,b)\ \overline{q}\ \overline{1}-U$,
4. $c(a,b)\ q\ U$ *if and only if* $c(a,b)\not\leq\overline{1}-U$.

Definition 2.9. *[5] Let $f:X\to Y$ be a function and $c(a,b)$ an intuitionistic fuzzy point in X. Then the image of $c(a,b)$ under f, denoted by $f(c(a,b))$, is defined by $f(c(a,b))=\langle y,f(c)_a,1-f(c)_{1-b}\rangle$.*

Proposition 2.10. *[7] Let $f:X\to Y$ be a function and $c(a,b)$ an intuitionistic fuzzy point in X.*

1. *If for intuitionistic fuzzy set V in Y we have $f(c(a,b))\ q\ V$, then $c(a,b)\ q\ f^{-1}(V)$.*
2. *If for intuitionistic fuzzy set U in X we have $c(a,b)\ q\ U$, then $f(c(a,b))\ q\ f(U)$.*

Definition 2.11. *Let (X,τ) be an intuitionistic fuzzy topological space on X and $c(a,b)$ an intuitionistic fuzzy point in X. An intuitionistic fuzzy set A is called q-neighbourhood of $c(a,b)$, denoted by $\mathcal{N}_q(c(a,b))$, if there exists an intuitionistic fuzzy open set U in X such that $c(a,b)\ q\ U$ and $U\leq A$.*

Definition 2.12. *An intuitionistic fuzzy set λ of an intuitionistic fuzzy topological space $(X,\tau,\tau^\star)$ is said to be*

1. *intuitionistic r-fuzzy regular open [6] if $\lambda=I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r)$.*
2. *intuitionistic r-fuzzy regular closed [6] if $\lambda=C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(\lambda,r),r)$.*
3. *intuitionistic r-fuzzy preopen [6] if $\lambda\leq I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r)$.*
4. *intuitionistic r-fuzzy semiopen [1] if $\lambda\leq C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(\lambda,r),r)$.*

3. Intuitionistic fuzzy θ-closed sets

Definition 3.1. *An intuitionistic fuzzy point $c(\alpha,\beta)$ is said to be an intuitionistic r-fuzzy θ-cluster point of an intuitionistic fuzzy set λ if for each q-neighborhood μ of $c(\alpha,\beta)$, $\lambda\ q\ C_{\tau,\tau^\star}(\mu,r)$. The set of all intuitionistic r-fuzzy θ-cluster points of λ is called intuitionistic r-fuzzy θ-closure of λ and is denoted by $\theta C_{\tau,\tau^\star}(\lambda,r)$. An intuitionistic fuzzy set λ is called intuitionistic r-fuzzy θ-closed if $\lambda=\theta C_{\tau,\tau^\star}(\lambda,r)$. The complement of an intuitionistic r-fuzzy θ-closed set is called an intuitionistic r-fuzzy θ-open set. The r-fuzzy θ-interior of λ denoted by $\theta I_{\tau,\tau^\star}(\lambda,r)$ is defined by $\theta I_{\tau,\tau^\star}(\lambda,r)=\overline{1}-\theta C_{\tau,\tau^\star}(\overline{1}-\lambda,r)$.*

Remark 3.2. *For an intuitionistic smooth fuzzy topological space $(X,\tau,\tau^\star)$, we have*

1. *every intuitionistic r-fuzzy θ-open set is intuitionistic r-fuzzy open.*

2. *every intuitionistic fuzzy r-regular open set is intuitionistic fuzzy r-open.*
3. *For any* $\lambda \in I^X$, $C_{\tau,\tau^\star}(\lambda,r) \leq \theta C_{\tau,\tau^\star}(\lambda,r)$.
4. $\theta C_{\tau,\tau^\star}(\overline{1}-\lambda,r) = \overline{1}-\theta I_{\tau,\tau^\star}(\lambda,r)$.
5. $\overline{1}-\theta C_{\tau,\tau^\star}(\lambda,r) = \theta I_{\tau,\tau^\star}(\overline{1}-\lambda,r)$.

The following examples show the converse of (1) and (2) are not true in general.

Example 3.3. *Let* $X=\{a,b\}$. *Define the intuitionistic fuzzy subset* λ_1 *as* $\lambda_1 = \langle x, \left(\frac{a}{0.4},\frac{b}{0.5}\right), \left(\frac{a}{0.4},\frac{b}{0.4}\right)\rangle$. *Let* $\tau,\tau^\star : I^X \to I$ *be defined as follows:*

$$\tau(\lambda)=\begin{cases}1 & \text{if } \lambda=\bar{0} \text{ or } \bar{1}\\ \frac{1}{2} & \text{if } \lambda=\lambda_1\\ 0 & \text{otherwise,}\end{cases} \qquad \tau^\star(\lambda)=\begin{cases}0 & \text{if } \lambda=\bar{0} \text{ or } \bar{1}\\ \frac{1}{2} & \text{if } \lambda=\lambda_1\\ 1 & \text{otherwise.}\end{cases}$$

Let $r=\frac{1}{2}$. *Clearly,* λ_1 *is intuitionistic* $\frac{1}{2}$*-fuzzy open but not intuitionistic* $\frac{1}{2}$*-fuzzy regular open.*

Example 3.4. *Let* $X=\{a,b,c\}$. *Define the intuitionistic fuzzy subsets* λ_1 *and* λ_2 *as* $\lambda_1 = \langle x, \left(\frac{a}{0.0},\frac{b}{0.0},\frac{c}{0.5}\right), \left(\frac{a}{1.0},\frac{b}{1.0},\frac{c}{0.0}\right)\rangle$ *and* $\lambda_2 = \langle x, \left(\frac{a}{1.0},\frac{b}{0.5},\frac{c}{0.0}\right), \left(\frac{a}{0.0},\frac{b}{0.5},\frac{c}{1.0}\right)\rangle$.
Let $\tau,\tau^\star : I^X \to I$ *be defined as follows:*

$$\tau(\lambda)=\begin{cases}1 & \text{if } \lambda=\bar{0} \text{ or } \bar{1}\\ \frac{1}{4} & \text{if } \lambda=\lambda_1\\ \frac{1}{3} & \text{if } \lambda=\lambda_2\\ \frac{1}{2} & \text{if } \lambda=\lambda_1\vee\lambda_2\\ 0 & \text{otherwise,}\end{cases} \qquad \tau^\star(\lambda)=\begin{cases}0 & \text{if } \lambda=\bar{0} \text{ or } \bar{1}\\ \frac{3}{4} & \text{if } \lambda=\lambda_1\\ \frac{2}{3} & \text{if } \lambda=\lambda_2\\ \frac{1}{2} & \text{if } \lambda=\lambda_1\vee\lambda_2\\ 1 & \text{otherwise.}\end{cases}$$

Clearly, λ_1 *is intuitionistic* $\frac{1}{2}$*-fuzzy regular open but not intuitionistic* $\frac{1}{2}$*-fuzzy* θ*-open.*

Theorem 3.5. *If* λ *is an intuitionistic r-fuzzy open set in an intuitionistic smooth fuzzy topological space* $(X,\tau,\tau^\star)$*, then* $C_{\tau,\tau^\star}(\lambda,r) = \theta C_{\tau,\tau^\star}(\lambda,r)$.

Proof. It is enough to prove $\theta C_{\tau,\tau^\star}(\lambda,r) \leq C_{\tau,\tau^\star}(\lambda,r)$. Let $c(\alpha,\beta)$ be an intuitionistic fuzzy point in X such that $c(\alpha,\beta) \notin C_{\tau,\tau^\star}(\lambda,r)$, then there exists $\mu \in \mathcal{N}_q(c(\alpha,\beta))$ such that $\mu\,\overline{q}\,\lambda$ and hence $\mu \leq \lambda$. Thus $C_{\tau,\tau^\star}(\mu,r) \leq I_{\tau,\tau^\star}(\lambda,r) \leq \lambda$, since λ is an intuitionistic r-fuzzy open set in X. Hence $C_{\tau,\tau^\star}(\mu,r)\,\overline{q}\,\lambda$ which implies $c(\alpha,\beta) \notin \theta C_{\tau,\tau^\star}(\lambda,r)$. Therefore $\theta C_{\tau,\tau^\star}(\lambda,r) \leq C_{\tau,\tau^\star}(\lambda,r)$. Thus $C_{\tau,\tau^\star}(\lambda,r) = \theta C_{\tau,\tau^\star}(\lambda,r)$. □

Theorem 3.6. *In an intuitionistic smooth fuzzy topological space* $(X,\tau,\tau^\star)$*, the following hold:*

1. *Finite union and arbitrary intersection of intuitionistic r-fuzzy* θ*-closed sets is an intuitionistic r-fuzzy* θ*-closed set.*
2. *The intuitionistic fuzzy sets* $\overline{0}$ *and* $\overline{1}$ *are intuitionistic r-fuzzy* θ*-closed.*

Proof. Straightforward. □

Theorem 3.7. *Let* λ *and* μ *be two intuitionistic fuzzy sets in an intuitionistic smooth fuzzy topological space* $(X,\tau,\tau^\star)$*. Then we have the following:*

1. $\theta C_{\tau,\tau^\star}(\overline{0},r) = \overline{0}$, $\theta C_{\tau,\tau^\star}(\overline{1},r) = \overline{1}$,
2. $\lambda \leq \theta C_{\tau,\tau^\star}(\lambda,r)$,
3. $\lambda \leq \mu \Rightarrow \theta C_{\tau,\tau^\star}(\lambda,r) \leq \theta C_{\tau,\tau^\star}(\mu,r)$,
4. $\theta C_{\tau,\tau^\star}(\lambda,r) \vee \theta C_{\tau,\tau^\star}(\mu,r) = \theta C_{\tau,\tau^\star}(\lambda\vee\mu)$,
5. $\theta C_{\tau,\tau^\star}(\lambda\wedge\mu) \leq \theta C_{\tau,\tau^\star}(\lambda,r) \wedge \theta C_{\tau,\tau^\star}(\mu)$.

Proof. (1). Obvious.
(2). Suppose that there is an intuitionistic fuzzy point $x_{(\alpha,\beta)}$ in X such that $x_{(\alpha,\beta)} \notin \theta C_{\tau,\tau^\star}(\lambda,r)$ and $x_{(\alpha,\beta)} \in \lambda$. Then there is a q-neighborhood μ of $x_{(\alpha,\beta)}$ such that $C_{\tau,\tau^\star}(\mu,r)\,\overline{q}\,\lambda$. Thus $\mu \leq \overline{1}-\lambda$. Since μ is a q-neighborhood of $x_{(\alpha,\beta)}$, there is an intuitionistic r-fuzzy open set γ such that $x_{(\alpha,\beta)}\ q\ \gamma \leq \mu$. Since $\mu \leq \overline{1}-\lambda$, $x_{(\alpha,\beta)}\ q\ \overline{1}-\lambda$, and hence $x_{(\alpha,\beta)} \nleq \lambda$. On the other hand we have $x_{(\alpha,\beta)} \leq \lambda$, because $x_{(\alpha,\beta)} \in \lambda$. It is a contradiction.
(3). Let $x_{(\alpha,\beta)}$ be an intuitionistic fuzzy point in X such that $x_{(\alpha,\beta)} \notin \theta C_{\tau,\tau^\star}(\mu,r)$. Then there is a q-neighborhood γ of $x_{(\alpha,\beta)}$ such that $C_{\tau,\tau^\star}(\gamma,r)\,\overline{q}\,\mu$. Since $\lambda \leq \mu$, we have $C_{\tau,\tau^\star}(\gamma,r)\,\overline{q}\,\lambda$. Therefore $x_{(\alpha,\beta)} \notin \theta C_{\tau,\tau^\star}(\lambda,r)$.
(4). Since $\lambda \leq \lambda\vee\mu$, $\theta C_{\tau,\tau^\star}(\lambda,r) \leq \theta C_{\tau,\tau^\star}(\lambda\vee\mu,r)$. Similarly, $\theta C_{\tau,\tau^\star}(\mu,r) \leq \theta C_{\tau,\tau^\star}(\lambda\vee\mu,r)$. Hence $\theta C_{\tau,\tau^\star}(\lambda,r) \vee \theta C_{\tau,\tau^\star}(\mu,r) \leq \theta C_{\tau,\tau^\star}(\lambda\vee\mu,r)$. On the other hand, take any $x_{(\alpha,\beta)} \in \theta C_{\tau,\tau^\star}(\lambda\vee\mu,r)$. Then for any q-neighborhood γ of $x_{(\alpha,\beta)}$, $C_{\tau,\tau^\star}(\gamma,r)\ q\ (\lambda\vee\mu)$. By Lemma 3.4, $C_{\tau,\tau^\star}(\gamma,r)\ q\ \lambda$ or $C_{\tau,\tau^\star}(\gamma,r)\ q\ \mu$. Therefore $x_{(\alpha,\beta)} \in \theta C_{\tau,\tau^\star}(\lambda,r)$ or $x_{(\alpha,\beta)} \in \theta C_{\tau,\tau^\star}(\mu,r)$. Hence $\theta C_{\tau,\tau^\star}(\lambda\vee\mu,r) \leq \theta C_{\tau,\tau^\star}(\lambda,r) \vee \theta C_{\tau,\tau^\star}(\mu,r)$.
(5). Since $\lambda\wedge\mu \leq \lambda$, $\theta C_{\tau,\tau^\star}(\lambda\wedge\mu,r) \leq \theta C_{\tau,\tau^\star}(\lambda,r)$. Similarly, $\theta C_{\tau,\tau^\star}(\lambda\wedge\mu,r) \leq \theta C_{\tau,\tau^\star}(\mu,r)$. Therefore $\theta C_{\tau,\tau^\star}(\lambda\wedge\mu,r) \leq \theta C_{\tau,\tau^\star}(\lambda,r) \wedge \theta C_{\tau,\tau^\star}(\mu,r)$. □

Remark 3.8. *For an intuitionistic fuzzy set λ in an intuitionistic smooth fuzzy topological space $(X,\tau,\tau^\star)$, the intuitionistic r-fuzzy θ-closure, $\theta C_{\tau,\tau^\star}(\lambda,r)$, is not necessarily an intuitionistic r-fuzzy θ-closed, and hence $\theta C_{\tau,\tau^\star}(\theta C_{\tau,\tau^\star}(\lambda,r),r)\neq\theta C_{\tau,\tau^\star}(\lambda,r)$, which is shown in the following example. Thus $\theta C_{\tau,\tau^\star}$ operator does not satisfies the Kuratowski closure axioms.*

Example 3.9. *Let $X=\{a,b,c\}$. Define the intuitionistic fuzzy subsets λ_1,λ_2 and λ_3 as $\lambda_1=\langle x,\left(\frac{a}{0.5},\frac{b}{0.6},\frac{c}{0.2}\right),\left(\frac{a}{0.4},\frac{b}{0.4},\frac{c}{0.1}\right)\rangle$, $\lambda_2=\langle x,\left(\frac{a}{0.4},\frac{b}{0.3},\frac{c}{0.2}\right),\left(\frac{a}{0.5},\frac{b}{0.6},\frac{c}{0.3}\right)\rangle$ and $\lambda_3=\langle x,\left(\frac{a}{0.3},\frac{b}{0.3},\frac{c}{0.0}\right),\left(\frac{a}{0.6},\frac{b}{0.7},\frac{c}{0.1}\right)\rangle$.*
Let $\tau_1,\tau_1^\star:I^X\to I$ be defined as follows:

$$\tau_1(\lambda)=\begin{cases}1 & \text{if } \lambda=\overline{0} \text{ or } \overline{1}\\ \frac{1}{4} & \text{if } \lambda=\lambda_1\\ \frac{1}{2} & \text{if } \lambda=\lambda_2\\ 0 & \text{otherwise}\end{cases}\qquad \tau_1^\star(\lambda)=\begin{cases}0 & \text{if } \lambda=\overline{0} \text{ or } \overline{1}\\ \frac{3}{4} & \text{if } \lambda=\lambda_1\\ \frac{1}{2} & \text{if } \lambda=\lambda_2\\ 1 & \text{otherwise}\end{cases}$$

Then an intuitionistic fuzzy $a(0.6,0.3)\in\theta C_{\tau,\tau^\star}(\lambda_3,\frac{1}{2})$ (because $a(0.6,0.3)\ q\ \lambda_1$, $C_{\tau,\tau^\star}(\lambda_3,\frac{1}{2})=\overline{1}\ q\ \lambda_3$), also $a(0.8,0.1)\notin\theta C_{\tau,\tau^\star}(\lambda_3,\frac{1}{2})$ (because $a(0.8,0.1)\ q\ \lambda_2$, $C_{\tau,\tau^\star}(\lambda_2,\frac{1}{2})=\overline{1}-\lambda_2\ \overline{q}\ \lambda_3$). But $a(0.8,0.1)\in\theta C_{\tau,\tau^\star}(a(0.6,0.3),\frac{1}{2},\frac{1}{2})\leq\theta C_{\tau,\tau^\star}(\theta C_{\tau,\tau^\star}(\lambda_3,\frac{1}{2},\frac{1}{2}),\frac{1}{2},\frac{1}{2})$. Hence $\theta C_{\tau,\tau^\star}(\lambda_3,\frac{1}{2})$ is not an intuitionistic $\frac{1}{2}$-fuzzy θ-closed set.

Remark 3.10. *Clearly, λ is an intuitionistic r-fuzzy θ-open set if and only if $\theta I_{\tau,\tau^\star}(\lambda,r)=\lambda$. Also we have following properties for the interior operator.*

Theorem 3.11. *Let λ and μ be two intuitionistic fuzzy sets in an intuitionistic smooth fuzzy topological space $(X,\tau,\tau^\star)$. Then we have the following:*

1. $\theta I_{\tau,\tau^\star}(\overline{1},r)=\overline{1}$,
2. $\theta I_{\tau,\tau^\star}(\lambda,r)\leq\lambda$,
3. $\lambda\leq\mu\Rightarrow\theta I_{\tau,\tau^\star}(\lambda,r)\leq\theta I_{\tau,\tau^\star}(\mu,r)$,
4. $\theta I_{\tau,\tau^\star}(\lambda\wedge\mu,r)=\theta I_{\tau,\tau^\star}(\lambda,r)\wedge\theta I_{\tau,\tau^\star}(\mu,r)$,
5. $\theta I_{\tau,\tau^\star}(\lambda,r)\vee\theta I_{\tau,\tau^\star}(\mu,r)\leq\theta I_{\tau,\tau^\star}(\lambda\vee\mu,r)$.

Proof. (1). Obvious.
(2). Let $x_{(\alpha,\beta)}\in\theta I_{\tau,\tau^\star}(\lambda,r)$. From the fact that $\theta I_{\tau,\tau^\star}(\lambda,r)=\overline{1}-\theta C_{\tau,\tau^\star}(\overline{1}-\lambda,r)=\langle x,\gamma_{\theta C_{\tau,\tau^\star}(1-\lambda,r)},\mu_{\theta C_{\tau,\tau^\star}(1-\lambda,r)}\rangle$, $\alpha\leq\gamma_{\theta C_{\tau,\tau^\star}(1-\lambda,r)}(x)$ and $\beta\geq\mu_{\theta C_{\tau,\tau^\star}(1-\lambda,r)}(x)$. Since $\overline{1}-\lambda\leq\theta C_{\tau,\tau^\star}(\overline{1}-\lambda,r)$, $\mu_{\overline{1}-\lambda}\leq\gamma_{\theta C_{\tau,\tau^\star}(\overline{1}-\lambda,r)}$ and $\gamma_{\overline{1}-\lambda}\geq\mu_{\theta C_{\tau,\tau^\star}(\overline{1}-\lambda,r)}$. Thus $\alpha\leq\gamma_{\theta C_{\tau,\tau^\star}(\overline{1}-\lambda,r)}(x)=\mu_\lambda(x)$ and $\beta\leq\mu_{\theta C_{\tau,\tau^\star}(\overline{1}-\lambda,r)}(x)=\gamma_\lambda(x)$. Hence $x_{(\alpha,\beta)}\in\lambda$.
(3). Let $\lambda\leq\mu$. Then $\overline{1}-\lambda\geq\overline{1}-\mu$. By Theorem 3.7 (3), $\theta C_{\tau,\tau^\star}(\overline{1}-\lambda,r)\geq\theta C_{\tau,\tau^\star}(\overline{1}-\mu,r)$. Thus $\overline{1}-\theta C_{\tau,\tau^\star}(\overline{1}-\lambda,r)\leq\overline{1}-\theta C_{\tau,\tau^\star}(\overline{1}-\mu,r)$. Hence $\theta I_{\tau,\tau^\star}(\lambda,r)=\overline{1}-\theta C_{\tau,\tau^\star}(\overline{1}-\lambda,r)\leq\overline{1}-\theta C_{\tau,\tau^\star}(\overline{1}-\mu,r)=\theta I_{\tau,\tau^\star}(\mu,r)$.
(4). $\theta I_{\tau,\tau^\star}(\lambda\wedge\mu,r)=\overline{1}-\theta C_{\tau,\tau^\star}(\overline{1}-(\lambda\wedge\mu),r)=\overline{1}-\theta C_{\tau,\tau^\star}(\overline{1}-\lambda\vee\overline{1}-\mu),r)=\overline{1}-(\theta C_{\tau,\tau^\star}(\overline{1}-\lambda,r)\vee\theta C_{\tau,\tau^\star}(\overline{1}-\mu,r))=\overline{1}-(\theta C_{\tau,\tau^\star}(\overline{1}-\lambda,r))\wedge\overline{1}-(\theta C_{\tau,\tau^\star}(\overline{1}-\mu,r))=\theta I_{\tau,\tau^\star}(\lambda,r)\wedge\theta I_{\tau,\tau^\star}(\mu,r)$.
(5). Since $\lambda\leq\lambda\vee\mu$, we have $\theta I_{\tau,\tau^\star}(\lambda,r)\leq\theta I_{\tau,\tau^\star}(\lambda\vee\mu,r)$. Since $\mu\leq\lambda\vee\mu$, $\theta I_{\tau,\tau^\star}(\mu,r)\leq\theta I_{\tau,\tau^\star}(\lambda\vee\mu,r)$. Therefore $\theta I_{\tau,\tau^\star}(\lambda,r)\vee\theta I_{\tau,\tau^\star}(\mu,r)\leq\theta I_{\tau,\tau^\star}(\lambda\vee\mu,r)$. □

Corollary 3.12. *For an intuitionistic fuzzy set λ, $\theta I_{\tau,\tau^\star}(\lambda,r)\leq I_{\tau,\tau^\star}(\lambda,r)$.*

Proof. Let $\lambda\in I^X$. Thus $C_{\tau,\tau^\star}(\overline{1}-\lambda,r)\leq\theta C_{\tau,\tau^\star}(\overline{1}-\lambda,r)$ by Remark 3.2 (3). Hence $\theta I_{\tau,\tau^\star}(\lambda,r)=\overline{1}-\theta C_{\tau,\tau^\star}(\overline{1}-\lambda,r)\leq\overline{1}-C_{\tau,\tau^\star}(\overline{1}-\lambda,r)=I_{\tau,\tau^\star}(\lambda,r)$. □

Theorem 3.13. *If λ is an intuitionistic r-fuzzy closed set in an intuitionistic smooth fuzzy topological space $(X,\tau,\tau^\star)$, then $\theta I_{\tau,\tau^\star}(\lambda,r)=I_{\tau,\tau^\star}(\lambda,r)$.*

Proof. Let λ be an intuitionistic r-fuzzy closed set. Then $\overline{1}-\lambda$ is an intuitionistic r-fuzzy open set. Thus $C_{\tau,\tau^\star}(\overline{1}-\lambda,r)=\theta C_{\tau,\tau^\star}(\overline{1}-\lambda,r)$ by Theorem 3.7. Hence $\theta I_{\tau,\tau^\star}(\lambda,r)=\overline{1}-\theta C_{\tau,\tau^\star}(\overline{1}-\lambda,r)=\overline{1}-C_{\tau,\tau^\star}(\overline{1}-\lambda,r)=I_{\tau,\tau^\star}(\lambda,r)$. □

Theorem 3.14. *Let λ be an intuitionistic fuzzy set in an intuitionistic smooth fuzzy topological space $(X,\tau,\tau^\star)$. Then $\theta I_{\tau,\tau^\star}(\lambda,r)=\bigvee\{\theta I_{\tau,\tau^\star}(\mu,r):\tau(\overline{1}-\mu)\geq r,\tau^\star(\overline{1}-\mu)\leq\overline{1}-r,\mu\leq\lambda\}=\bigvee\{I_{\tau,\tau^\star}(\mu,r):\tau(\overline{1}-\mu)\geq r,\tau^\star(\overline{1}-\mu)\leq\overline{1}-r,\mu\leq\lambda\}$.*

Proof. We have

$$\begin{aligned}\theta I_{\tau,\tau^\star}(\lambda,r) &= \overline{1}-\theta C_{\tau,\tau^\star}(\overline{1}-\lambda,r)\\ &= \overline{1}-\bigwedge\{\theta C_{\tau,\tau^\star}(\gamma,r):\tau(\gamma)\geq r,\tau^\star(\gamma)\leq\overline{1}-r,\overline{1}-\lambda\leq\gamma\}\\ &= \bigvee\{\overline{1}-\theta C_{\tau,\tau^\star}(\gamma,r):\tau(\gamma)\geq r,\tau^\star(\gamma)\leq\overline{1}-r,\overline{1}-\lambda\leq\gamma\}\\ &= \bigvee\{\overline{1}-\theta I_{\tau,\tau^\star}(\gamma,r):\tau(\gamma)\geq r,\tau^\star(\gamma)\leq\overline{1}-r,\overline{1}-\lambda\leq\gamma\}.\end{aligned}$$

Let $\mu=\overline{1}-\gamma$. Then $\theta I_{\tau,\tau^\star}(\lambda,r)=\bigvee\{\theta I_{\tau,\tau^\star}(\mu,r):\tau(\overline{1}-\mu)\geq r,\tau^\star(\overline{1}-\mu)\leq\overline{1}-r,\mu\leq\lambda\}$. The second equality holds from Theorem 3.13. □

Corollary 3.15. *For an intuitionistic fuzzy set λ in an intuitionistic smooth fuzzy topological space $(X,\tau,\tau^\star)$, $\theta I_{\tau,\tau^\star}(\lambda,r)$ is an intuitionistic r-fuzzy open set.*

Remark 3.16. *For an intuitionistic fuzzy set λ in an intuitionistic smooth fuzzy topological space $(X,\tau,\tau^{\star})$, $\theta I_{\tau,\tau^{\star}}(\lambda,r)$ is not necessarily intuitionistic r-fuzzy θ-open set.*

Definition 3.17. *An intuitionistic fuzzy topological space $(X,\tau,\tau^{\star})$ is said to be intuitionistic r-fuzzy extremely disconnected if $C_{\tau,\tau^{\star}}(\lambda,r)$ is intuitionistic r-fuzzy open for every intuitionistic r-fuzzy open set λ of $(X,\tau,\tau^{\star})$.*

Lemma 3.18. *If λ,μ are intuitionistic r-fuzzy open sets in an intuitionistic r-fuzzy extremally disconnected space X, then $C_{\tau,\tau^{\star}}(\mu,r)\ \overline{q}\ \lambda \leq C_{\tau,\tau^{\star}}(\mu,r)\ \overline{q}\ \theta C_{\tau,\tau^{\star}}(\lambda,r)$.*

Proof. Let $C_{\tau,\tau^{\star}}(\mu,r)\ \overline{q}\ \lambda \Rightarrow \lambda \leq C_{\tau,\tau^{\star}}(\mu,r) \Rightarrow C_{\tau,\tau^{\star}}(\lambda,r) \leq C_{\tau,\tau^{\star}}(\mu,r)$ since X is an intuitionistic r-fuzzy extremally disconnected space. Hence $C_{\tau,\tau^{\star}}(\mu,r)\ \overline{q}\ \lambda \Rightarrow C_{\tau,\tau^{\star}}(\mu,r)\ \overline{q}\ C_{\tau,\tau^{\star}}(\lambda,r) \Rightarrow C_{\tau,\tau^{\star}}(\mu,r)\ \overline{q}\ \theta C_{\tau,\tau^{\star}}(\lambda,r)$ by Remark 3.2. □

Theorem 3.19. *If λ is an intuitionistic r-fuzzy open set in an intuitionistic r-fuzzy extremally disconnected space $(X,\tau,\tau^{\star})$, then $\theta C_{\tau,\tau^{\star}}(\lambda,r)$ is an intuitionistic r-fuzzy θ-closed set in X.*

Proof. Let $c(\alpha,\beta)$ be an intuitionistic fuzzy point in X and let $c(\alpha,\beta) \notin \theta C_{\tau,\tau^{\star}}(\lambda,r)$. Then there is $\mu \in \mathscr{N}_q(c(\alpha,\beta))$ such that $C_{\tau,\tau^{\star}}(\mu,r)\ \overline{q}\ \lambda$. By Lemma 3.18, $C_{\tau,\tau^{\star}}(\mu,r)\ \overline{q}\ \theta C_{\tau,\tau^{\star}}(\lambda,r)$ and hence $c(\alpha,\beta) > \theta C_{\tau,\tau^{\star}}(\lambda,r)$ implies $c(\alpha,\beta) \notin \theta C_{\tau,\tau^{\star}}(\theta C_{\tau,\tau^{\star}}(\lambda,r),r)$. Then $\theta C_{\tau,\tau^{\star}}(\theta C_{\tau,\tau^{\star}}(\lambda,r),r) \leq \theta C_{\tau,\tau^{\star}}(\lambda,r)$. But $\theta C_{\tau,\tau^{\star}}(\lambda,r) \leq \theta C_{\tau,\tau^{\star}}(\theta C_{\tau,\tau^{\star}}(\lambda,r),r)$, then $\theta C_{\tau,\tau^{\star}}(\lambda,r) = \theta C_{\tau,\tau^{\star}}(\theta C_{\tau,\tau^{\star}}(\lambda,r),r)$. Thus $\theta C_{\tau,\tau^{\star}}(\lambda,r)$ is an intuitionistic r-fuzzy θ-closed set. □

Theorem 3.20. *In an intuitionistic r-fuzzy extremally disconnected space $(X,\tau,\tau^{\star})$, every intuitionistic r-fuzzy regular open set is an intuitionistic r-fuzzy θ-open.*

Proof. Let λ be an intuitionistic r-fuzzy regular open set in an intuitionistic r-fuzzy extremally disconnected space $(X,\tau,\tau^{\star})$. Then $\lambda = I_{\tau,\tau^{\star}}(C_{\tau,\tau^{\star}}(\lambda,r),r) = C_{\tau,\tau^{\star}}(\lambda,r) = I_{\tau,\tau^{\star}}(\lambda,r)$. Since λ is an intuitionistic r-fuzzy closed set, λ is an intuitionistic r-fuzzy open set and by Theorem 3.5, $C_{\tau,\tau^{\star}}(\lambda,r) = \theta C_{\tau,\tau^{\star}}(\lambda,r)$. Now $C_{\tau,\tau^{\star}}(\lambda,r) = \theta C_{\tau,\tau^{\star}}(\lambda,r)$, $I_{\tau,\tau^{\star}}(\lambda,r) = \theta I_{\tau,\tau^{\star}}(\lambda,r)$. Thus $\lambda = I_{\tau,\tau^{\star}}(\lambda,r) = \theta I_{\tau,\tau^{\star}}(\lambda,r)$, and hence λ is an intuitionistic r-fuzzy θ-open set in (X,τ). □

Theorem 3.21. *An intuitionistic fuzzy set λ in an intuitionistic smooth fuzzy topological space X is intuitionistic r-fuzzy θ-open if and only if for each intuitionistic fuzzy point $c(\alpha,\beta)$ in X with $c(\alpha,\beta)\ q\ \lambda$, λ is an q-neighbourhood of $c(\alpha,\beta)$.*

Proof. Let λ be an intuitionistic r-fuzzy θ-open set and $c(\alpha,\beta)$ be an intuitionistic fuzzy point in X with $c(\alpha,\beta)\ q\ \lambda$. Then $c(\alpha,\beta) \nleq \lambda$. Since λ is an intuitionistic r-fuzzy θ-closed set, $c(\alpha,\beta) \nleq \lambda = \theta C_{\tau,\tau^{\star}}(\lambda,r)$. Then there exists q-neighbourhood μ of $c(\alpha,\beta)$ such that $C_{\tau,\tau^{\star}}(\mu,r)\ \overline{q}\ \lambda$. Hence λ is an q-neighbourhood of $c(\alpha,\beta)$. Conversely, if $c(\alpha,\beta) \nleq \lambda$, then $c(\alpha,\beta)\ q\ \lambda$. Since λ is an q-neighbourhood of $c(\alpha,\beta)$, then there exists q-neighbourhood μ of $c(\alpha,\beta)$ such that $C_{\tau,\tau^{\star}}(\mu,r)\ \overline{q}\ \lambda$ and so $c(\alpha,\beta) \leq \theta C_{\tau,\tau^{\star}}(\lambda,r)$. Hence λ is an intuitionistic r-fuzzy θ-closed set and then λ is an intuitionistic r-fuzzy θ-open set. □

Theorem 3.22. *For any intuitionistic fuzzy set λ in an intuitionistic smooth fuzzy topological space $(X,\tau,\tau^{\star})$, $\theta C_{\tau,\tau^{\star}}(\lambda,r) = \bigwedge\{\theta C_{\tau,\tau^{\star}}(\mu,r) : \tau(\mu) \geq r, \tau^{\star}(\mu) \leq \overline{1}-r, \lambda \leq \mu\}$.*

Proof. Obviously that $\theta C_{\tau,\tau^{\star}}(\lambda,r) \leq \bigwedge\{\theta C_{\tau,\tau^{\star}}(\mu,r) : \tau(\mu) \geq r, \tau^{\star}(\mu) \leq \overline{1}-r, \lambda \leq \mu\}$. Now, let $c(\alpha,\beta) \in \bigwedge\{\theta C_{\tau,\tau^{\star}}(\mu,r) : \tau(\mu) \geq r, \tau^{\star}(\mu) \leq \overline{1}-r, \lambda \leq \mu\}$ but $c(\alpha,\beta) \notin \theta C_{\tau,\tau^{\star}}(\lambda,r)$. Then there exists an q-neighbourhood γ of $c(\alpha,\beta)$ such that $C_{\tau,\tau^{\star}}(\gamma,r)\ \overline{q}\ \lambda$ and hence $\lambda \leq C_{\tau,\tau^{\star}}(\gamma,r)$. Then $c(\alpha,\beta) \in \theta C_{\tau,\tau^{\star}}(C_{\tau,\tau^{\star}}(\gamma,r),r)$ and consequently, we have $C_{\tau,\tau^{\star}}(\gamma,r)\ q\ C_{\tau,\tau^{\star}}(\gamma,r)$, which is not true. Hence the result. □

Definition 3.23. *An intuitionistic smooth fuzzy topological space $(X,\tau,\tau^{\star})$ is said to be intuitionistic r-fuzzy regular if and only if for each intuitionistic fuzzy point $c(\alpha,\beta)$ in X and each q-neighbourhood λ of $c(\alpha,\beta)$, there exists q-neighbourhood μ of $c(\alpha,\beta)$ such that $C_{\tau,\tau^{\star}}(\mu) \leq \lambda$.*

Theorem 3.24. *An intuitionistic smooth fuzzy topological space $(X,\tau,\tau^{\star})$ is intuitionistic r-fuzzy regular if and only if for any intuitionistic fuzzy set λ in X, $C_{\tau,\tau^{\star}}(\lambda,r) = \theta C_{\tau,\tau^{\star}}(\lambda,r)$.*

Proof. Let (X,τ) be an intuitionistic r-fuzzy regular space. It is always true that $C_{\tau,\tau^{\star}}(\lambda,r) \leq \theta C_{\tau,\tau^{\star}}(\lambda,r)$ for any intuitionistic fuzzy set λ. Now, let $c(\alpha,\beta)$ be an intuitionistic fuzzy point in X with $c(\alpha,\beta) \in \theta C_{\tau,\tau^{\star}}(\lambda,r)$ and let μ be an q-neighbourhood of $c(\alpha,\beta)$. Then by intuitionistic r-fuzzy regular space (X,τ), there exists q-neighbourhood γ of $c(\alpha,\beta)$ such that $C_{\tau,\tau^{\star}}(\gamma,r) \leq \mu$. Now $c(\alpha,\beta) \in \theta C_{\tau,\tau^{\star}}(\lambda,r)$ implies $C_{\tau,\tau^{\star}}(\gamma,r)\ q\ \lambda$ implies $\mu\ q\ \lambda$ implies $c(\alpha,\beta) \in C_{\tau,\tau^{\star}}(\lambda,r)$. Hence $\theta C_{\tau,\tau^{\star}}(\lambda,r) \leq C_{\tau,\tau^{\star}}(\lambda,r)$. Thus $\theta C_{\tau,\tau^{\star}}(\lambda,r) = C_{\tau,\tau^{\star}}(\lambda,r)$. Conversely, let $c(\alpha,\beta)$ be an intuitionistic fuzzy point in X and λ an q-neighbourhood of $c(\alpha,\beta)$. Then $c(\alpha,\beta) \notin \lambda = C_{\tau,\tau^{\star}}(\lambda,r) = \theta C_{\tau,\tau^{\star}}(\lambda,r)$. Thus there exists an q-neighbourhood γ of $c(\alpha,\beta)$ such that $C_{\tau,\tau^{\star}}(\gamma,r)\ \overline{q}\ \lambda$ and then $C_{\tau,\tau^{\star}}(\gamma,r) \leq \lambda$. Hence (X,τ) is intuitionistic r-fuzzy regular. □

Corollary 3.25. *In an intuitionistic r-fuzzy regular space (X,τ) an intuitionistic r-fuzzy closed set is an intuitionistic r-fuzzy θ-closed set and hence for any intuitionistic fuzzy set λ in X, $\theta C_{\tau,\tau^{\star}}(\lambda,r)$ is an intuitionistic r-fuzzy θ-closed set.*

4. Characterizations for some types of functions

Definition 4.1. *A function $f:(X,\tau,\tau^\star)\to(Y,\sigma,\sigma^\star)$ is said to be intuitionistic r-fuzzy strongly θ-continuous if for each intuitionistic fuzzy point $x_{(\alpha,\beta)}$ in X and $\mu\in\mathscr{N}_q(f(x_{(\alpha,\beta)})$, there exists $\lambda\in\mathscr{N}_q(x_{(\alpha,\beta)})$ such that $f(C_{\tau,\tau^\star}(\lambda,r))\le\mu$.*

Theorem 4.2. *For a function $f:(X,\tau,\tau^\star)\to(Y,\sigma,\sigma^\star)$, the following statements are equivalent:*

1. *f is intuitionistic r-fuzzy strongly θ-continuous.*
2. *$f(\theta C_{\tau,\tau^\star}(\lambda,r)))\le C_{\sigma,\sigma^\star}(f(\lambda),r)$ for each intuitionistic fuzzy set λ in X.*
3. *$\theta C_{\tau,\tau^\star}(f^{-1}(\mu),r)\le f^{-1}(C_{\sigma,\sigma^\star}(\mu,r))$ for each intuitionistic fuzzy set μ inY.*
4. *$f^{-1}(\mu)$ is an intuitionistic r-fuzzy θ-closed in X for each intuitionistic r-fuzzy closed set μ in Y.*
5. *$f^{-1}(\mu)$ is an intuitionistic r-fuzzy θ-open set in X for each intuitionistic r-fuzzy open set μ in Y.*
6. *$f^{-1}(I_{\sigma,\sigma^\star}(\mu,r))\le\theta I_{\tau,\tau^\star}(f^{-1}(\mu),r)$ for each intuitionistic fuzzy set μ of Y.*

Proof. (1) $\Rightarrow$ (2): Let $c(\alpha,\beta)\in\theta C_{\tau,\tau^\star}(\lambda,r)$ and $\gamma\in\mathscr{N}_q(f(c(\alpha,\beta)))$. By (1), there exists $\eta\in\mathscr{N}_q(c(\alpha,\beta))$ such that $f(C_{\tau,\tau^\star}(\eta,r))\le\gamma$. Now we have $c(\alpha,\beta)\in\theta C_{\tau,\tau^\star}(\lambda,r)\Rightarrow C_{\tau,\tau^\star}(\eta,r)\,q\,\lambda\Rightarrow f(C_{\tau,\tau^\star}(\eta,r))\,q\,f(\lambda)\Rightarrow\gamma\,q\,f(\lambda)\Rightarrow f(c(\alpha,\beta))\in C_{\sigma,\sigma^\star}(f(\lambda),r)\Rightarrow c(\alpha,\beta)\in f^{-1}(C_{\sigma,\sigma^\star}(f(\lambda),r))$. Hence $\theta C_{\tau,\tau^\star}(\lambda,r)\le f^{-1}(C_{\sigma,\sigma^\star}(f(\lambda),r))$ and so $f(\theta C_{\tau,\tau^\star}(\lambda,r)\le C_{\sigma,\sigma^\star}(f(\lambda),r)$.
(2) $\Rightarrow$ (3): Obvious by putting $\lambda=f^{-1}(\mu)$.
(3) $\Rightarrow$ (4): Let μ be an intuitionistic r-fuzzy closed set in Y. By (3), we have $\theta C_{\tau,\tau^\star}(f^{-1}(\mu),r)\le f^{-1}(C_{\sigma,\sigma^\star}(\mu,r))=f^{-1}(\mu)$ which implies that $f^{-1}(\mu)=\theta C_{\tau,\tau^\star}(\mu,r)$. Hence $f^{-1}(\mu)$ is an intuitionistic r-fuzzy θ-closed set in X.
(4) $\Rightarrow$ (5): By taking the complement.
(5) $\Rightarrow$ (1): Let $c(\alpha,\beta)$ be an intuitionistic fuzzy point and $\gamma\in\mathscr{N}_q(f(c(\alpha,\beta)))$. By (5), $f^{-1}(\gamma)$ is an intuitionistic r-fuzzy θ-open set in X. Now we have $f(c(\alpha,\beta))\,q\,\gamma\Rightarrow c(\alpha,\beta)\,q\,f^{-1}(\gamma)\Rightarrow c(\alpha,\beta)\notin f^{-1}(\gamma)$. Hence $f^{-1}(\gamma)$ is an intuitionistic r-fuzzy θ-closed set such that $c(\alpha,\beta)\notin f^{-1}(\gamma)$. Then there exists $\eta\in\mathscr{N}_q(c(\alpha,\beta))$ such that $C_{\tau,\tau^\star}(\eta,r)\,\overline{q}\,f^{-1}(\gamma)$ which implies that $f(C_{\tau,\tau^\star}(\eta,r))\le\gamma$. Hence f is an intuitionistic r-fuzzy strongly θ-continuous.
(3) $\Leftrightarrow$ (6): Let $\mu\in I^Y$. Since f is an intuitionistic r-fuzzy strongly θ-continuous function, $\theta C_{\tau,\tau^\star}(f^{-1}(\overline{1}-\mu),r)\le f^{-1}(C_{\sigma,\sigma^\star}(\overline{1}-\mu),r)$. Thus $f^{-1}(I_{\sigma,\sigma^\star}(\mu,r))=f^{-1}(\overline{1}-(C_{\sigma,\sigma^\star}(\overline{1}-\mu),r))=\overline{1}-f^{-1}(C_{\sigma,\sigma^\star}(\overline{1}-\mu),r))\le\overline{1}-\theta C_{\tau,\tau^\star}(f^{-1}(\overline{1}-\mu),r)=\overline{1}-\theta C_{\tau,\tau^\star}(\overline{1}-f^{-1}(\mu),r)=\theta I_{\tau,\tau^\star}(f^{-1}(\mu),r)$. Conversely, let $\mu\in I^Y$. By the hypothesis, $f^{-1}(I_{\sigma,\sigma^\star}(\overline{1}-\mu,r))\le\theta I_{\tau,\tau^\star}(f^{-1}(\overline{1}-\mu),r)$. Thus $\theta C_{\tau,\tau^\star}(f^{-1}(\mu),r)=\overline{1}-\theta I_{\tau,\tau^\star}(\overline{1}-f^{-1}(\mu),r)=\overline{1}-\theta I_{\tau,\tau^\star}(f^{-1}(\overline{1}-\mu),r)\le\overline{1}-f^{-1}(I_{\sigma,\sigma^\star}(\overline{1}-\mu,r))=f^{-1}(\overline{1}-I_{\sigma,\sigma^\star}(\overline{1}-\mu,r))=f^{-1}(C_{\sigma,\sigma^\star}(\mu,r))$. □

Theorem 4.3. *For a function $f:(X,\tau,\tau^\star)\to(Y,\sigma,\sigma^\star)$, the following statements are equivalent:*

1. *f is an intuitionistic r-fuzzy strongly θ-continuous function.*
2. *$f^{-1}(I_{\sigma,\sigma^\star}(\mu,r))\le\theta I_{\tau,\tau^\star}(f^{-1}(\mu),r)$ for each intuitionistic fuzzy set μ of Y.*
3. *$I_{\sigma,\sigma^\star}(f(\lambda),r)\le f(\theta I_{\sigma,\sigma^\star}(\lambda,r))$ for each intuitionistic fuzzy set λ in X.*

Proof. By Theorem 4.2, it suffices to show that (2) is equivalent to (3).
(2) $\Rightarrow$ (3): Let $\lambda\in I^X$. Then $f(\lambda)\in I^Y$. By the hypothesis, $f^{-1}(I_{\sigma,\sigma^\star}(f(\lambda),r)\le\theta I_{\tau,\tau^\star}(f^{-1}(f(\lambda)),r)$. Since f is one-to-one, we have $f^{-1}(I_{\sigma,\sigma^\star}(f(\lambda),r))\le\theta I_{\tau,\tau^\star}(f^{-1}(f(\lambda)),r)=\theta I_{\tau,\tau^\star}(\lambda,r)$. Since f is onto, $I_{\sigma,\sigma^\star}(f(\lambda),r)=f(f^{-1}(I_{\sigma,\sigma^\star}(f(\lambda),r)))\le f(\theta I_{\tau,\tau^\star}(\lambda,r))$.
(3) $\Rightarrow$ (2): Let $\mu\in I^Y$. Then $f^{-1}(\mu)\in I^X$. By the hypothesis, $I_{\sigma,\sigma^\star}(f(f^{-1}(\mu)),r)\le f(\theta I_{\tau,\tau^\star}(f^{-1}(\mu),r))$. Since f is onto, $I_{\sigma,\sigma^\star}(\mu,r)\le f(\theta I_{\tau,\tau^\star}(f^{-1}(\mu),r))$. Since f is one-to-one, we have $f^{-1}(I_{\sigma,\sigma^\star}(\mu,r))\le f^{-1}(f(\theta I_{\tau,\tau^\star}(f^{-1}(\mu),r))=\theta I_{\tau,\tau^\star}(f^{-1}(\mu),r)$. □

Definition 4.4. *A function $f:(X,\tau,\tau^\star)\to(Y,\sigma,\sigma^\star)$ is said to be an intuitionistic r-fuzzy θ-continuous if and only if for each intuitionistic fuzzy point $x_{(\alpha,\beta)}$ in X and $\mu\in\mathscr{N}_q(f(x_{(\alpha,\beta)})$, there exists $\lambda\in\mathscr{N}_q(x_{(\alpha,\beta)})$ such that $f(C_{\tau,\tau^\star}(\lambda,r))\le C_{\sigma,\sigma^\star}(\mu,r)$.*

Theorem 4.5. *For a function $f:(X,\tau,\tau^\star)\to(Y,\sigma,\sigma^\star)$, the following statements are equivalent:*

1. *f is intuitionistic r-fuzzy θ-continuous.*
2. *$f(\theta C_{\tau,\tau^\star}(\lambda,r))\le\theta C_{\sigma,\sigma^\star}(f(\lambda),r)$ for each intuitionistic fuzzy set λ in X.*
3. *$\theta C_{\tau,\tau^\star}(f^{-1}(\mu),r)\le f^{-1}(\theta C_{\sigma,\sigma^\star}(\mu,r))$ for each intuitionistic fuzzy set μ in Y.*
4. *$\theta C_{\tau,\tau^\star}(f^{-1}(\mu),r)\le f^{-1}(C_{\sigma,\sigma^\star}(\mu,r))$ for each intuitionistic r-fuzzy open set μ in Y.*
5. *$f^{-1}(\theta I_{\sigma,\sigma^\star}(\mu,r))\le\theta I_{\tau,\tau^\star}(f^{-1}(\mu),r)$ for each intuitionistic fuzzy set μ of Y.*

Proof. (1) $\Rightarrow$ (2): Let $c(\alpha,\beta)\in\theta C_{\tau,\tau^\star}(\lambda,r)$ and $\eta\in\mathscr{N}_q(f(c(\alpha,\beta)))$. By (1), there is $\gamma\in\mathscr{N}_q(c(\alpha,\beta))$ such that $f(C_{\tau,\tau^\star}(\gamma,r))\le C_{\sigma,\sigma^\star}(\eta,r)$. Now, if $c(\alpha,\beta)\in\theta C_{\tau,\tau^\star}(\lambda,r)$, then $C_{\tau,\tau^\star}(\gamma,r)\,q\,\lambda$ so that $f(C_{\tau,\tau^\star}(\gamma,r))\,q\,f(\lambda)$ and hence $C_{\sigma,\sigma^\star}(\eta,r)\,q\,f(\lambda)$. Therefore $f(c(\alpha,\beta))\in\theta C_{\sigma,\sigma^\star}(f(\lambda),r)$ and it follows that $c(\alpha,\beta)\in f^{-1}(\theta C_{\sigma,\sigma^\star}(f(\lambda),r))$. Thus $\theta C_{\tau,\tau^\star}(\lambda,r)\le f^{-1}(\theta C_{\sigma,\sigma^\star}(f(\lambda),r))$ and hence $f(\theta C_{\tau,\tau^\star}(\lambda,r)\le\theta C_{\sigma,\sigma^\star}(f(\lambda),r)$.
(2) $\Rightarrow$ (3): By (2), if $f(\theta C_{\tau,\tau^\star}(f^{-1}(\mu),r))\le\theta C_{\sigma,\sigma^\star}(f(f^{-1}(\mu)),r)\le\theta C_{\sigma,\sigma^\star}(\mu,r)$, then it follows that $\theta C_{\tau,\tau^\star}(f^{-1}(\mu),r)\le f^{-1}(\theta C_{\sigma,\sigma^\star}(\mu,r))$.
(3) $\Rightarrow$ (4): Clear by Theorem 3.5.
(4) $\Rightarrow$ (1): Let $c(\alpha,\beta)$ be an intuitionistic fuzzy point in X and $\mu\in\mathscr{N}_q(f(c(\alpha,\beta)))$. Then $f(c(\alpha,\beta))\notin C_{\sigma,\sigma^\star}(C_{\sigma,\sigma^\star}(\mu,r),r)$ and $c(\alpha,\beta)\notin f^{-1}(C_{\tau,\tau^\star}(\mu,r),r))$. By (4), $c(\alpha,\beta)\notin\theta C_{\tau,\tau^\star}(f^{-1}(C_{\sigma,\sigma^\star}(\mu,r)))$ and there exists $\lambda\in\mathscr{N}_q(c(\alpha,\beta))$ such that $C_{\tau,\tau^\star}(\lambda,r)\,\overline{q}\,f^{-1}(C_{\sigma,\sigma^\star}(\mu,r))=f^{-1}(C_{\sigma,\sigma^\star}(\mu,r))$ which implies $f(C_{\tau,\tau^\star}(\lambda,r))\le C_{\sigma,\sigma^\star}(\mu,r)$. Thus f is an intuitionistic r-fuzzy θ-continuous.
(3) $\Leftrightarrow$ (5): Let $\mu\in I^Y$. Since f is intuitionistic r-fuzzy θ-continuous, by hypothesis, $\theta C_{\tau,\tau^\star}(f^{-1}(\overline{1}-\mu),r)\le f^{-1}(\theta C_{\sigma,\sigma^\star}(\overline{1}-\mu,r))$. Thus $f^{-1}(\theta I_{\sigma,\sigma^\star}(\mu,r))=f^{-1}(\overline{1}-\theta C_{\sigma,\sigma^\star}(\overline{1}-\mu,r))=\overline{1}-f^{-1}(\theta C_{\sigma,\sigma^\star}(\overline{1}-\mu,r))\le\overline{1}-\theta C_{\tau,\tau^\star}(f^{-1}(\overline{1}-\mu),r)=\overline{1}-\theta C_{\tau,\tau^\star}(\overline{1}-f^{-1}(\mu),r)=\theta I_{\tau,\tau^\star}(f^{-1}(\mu),r)$. Conversely, let $\mu\in I^Y$. By the hypothesis, $f^{-1}(\theta I_{\sigma,\sigma^\star}(\overline{1}-\mu,r))\le\theta I_{\tau,\tau^\star}(f^{-1}(\overline{1}-\mu),r)$. Thus $\theta C_{\tau,\tau^\star}(f^{-1}(\mu),r)=\overline{1}-\theta I_{\tau,\tau^\star}(\overline{1}-f^{-1}(\mu),r)=\overline{1}-\theta I_{\tau,\tau^\star}(f^{-1}(\overline{1}-\mu),r)\le\overline{1}-f^{-1}(\theta I_{\sigma,\sigma^\star}(\overline{1}-\mu,r))=f^{-1}(\overline{1}-\theta I_{\sigma,\sigma^\star}(\overline{1}-\mu,r))=f^{-1}(\theta C_{\sigma,\sigma^\star}(\mu,r))$. □

Theorem 4.6. *For a bijective function $f:(X,\tau,\tau^\star)\to(Y,\sigma,\sigma^\star)$, the following statements are equivalent:*

1. *f is intuitionistic r-fuzzy θ-continuous.*
2. *$f^{-1}(\theta I_{\sigma,\sigma^\star}(\mu,r))\le\theta I_{\tau,\tau^\star}(f^{-1}(\mu),r)$ for each intuitionistic fuzzy set μ of Y.*

3. $\theta I_{\sigma,\sigma^\star}(f(\lambda),r) \le f(\theta I_{\tau,\tau^\star}(\lambda,r))$ *for each intuitionistic fuzzy set* λ *in* X.

Proof. By Theorem 4.5, it suffices to show that (2) isequivalent to (3).
(2) ⇒ (3): Let $\lambda \in I^X$. Then $f(\lambda) \in I^Y$. By the hypothesis, $f^{-1}(\theta I_{\sigma,\sigma^\star}(f(\lambda),r)) \le \theta I_{\tau,\tau^\star}(f^{-1}(f(\lambda)),r)$. Since f is one-to-one, we have $f^{-1}(\theta I_{\sigma,\sigma^\star}(f(\lambda),r)) \le \theta I_{\tau,\tau^\star}(f^{-1}(f(\lambda)),r) = \theta I_{\tau,\tau^\star}(\lambda,r)$. Since f is onto, $\theta I_{\sigma,\sigma^\star}(f(\lambda),r) = f(f^{-1}(I_{\sigma,\sigma^\star}(f(\lambda),r))) \le f(I_{\tau,\tau^\star}(\lambda,r))$.
(3) ⇒ (2): Let μ be an intuitionistic fuzzy set in Y. Then $f^{-1}(\mu)$ is an intuitionistic fuzzy set in X. By the hypothesis, $\theta I_{\sigma,\sigma^\star}(f(f^{-1}(\mu)),r) \le f(\theta I_{\tau,\tau^\star}(f^{-1}(\mu),r))$. Since f is onto, $\theta I_{\sigma,\sigma^\star}(\mu,r) = \theta I_{\sigma,\sigma^\star}(f(f^{-1}(\mu)),r) \le f(\theta I_{\tau,\tau^\star}(f^{-1}(\mu),r))$. Since f is one-to-one, we have $f^{-1}(\theta I_{\sigma,\sigma^\star}(\mu,r)) \le f^{-1}(f(\theta I_{\tau,\tau^\star}(f^{-1}(\mu),r))) = \theta I_{\tau,\tau^\star}(f^{-1}(\mu),r)$. □

Theorem 4.7. *Let* $f:(X,\tau,\tau^\star) \to (Y,\sigma,\sigma^\star)$ *be a function. If* $(X,\tau,\tau^\star)$ *is an intuitionistic fuzzy extremally disconnected space, then the following are equivalent:*

1. *f is an intuitionistic r-fuzzy θ-continuous.*
2. *$f^{-1}(\mu)$ is an intuitionistic r-fuzzy θ-closed set in X for each intuitionistic r-fuzzy θ-closed set μ in Y.*
3. *$f^{-1}(\mu)$ is an intuitionistic r-fuzzy θ-open set in X for each intuitionistic r-fuzzy θ-open set μ in Y.*

Proof. (1) ⇒ (2): Let μ be an intuitionistic r-fuzzy θ-closed set in Y. Since f is an intuitionistic r-fuzzy θ-continuous, then by (3) in Theorem 4.5, we have $\theta C_{\tau,\tau^\star}(f^{-1}(\mu),r) \le f^{-1}(\theta C_{\sigma,\sigma^\star}(\mu,r)) = f^{-1}(\mu)$ which implies that $f^{-1}(\mu) = \theta C_{\tau,\tau^\star}(f^{-1}(\mu),r)$. Hence $f^{-1}(\mu)$ is an intuitionistic r-fuzzy θ-closed set in X.
(2) ⇔ (3): Obvious.
(2) ⇒ (1): Let μ be an intuitionistic r-fuzzy open set in Y. Then by Theorem 3.5 $C_{\sigma,\sigma^\star}(\mu,r) = \theta C_{\sigma,\sigma^\star}(\mu,r)$, which is an intuitionistic r-fuzzy θ-closed set. From (2), $f^{-1}(C_{\sigma,\sigma^\star}(\mu,r)) = f^{-1}(\theta C_{\sigma,\sigma^\star}(\mu,r))$ is an intuitionistic r-fuzzy θ-closed set in X. Since $f^{-1}(\mu) \le f^{-1}(C_{\sigma,\sigma^\star}(\mu,r))$, then $\theta C_{\tau,\tau^\star}(f^{-1}(\mu),r) \le f^{-1}(C_{\sigma,\sigma^\star}(\mu,r))$. Hence f is an intuitionistic r-fuzzy θ-continuous. □

Definition 4.8. *A function* $f:(X,\tau,\tau^\star) \to (Y,\sigma,\sigma^\star)$ *is said to be an intuitionistic r-fuzzy weakly continuous if for each intuitionistic r-fuzzy open set* μ *in* Y, $f^{-1}(\mu) \le I_{\tau,\tau^\star}(f^{-1}(C_{\sigma,\sigma^\star}(\mu,r)))$.

Proposition 4.9.
1. *Every intuitionistic r-fuzzy strongly θ-continuous function is intuitionistic fuzzy continuous.*
2. *Every intuitionistic fuzzy continuous function is intuitionistic r-fuzzy weakly continuous.*
3. *Every intuitionistic fuzzy continuous function is intuitionistic r-fuzzy θ-continuous.*

The following example shows that the converses of the above proposition are not true.

Example 4.10. *Let* $X = \{a,b,c\}$. *Define the intuitionistic fuzzy subsets* $\lambda_1,\lambda_2,\lambda_3$ *and* λ_4 *as* $\lambda_1 = \langle x, \left(\frac{a}{0.4},\frac{b}{0.5},\frac{c}{0.5}\right), \left(\frac{a}{0.3},\frac{b}{0.4},\frac{c}{0.4}\right)\rangle$, $\lambda_2 = \langle x, \left(\frac{a}{0.5},\frac{b}{0.5},\frac{c}{0.5}\right), \left(\frac{a}{0.2},\frac{b}{0.3},\frac{c}{0.1}\right)\rangle$, $\lambda_3 = \langle x, \left(\frac{a}{0.5},\frac{b}{0.4},\frac{c}{0.5}\right), \left(\frac{a}{0.4},\frac{b}{0.4},\frac{c}{0.3}\right)\rangle$ *and* $\lambda_4 = \langle x, \left(\frac{a}{0.4},\frac{b}{0.2},\frac{c}{0.4}\right), \left(\frac{a}{0.5},\frac{b}{0.4},\frac{c}{0.5}\right)\rangle$.
Let $\tau,\tau^\star,\sigma,\sigma^\star : I^X \to I$ *defined as follows:*

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0} \text{ or } \bar{1} \\ \frac{1}{4} & \text{if } \lambda = \lambda_1 \\ \frac{1}{2} & \text{if } \lambda = \lambda_2 \\ 0 & \text{otherwise,} \end{cases} \qquad \tau^\star(\lambda) = \begin{cases} 0 & \text{if } \lambda = \bar{0} \text{ or } \bar{1} \\ \frac{3}{4} & \text{if } \lambda = \lambda_1 \\ \frac{1}{2} & \text{if } \lambda = \lambda_2 \\ 1 & \text{otherwise,} \end{cases}$$

$$\sigma(\lambda) = \begin{cases} 1 & \text{if } \lambda = \bar{0} \text{ or } \bar{1} \\ \frac{1}{4} & \text{if } \lambda = \lambda_3 \\ \frac{1}{2} & \text{if } \lambda = \lambda_4 \\ 0 & \text{otherwise,} \end{cases} \qquad \sigma^\star(\lambda) = \begin{cases} 0 & \text{if } \lambda = \bar{0} \text{ or } \bar{1} \\ \frac{3}{4} & \text{if } \lambda = \lambda_3 \\ \frac{1}{2} & \text{if } \lambda = \lambda_4 \\ 1 & \text{otherwise.} \end{cases}$$

Let $r = \frac{1}{2}$. *Define a function* $f:(X,\tau,\tau^\star) \to (Y,\sigma,\sigma^\star)$ *by* $f(a) = b$, $f(b) = c$ *and* $f(c) = a$. *Then* $f^{-1}(\lambda_3) \le I_{\tau,\tau^\star}(f^{-1}(C_{\tau,\tau^\star}(\lambda_3,\frac{1}{2})),\frac{1}{2}) = 1$ *and* $f^{-1}(\lambda_4) \le I_{\tau,\tau^\star}(f^{-1}(C_{\tau,\tau^\star}(\lambda_4,\frac{1}{2})),\frac{1}{2}) = \lambda_1$. *Thus* f *is intuitionistic* $\frac{1}{2}$*-fuzzy weakly continuous but not intuitionistic* $\frac{1}{2}$*-fuzzy continuous. From this example, one can show that intuitionistic* $\frac{1}{2}$*-fuzzy weakly continuous does not implies each of the concepts intuitionistic* $\frac{1}{2}$*-fuzzy strongly* θ*-continuous and intuitionistic* $\frac{1}{2}$*-fuzzy* θ*-continuous.*

Theorem 4.11. *For a function* $f:(X,\tau,\tau^\star) \to (Y,\sigma,\sigma^\star)$, *the following statements are equivalent:*

1. *f is an intuitionistic r-fuzzy weakly continuous function.*
2. *$f(C_{\tau,\tau^\star}(\lambda,r))) \le \theta C_{\sigma,\sigma^\star}(f(\lambda),r)$ for each intuitionistic fuzzy set λ in X.*
3. *$C_{\tau,\tau^\star}(f^{-1}(\mu),r) \le f^{-1}(\theta C_{\sigma,\sigma^\star}(\mu,r))$ for each intuitionistic fuzzy set μ in Y.*
4. *$C_{\tau,\tau^\star}(f^{-1}(\mu),r) \le f^{-1}(C_{\sigma,\sigma^\star}(\mu,r))$ for each intuitionistic r-fuzzy open set μ of Y.*
5. *$f^{-1}(\theta I_{\sigma,\sigma^\star}(\mu,r)) \le I_{\tau,\tau^\star}(f^{-1}(\mu),r)$ for each intuitionistic fuzzy set μ of Y.*

Proof. (1) ⇒ (2): Let f be intuitionistic r-fuzzy weakly continuous and $\lambda \in I^X$. Suppose $c(\alpha,\beta) \in C_{\tau,\tau^\star}(\lambda,r)$, then $f(c(\alpha,\beta)) \in f(C_{\tau,\tau^\star}(\lambda,r))$. It is enough to show that $f(c(\alpha,\beta)) \in \theta C_{\sigma,\sigma^\star}(f(\lambda))$. Let $\gamma \in \mathcal{N}_q(f(c(\alpha,\beta)))$. Then we have $f^{-1}(\gamma)\ q\ c(\alpha,\beta)$. By intuitionistic r-fuzzy weakly continuous of f, $f^{-1}(\gamma) \le I_{\tau,\tau^\star}(f^{-1}(C_{\sigma,\sigma^\star}(\gamma,r)),r)$ and $I_{\tau,\tau^\star}(f^{-1}(C_{\sigma,\sigma^\star}(\gamma,r)),r) \in \mathcal{N}_q(c(\alpha,\beta))$. Since $c(\alpha,\beta) \in C_{\tau,\tau^\star}(\lambda,r)$, $I_{\tau,\tau^\star}(f^{-1}(C_{\tau,\tau^\star}(\gamma,r)),r)\ q\ \lambda$ and hence $C_{\sigma,\sigma^\star}(\gamma)\ q\ f(\lambda)$. Thus $f(c(\alpha,\beta)) \in \theta C_{\sigma,\sigma^\star}(f(\lambda),r)$.
(2) ⇒ (3): Let $\mu \in I^Y$. By (2), we have $f(C_{\tau,\tau^\star}(f^{-1}(\mu),r)) \le \theta C_{\sigma,\sigma^\star}(f(f^{-1}(\mu)),r) \le \theta C_{\sigma,\sigma^\star}(\mu,r)$. Hence $C_{\tau,\tau^\star}(f^{-1}(\mu),r) \le f^{-1}(\theta C_{\sigma,\sigma^\star}(\mu,r))$.

(3) ⇒ (4): Let μ be an intuitionistic r-fuzzy open set in Y. By Theorem 3.5, $C_{\sigma,\sigma^\star}(\mu,r) = \theta C_{\sigma,\sigma^\star}(\mu,r)$ and by (3), we have $C_{\tau,\tau^\star}(f^{-1}(\mu),r) \leq f^{-1}(C_{\sigma,\sigma^\star}(\mu,r))$.
(4) ⇒ (1): Let μ be an intuitionistic r-fuzzy open set in Y, and $C_{\tau,\tau^\star}(f^{-1}(\mu),r) \leq f^{-1}(C_{\sigma,\sigma^\star}(\mu,r))$. Then from $f^{-1}(\mu) \leq C_{\tau,\tau^\star}(f^{-1}(\mu),r)$ and the fact that μ be an intuitionistic r-fuzzy open set it follows that $f^{-1}(\mu) = I_{\tau,\tau^\star}(f^{-1}(\mu),r) \leq I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(f^{-1}(\mu),r),r) \leq I_{\tau,\tau^\star}(f^{-1}(C_{\sigma,\sigma^\star}(\mu,r)),r)$. Hence f is an intuitionistic r-fuzzy weakly continuous.
(3) ⇔ (5): Let $\mu \in I^Y$. Since f is an intuitionistic r-fuzzy weakly continuous function, by the hypothesis, $C_{\tau,\tau^\star}(f^{-1}(\overline{1}-\mu),r) \leq f^{-1}(\theta C_{\sigma,\sigma^\star}(\overline{1}-\mu,r))$. Thus $f^{-1}(\theta I_{\sigma,\sigma^\star}(\mu,r)) = f^{-1}(\overline{1}-\theta C_{\sigma,\sigma^\star}(\overline{1}-\mu,r)) = \overline{1}-f^{-1}(\theta C_{\sigma,\sigma^\star}(\overline{1}-\mu,r)) \leq \overline{1}-C_{\tau,\tau^\star}(f^{-1}(\overline{1}-\mu),r) = \overline{1}-C_{\tau,\tau^\star}(\overline{1}-f^{-1}(\mu),r) = I_{\tau,\tau^\star}(f^{-1}(\mu),r)$. Conversely, let $\mu \in I^Y$. By the hypothesis, $f^{-1}(\theta I_{\sigma,\sigma^\star}(\overline{1}-\mu)) \leq int(f^{-1}(\overline{1}-\mu))$. Thus $C_{\tau,\tau^\star}(f^{-1}(\mu)) = \overline{1}-I_{\tau,\tau^\star}(\overline{1}-f^{-1}(\mu),r) = \overline{1}-I_{\tau,\tau^\star}(f^{-1}(\overline{1}-\mu),r) \leq \overline{1}-f^{-1}(\theta I_{\sigma,\sigma^\star}(\overline{1}-\mu),r) = f^{-1}(\overline{1}-\theta I_{\sigma,\sigma^\star}(\overline{1}-\mu,r)) = f^{-1}(\theta C_{\sigma,\sigma^\star}(\mu,r))$. □

Theorem 4.12. *For a bijective function $f:(X,\tau,\tau^\star) \to (Y,\sigma,\sigma^\star)$, the following statements are equivalent:*

1. *f is an intuitionistic r-fuzzy weakly continuous function.*
2. *$f^{-1}(\theta I_{\sigma,\sigma^\star}(\mu,r)) \leq I_{\tau,\tau^\star}(f^{-1}(\mu),r)$ for each intuitionistic fuzzy set μ of Y.*
3. *$\theta I_{\sigma,\sigma^\star}(f(\lambda),r) \leq f(I_{\tau,\tau^\star}(\lambda,r))$ for each intuitionistic fuzzy set λ in X.*

Proof. By Theorem 4.11, it suffices to show that (2) is equivalent to (3).
(2) ⇒ (3): Let λ be an intuitionistic fuzzy set in X. Then $f(\lambda)$ is an intuitionistic fuzzy set in Y. By the hypothesis, $f^{-1}(\theta I_{\sigma,\sigma^\star}(f(\lambda),r)) \leq I_{\tau,\tau^\star}(f^{-1}(f(\lambda)),r)$. Since f is one-to-one, we have $f^{-1}(\theta I_{\sigma,\sigma^\star}(f(\lambda),r)) \leq I_{\tau,\tau^\star}(f^{-1}(f(\lambda)),r) = I_{\tau,\tau^\star}(\lambda,r)$. Since f is onto, $\theta I_{\sigma,\sigma^\star}(f(\lambda),r) = f(f^{-1}(I_{\tau,\tau^\star}(f(\lambda),r))) \leq f(I_{\tau,\tau^\star}(\lambda),r)$.
(3) ⇒ (2): Let μ be an intuitionistic fuzzy set in Y. Then $f^{-1}(\mu)$ is an intuitionistic fuzzy set in X. By the hypothesis, $\theta I_{\sigma,\sigma^\star}(f(f^{-1}(\mu)),r) \leq f(I_{\tau,\tau^\star}(f^{-1}(\mu),r))$. Since f is onto, $\theta I_{\sigma,\sigma^\star}(\mu,r) \leq f(I_{\tau,\tau^\star}(f^{-1}(\mu),r))$. Since f is one-to-one, $f^{-1}(\theta I_{\sigma,\sigma^\star}(\mu,r)) \leq f^{-1}(f(I_{\tau,\tau^\star}(f^{-1}(\mu)),r)) = I_{\tau,\tau^\star}(f^{-1}(\mu),r)$. □

Theorem 4.13. *Let $f:(X,\tau,\tau^\star) \to (Y,\sigma,\sigma^\star)$ be an intuitionistic r-fuzzy weakly continuous function, then*

1. *$f^{-1}(\mu)$ is an intuitionistic r-fuzzy closed set in X for each intuitionistic r-fuzzy θ-closed set μ in Y.*
2. *$f^{-1}(\mu)$ is an intuitionistic r-fuzzy open set in X for each intuitionistic r-fuzzy θ-open set μ in Y.*

Proof. (1). Let μ be an intuitionistic r-fuzzy θ-closed set in Y, then $\mu = \theta C_{\sigma,\sigma^\star}(\mu,r)$. By Theorem 4.11 (3), we have $C_{\tau,\tau^\star}(f^{-1}(\mu),r) \leq f^{-1}(\theta C_{\sigma,\sigma^\star}(\mu,r)) = f^{-1}(\mu)$. Hence $f^{-1}(\mu)$ is an intuitionistic r-fuzzy closed set in X.
(2). Obvious. □

5. Intuitionistic Fuzzy δ-closure and δ-interior

Definition 5.1. *Let $(X,\tau,\tau^\star)$ be an intuitionistic smooth fuzzy topological space. An intuitionistic fuzzy point $x(\alpha,\beta)$ is said to be an intuitionistic fuzzy δ-cluster point of an intuitionistic fuzzy set λ if $\mu\ q\ \lambda$ for each intuitionistic r-fuzzy regular open q-neighborhood μ of $x(\alpha,\beta)$. The set of all intuitionistic fuzzy δ-cluster points of λ is called the intuitionistic fuzzy δ-closure of λ and denoted by $\delta C_{\tau,\tau^\star}(\lambda,r)$. An intuitionistic fuzzy set λ is said to be an intuitionistic r-fuzzy δ-closed set if $\lambda = \delta C_{\tau,\tau^\star}(\lambda,r)$. The complement of an intuitionistic r-fuzzy δ-closed set is said to be an intuitionistic r-fuzzy δ-open set.*

Definition 5.2. *Let $(X,\tau,\tau^\star)$ be an intuitionistic smooth fuzzy topological space, and let λ be an intuitionistic fuzzy set in X. The intuitionistic fuzzy δ-interior of λ is denoted and defined by $\delta I_{\tau,\tau^\star}(\lambda,r) = \overline{1}-\delta C_{\tau,\tau^\star}(\overline{1}-\lambda,r)$.*

From the above definition, we have the following relations:

1. $\delta C_{\tau,\tau^\star}(\overline{1}-\lambda,r) = \overline{1}-\delta I_{\tau,\tau^\star}(\lambda,r)$,
2. $\overline{1}-\delta C_{\tau,\tau^\star}(\lambda,r) = \delta I_{\tau,\tau^\star}(\overline{1}-\lambda,r)$.

Remark 5.3. *For any intuitionistic fuzzy set λ, λ is an intuitionistic r-fuzzy δ-open set if and only if $\delta I_{\tau,\tau^\star}(\lambda,r) = \lambda$ because λ is intuitionistic r-fuzzy δ-open if and only if $\overline{1}-\lambda$ is intuitionistic r-fuzzy δ-closed if and only if $\overline{1}-\lambda = \delta C_{\tau,\tau^\star}(\overline{1}-\lambda,r)$ if and only if $\lambda = \overline{1}-\delta C_{\tau,\tau^\star}(\overline{1}-\lambda,r) = \delta I_{\tau,\tau^\star}(\lambda,r)$.*

Lemma 5.4.
1. *For any intuitionistic fuzzy set λ in an intuitionistic smooth fuzzy topological space $(X,\tau,\tau^\star)$, $I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r)$ is an intuitionistic r-fuzzy regular open set.*
2. *For any intuitionistic r-fuzzy open set λ in an intuitionistic smooth fuzzy topological space $(X,\tau,\tau^\star)$ such that $x(\alpha,\beta)\ q\ \lambda$, $I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r)$ is an intuitionistic r-fuzzy regular open q-neighborhood of $x(\alpha,\beta)$.*

Proof. (1). Since $I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r) \leq C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r),r)$, we have $I_{\tau,\tau^\star}(I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r),r) \leq I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r),r),r)$. Thus $I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r) \leq I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r),r),r)$. Conversely, since $I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r) \leq C_{\tau,\tau^\star}(\lambda,r)$, $C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r),r) \leq C_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r) = C_{\tau,\tau^\star}(\lambda,r)$. Thus $I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r),r),r) \leq I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r)$. Hence $I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r)$ is an intuitionistic r-fuzzy regular open set.
(2) Clearly, $I_{\tau,\tau^\star}(\lambda,r) \leq I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r)$. Since λ is an intuitionistic r-fuzzy open set, we have $\lambda = I_{\tau,\tau^\star}(\lambda,r) \leq I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r)$. By (1), $I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r)$ is an intuitionistic r-fuzzy regular open set. Therefore $I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r)$ is an intuitionistic r-fuzzy regular open q-neighborhood of $x(\alpha,\beta)$. □

Theorem 5.5. *For any intuitionistic fuzzy set λ in an intuitionistic smooth fuzzy topological space $(X,\tau,\tau^\star)$, $C_{\tau,\tau^\star}(\lambda,r) \leq \delta C_{\tau,\tau^\star}(\lambda,r) \leq \theta C_{\tau,\tau^\star}(\lambda,r)$.*

Proof. Let $x(\alpha,\beta)\notin \delta C_{\tau,\tau^\star}(\lambda,r)$. Then there exists an intuitionistic r-fuzzy regular open q-neighborhood μ of $x(\alpha,\beta)$ such that $\mu\,\bar{q}\,\lambda$. Then μ is an intuitionistic fuzzy q-neighborhood of $x(\alpha,\beta)$ such that $\mu\,\bar{q}\,\lambda$. Then $x(\alpha,\beta)\notin C_{\tau,\tau^\star}(\lambda,r)$. Thus $C_{\tau,\tau^\star}(\lambda,r)\le \delta C_{\tau,\tau^\star}(\lambda,r)$. Let $x(\alpha,\beta)\in \delta C_{\tau,\tau^\star}(\lambda,r)$. Then for each intuitionistic r-fuzzy regular open q-neighborhood μ of $x(\alpha,\beta)$, $\mu\, q\,\lambda$. Suppose that there exists an intuitionistic r-fuzzy open q-neighborhood η of $x(\alpha,\beta)$ such that $C_{\tau,\tau^\star}(\eta,r)\,\bar{q}\,\lambda$. Put $I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\eta,r),r)=\gamma$. Then γ is an intuitionistic r-fuzzy regular open q-neighborhood of $x(\alpha,\beta)$. Since $\gamma=I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\eta,r),r)\le C_{\tau,\tau^\star}(\eta,r)\,\bar{q}\,\lambda$, $\gamma=I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\eta,r),r)\le C_{\tau,\tau^\star}(\eta,r)\le \bar{1}-\lambda$. γ is an intuitionistic r-fuzzy regular open q-neighborhood of $x(\alpha,\beta)$ such that $\gamma\,\bar{q}\,\lambda$. This is a contradiction. Therefore, for any intuitionistic r-fuzzy open q-neighborhood η of $x(\alpha,\beta)$, $C_{\tau,\tau^\star}(\eta,r)\; q\;\lambda$. Hence $x(a,b)\in \theta C_{\tau,\tau^\star}(\lambda,r)$. □

Corollary 5.6.
1. *If λ is an intuitionistic r-fuzzy δ-closed set in an intuitionistic smooth fuzzy topological space $(X,\tau,\tau^\star)$, then λ is an intuitionistic r-fuzzy closed set.*
2. *If λ is an intuitionistic r-fuzzy θ-closed set in an intuitionistic smooth fuzzy topological space $(X,\tau,\tau^\star)$, then λ is an intuitionistic r-fuzzy δ-closed set.*

Remark 5.7. *The following examples show that the converses of Corollary 5.6 do not hold.*

Example 5.8. *Let $X=\{a,b\}$. Define the intuitionistic fuzzy subset λ as $\lambda=\langle x,\left(\frac{a}{0.5},\frac{b}{0.3}\right),\left(\frac{a}{0.3},\frac{b}{0.5}\right)\rangle$. Let $\tau,\tau^\star:I^X\to I$ defined as follows:*

$$\tau(\mu)=\begin{cases}1 & if\ \mu=\bar{0}\ or\ \bar{1}\\ \frac{1}{2} & if\ \mu=\lambda\\ 0 & otherwise,\end{cases}\qquad \tau^\star(\mu)=\begin{cases}0 & if\ \mu=\bar{0}\ or\ \bar{1}\\ \frac{1}{2} & if\ \mu=\lambda\\ 1 & otherwise.\end{cases}$$

Clearly, $\bar{1}-\mu$ is an intuitionistic r-fuzzy closed set. Since $C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(\bar{1}-\mu,\frac{1}{2}),\frac{1}{2})=C_{\tau,\tau^\star}(0,\frac{1}{2})=0\neq\bar{1}-\mu$, $\bar{1}-\mu$ is not an intuitionistic r-fuzzy regular closed set. Hence 0 and 1 are the only intuitionistic r-fuzzy regular closed sets. Thus $\delta C_{\tau,\tau^\star}(\bar{1}-\mu,\frac{1}{2})=\bigwedge\{\gamma:\bar{1}-\mu\le\gamma,\ \gamma$ is regular closed $\}=1\neq\bar{1}-\mu$. Hence $\bar{1}-\mu$ is not intuitionistic r-fuzzy δ-closed. Therefore, $\bar{1}-\mu$ is an intuitionistic r-fuzzy closed set which is not intuitionistic r-fuzzy δ-closed.

Example 5.9. *Let $X=\{a,b\}$. Define the intuitionistic fuzzy subset λ as $\lambda=\langle x,\left(\frac{a}{0.5},\frac{b}{0.3}\right),\left(\frac{a}{0.5},\frac{b}{0.4}\right)\rangle$. Let $\tau,\tau^\star:I^X\to I$ defined as follows:*

$$\tau(\mu)=\begin{cases}1 & if\ \mu=\bar{0}\ or\ \bar{1}\\ \frac{1}{2} & if\ \mu=\lambda\\ 0 & otherwise,\end{cases}\qquad \tau^\star(\mu)=\begin{cases}0 & if\ \mu=\bar{0}\ or\ \bar{1}\\ \frac{1}{2} & if\ \mu=\lambda\\ 1 & otherwise.\end{cases}$$

Since $I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\mu,\frac{1}{2}),\frac{1}{2})=I_{\tau,\tau^\star}(\bar{1}-\mu,\frac{1}{2})=\mu$, μ is an intuitionistic r-fuzzy regular open set. Thus $\bar{1}-\mu$ is an intuitionistic r-fuzzy regular closed set, and consequently $\delta C_{\tau,\tau^\star}(\bar{1}-\mu,\frac{1}{2})=\bigwedge\{\gamma:\bar{1}-\mu\le\gamma,\ \gamma$ is regular closed $\}=\bar{1}-\mu$. Hence $\bar{1}-\mu$ is an intuitionistic r-fuzzy δ-closed set. But $\theta C_{\tau,\tau^\star}(\bar{1}-\mu,\frac{1}{2})=\bigwedge\{C_{\tau,\tau^\star}(\gamma,\frac{1}{2}):\gamma\in\tau,\bar{1}-\mu\le\gamma\}=\bar{1}\neq\bar{1}-\mu$, and hence $\bar{1}-\mu$ is not intuitionistic r-fuzzy θ-closed. Therefore, $\bar{1}-\mu$ is an intuitionistic r-fuzzy δ-closed set which is not intuitionistic r-fuzzy θ-closed.

Theorem 5.10. *If λ is an intuitionistic r-fuzzy open set in an intuitionistic smooth fuzzy topological space $(X,\tau,\tau^\star)$, then the intuitionistic fuzzy closure and intuitionistic fuzzy δ-closure are the same, i.e. $C_{\tau,\tau^\star}(\lambda,r)=\delta C_{\tau,\tau^\star}(\lambda,r)$.*

Proof. By Theorem 5.5, it is sufficient to show that $\delta C_{\tau,\tau^\star}(\lambda,r)\le C_{\tau,\tau^\star}(\lambda,r)$. Take any $x(\alpha,\beta)\in\delta C_{\tau,\tau^\star}(\lambda,r)$. Suppose that $x(\alpha,\beta)\notin C_{\tau,\tau^\star}(\lambda,r)$. Then there exists an intuitionistic fuzzy q-neighborhood γ of $x(\alpha,\beta)$ such that $\gamma\,\bar{q}\,\lambda$. Since $\gamma\,\bar{q}\,\lambda$, we have $\gamma\le\bar{1}-\lambda$. Since $\bar{1}-\lambda$ is an intuitionistic r-fuzzy closed set, $C_{\tau,\tau^\star}(\gamma,r)\le C_{\tau,\tau^\star}(\bar{1}-\lambda,r)=\bar{1}-\lambda$. Therefore, $I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\gamma,r),r)\le I_{\tau,\tau^\star}(\bar{1}-\lambda,r)\le\bar{1}-\lambda$, that is, $I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\gamma,r),r)\,\bar{q}\,\lambda$. By Lemma 5.4, $I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r)$ is an intuitionistic r-fuzzy regular open q-neighborhood of $x(\alpha,\beta)$ such that $I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r)\,\bar{q}\,\lambda$. Hence $x(\alpha,\beta)\notin\delta C_{\tau,\tau^\star}(\lambda,r)$. □

In fact, the intuitionistic fuzzy closure and the intuitionistic fuzzy δ-closure are the same for any intuitionistic fuzzy semi-open set as follows.

Theorem 5.11. *For any intuitionistic r-fuzzy semi-open set λ, $C_{\tau,\tau^\star}(\lambda,r)=\delta C_{\tau,\tau^\star}(\lambda,r)$.*

Proof. Enough to show that $\delta C_{\tau,\tau^\star}(\lambda,r)\le C_{\tau,\tau^\star}(\lambda,r)$. Take any $x(\alpha,\beta)\in\delta C_{\tau,\tau^\star}(\lambda,r)$. Suppose that $x(\alpha,\beta)\notin C_{\tau,\tau^\star}(\lambda,r)$. Then there exists an intuitionistic r-fuzzy open q-neighborhood μ of $x(\alpha,\beta)$ such that $\mu\,\bar{q}\,\lambda$. By definition of intuitionistic r-fuzzy semi-open set, there exists an intuitionistic r-fuzzy open set γ such that $\gamma\le\lambda\le C_{\tau,\tau^\star}(\gamma,r)$. Thus $\mu\le\bar{1}-\lambda\le\bar{1}-\gamma$. Hence $C_{\tau,\tau^\star}(\mu,r)\le C_{\tau,\tau^\star}(\bar{1}-\lambda,r)\le C_{\tau,\tau^\star}(\bar{1}-\gamma,r)=\bar{1}-\gamma$. Also $I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\mu,r),r)\le I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\bar{1}-\lambda,r),r)\le I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\bar{1}-\gamma,r),r)=I_{\tau,\tau^\star}(\bar{1}-\gamma,r)\le\bar{1}-\gamma$, i.e. $I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\mu,r),r)\le\bar{1}-\gamma$. Therefore $\gamma\le\bar{1}-I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\mu,r),r)$. Hence $\lambda\le C_{\tau,\tau^\star}(\gamma,r)\le C_{\tau,\tau^\star}(\bar{1}-(I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\mu,r),r),r))=\bar{1}-I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\mu,r),r)$ because $\bar{1}-I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\mu,r),r)$ is an intuitionistic r-fuzzy closed set. Then we have $I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\mu,r),r)\,\bar{q}\,\lambda$. By Lemma 5.4, $I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\mu,r),r)$ is an intuitionistic r-fuzzy regular open q-neighborhood of $x(\alpha,\beta)$ such that $I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\mu,r),r)\,\bar{q}\,\lambda$. Hence $x(\alpha,\beta)\notin\delta C_{\tau,\tau^\star}(\lambda,r)$. □

Theorem 5.12. *If λ is intuitionistic r-fuzzy preopen, then $C_{\tau,\tau^\star}(\lambda,r)=\delta C_{\tau,\tau^\star}(\lambda,r)=\theta C_{\tau,\tau^\star}(\lambda,r)$.*

Proof. Let λ be intuitionistic r-fuzzy preopen set. We only show that $\theta C_{\tau,\tau^\star}(\lambda,r)\le C_{\tau,\tau^\star}(\lambda,r)$. Since $\lambda\le I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\lambda,r),r)$, $\theta C_{\tau,\tau^\star}(\lambda,r)\le C_{\tau,\tau^\star}(\lambda,r)$. □

Corollary 5.13. *If λ is intuitionistic r-fuzzy preopen and intuitionistic r-fuzzy regular closed, then $\theta C_{\tau,\tau^\star}(\lambda,r)=\lambda$.*

Proof. Clear. □

Theorem 5.14. *Let λ and μ be two intuitionistic fuzzy sets in an intuitionistic smooth fuzzy topological space $(X,\tau,\tau^\star)$. Then we have the following properties:*

1. $\delta C_{\tau,\tau^\star}(\overline{0},r)=\overline{0}$,
2. $\lambda\leq\delta C_{\tau,\tau^\star}(\lambda,r)$,
3. $\lambda\leq\mu\Rightarrow\delta C_{\tau,\tau^\star}(\lambda,r)\leq\delta C_{\tau,\tau^\star}(\mu,r)$,
4. $\delta C_{\tau,\tau^\star}(\lambda,r)\vee\delta C_{\tau,\tau^\star}(\mu,r)=\delta C_{\tau,\tau^\star}(\lambda\vee\mu,r)$,
5. $\delta C_{\tau,\tau^\star}(\lambda\wedge\mu,r)\leq\delta C_{\tau,\tau^\star}(\lambda,r)\wedge\delta C_{\tau,\tau^\star}(\mu,r)$.

Proof. (1). Obvious.
(2). Since $\lambda\leq C_{\tau,\tau^\star}(\lambda,r)\leq\delta C_{\tau,\tau^\star}(\lambda,r)$, $\lambda\leq\delta C_{\tau,\tau^\star}(\lambda,r)$. (3). Let $x(\alpha,\beta)$ be an intuitionistic fuzzy point in X such that $x(\alpha,\beta)\notin\delta C_{\tau,\tau^\star}(\mu,r)$. Then there is an intuitionistic r-fuzzy regular open q-neighborhood γ of $x(\alpha,\beta)$ such that $\gamma\,\overline{q}\,\mu$. Since $\lambda\leq\mu$, we have $\gamma\,\overline{q}\,\lambda$. Therefore $x(\alpha,\beta)\notin\delta C_{\tau,\tau^\star}(\lambda,r)$.
(4). Since $\lambda\leq\lambda\vee\mu$, $\delta C_{\tau,\tau^\star}(\lambda,r)\leq\delta C_{\tau,\tau^\star}(\lambda\vee\mu,r)$. Similarly, $\delta C_{\tau,\tau^\star}(\mu,r)\leq\delta C_{\tau,\tau^\star}(\lambda\vee\mu,r)$. Hence $\delta C_{\tau,\tau^\star}(\lambda,r,a)\vee\delta C_{\tau,\tau^\star}(\mu,r)\leq\delta C_{\tau,\tau^\star}(\lambda\vee\mu,r)$. To show that $\delta C_{\tau,\tau^\star}(\lambda\vee\mu,r)\leq\delta C_{\tau,\tau^\star}(\lambda,r)\vee\delta C_{\tau,\tau^\star}(\mu,r)$, take any $x(\alpha,\beta)\in\delta C_{\tau,\tau^\star}(\lambda\vee\mu,r)$. Then for any intuitionistic r-fuzzy regular open q-neighborhood γ of $x(\alpha,\beta)$, $\gamma\;q\;(\lambda\vee\mu)$. Hence $\gamma\;q\;\lambda$ or $\gamma\;q\;\mu$. Therefore $x(\alpha,\beta)\in\delta C_{\tau,\tau^\star}(\lambda,r)$ or $x(\alpha,\beta)\in\delta C_{\tau,\tau^\star}(\mu,r)$. Hence $x(\alpha,\beta)\in\delta C_{\tau,\tau^\star}(\lambda,r)\vee\delta C_{\tau,\tau^\star}(\mu,r)$.
(5). Since $\lambda\wedge\mu\leq\lambda$, $\delta C_{\tau,\tau^\star}(\lambda\wedge\mu,r)\leq\delta C_{\tau,\tau^\star}(\lambda,r)$. Similarly, $\delta C_{\tau,\tau^\star}(\lambda\wedge\mu,r)\leq\delta C_{\tau,\tau^\star}(\mu,r)$. Therefore $\delta C_{\tau,\tau^\star}(\lambda\wedge\mu,r)\leq\delta C_{\tau,\tau^\star}(\lambda,r)\wedge\delta C_{\tau,\tau^\star}(\mu,r)$. □

In general, finite intersection of intuitionistic r-fuzzy regular closed sets is not intuitionistic r-fuzzy regular closed. However, intuitionistic r-fuzzy δ-closed sets have a nice properties as in the following theorem.

Theorem 5.15. *Let $(X,\tau,\tau^\star)$ be an intuitionistic smooth fuzzy topological space. Then the following hold:*

1. *Finite union of intuitionistic r-fuzzy δ-closed sets in X is an intuitionistic r-fuzzy δ-closed set in X.*
2. *Arbitrary intersection of intuitionistic r-fuzzy δ-closed sets in X is an intuitionistic fuzzy δ-closed set in X.*

Proof. (1). Let λ_1 and λ_2 be intuitionistic r-fuzzy δ-closed sets. Then $\delta C_{\tau,\tau^\star}(\lambda_1\vee\lambda_2,r)=\delta C_{\tau,\tau^\star}(\lambda_1,r)\vee\delta C_{\tau,\tau^\star}(\lambda_2,r)=\lambda_1\vee\lambda_2$. Thus $\lambda_1\vee\lambda_2$ is an intuitionistic r-fuzzy δ-closed set.
(2). Let λ_i be an intuitionistic r-fuzzy δ-closed set, for each $i\in I$. To show that $\delta C_{\tau,\tau^\star}(\lambda_i,r)\leq\lambda_i$, take any $x(\alpha,\beta)\in\delta C_{\tau,\tau^\star}(\lambda_i,r)$. Suppose that $x(\alpha,\beta)\notin\lambda_i$. Then there exists an $i_0\in I$ such that $x(\alpha,\beta)\notin\lambda_{i_0}$. Since λ_{i_0} is an intuitionistic r-fuzzy δ-closed set, $x(\alpha,\beta)\notin\delta C_{\tau,\tau^\star}(\lambda_{i_0},r)$. Then there exists an intuitionistic r-fuzzy regular open q-neighborhood γ of $x(\alpha,\beta)$ such that $\gamma\,\overline{q}\,\lambda_{i_0}$. Since $\gamma\,\overline{q}\,\lambda_{i_0}$ and $\lambda_i\leq\lambda_{i_0}$, we have $\gamma\,\overline{q}\,\lambda_i$. Thus $x(\alpha,\beta)\notin\delta C_{\tau,\tau^\star}(\lambda_i,r)$. This is a contradiction. Hence $\delta C_{\tau,\tau^\star}(\lambda_i,r)\leq\lambda_i$. □

Theorem 5.16. *Let γ be an intuitionistic fuzzy set in an intuitionistic fuzzy $(X,\tau,\tau^\star)$, then $\delta C_{\tau,\tau^\star}(\gamma)$ is the intersection of all intuitionistic r-fuzzy regular closed supersets of γ, or $\delta C_{\tau,\tau^\star}(\gamma,r)=\bigwedge\{\eta:\gamma\leq\eta=C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(\eta,r),r)\}$.*

Proof. Suppose that $x(\alpha,\beta)\notin\bigwedge\{\eta:\gamma\leq\eta=C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(\eta,r),r)\}$. Then there exists an intuitionistic r-fuzzy regular closed set η such that $x(\alpha,\beta)\notin\eta$ and $\gamma\leq\eta$. Since $x(\alpha,\beta)\notin\eta$, $x(\alpha,\beta)\;q\;\overline{1}-\eta$. Note that $\gamma\leq\eta$ if and only if $\gamma\,\overline{q}\,\overline{1}-\eta$. Thus $\overline{1}-\eta$ is an intuitionistic r-fuzzy regular open q-neighborhood of $x(\alpha,\beta)$ such that $\gamma\,\overline{q}\,\overline{1}-\eta$. Hence $x(\alpha,\beta)\notin\delta C_{\tau,\tau^\star}(\gamma,r)$. Let $x(\alpha,\beta)\in\bigwedge\{\eta:\gamma\leq\eta=C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(\eta,r),r)\}$. Suppose that $x(\alpha,\beta)\notin\delta C_{\tau,\tau^\star}(\gamma,r)$. Then there exists an intuitionistic r-fuzzy regular open q-neighborhood λ of $x(\alpha,\beta)$ such that $\gamma\,\overline{q}\,\lambda$. So, $\gamma\leq\overline{1}-\lambda$. Since $x(\alpha,\beta)\;q\;\lambda$, $x(\alpha,\beta)\notin\overline{1}-\lambda$. Therefore, there exists an intuitionistic fuzzy regular closed set $\overline{1}-\lambda$ such that $x(\alpha,\beta)\notin\overline{1}-\lambda$ and $\gamma\leq\overline{1}-\lambda$. Hence $x(\alpha,\beta)\notin\bigwedge\{\eta:\gamma\leq\eta=C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(\eta,r),r)\}$. This is a contradiction. Thus $x(\alpha,\beta)\in\delta C_{\tau,\tau^\star}(\gamma,r)$. □

Remark 5.17. *From the above theorem, for any intuitionistic fuzzy set λ, the intuitionistic fuzzy δ-closure $\delta C_{\tau,\tau^\star}(\lambda,r)$ is an intuitionistic r-fuzzy closed set. Moreover, $\delta C_{\tau,\tau^\star}(\lambda,r)$ becomes intuitionistic r-fuzzy δ-closed.*

Theorem 5.18. *If λ is an intuitionistic r-fuzzy regular closed set, then λ is an intuitionistic r-fuzzy δ-closed set.*

Proof. Let λ be an intuitionistic r-fuzzy regular closed set. Then $C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(\lambda,r),r)=\lambda$. By Theorem 5.14, $\delta C_{\tau,\tau^\star}(\lambda,r)=\bigwedge\{\mu:\lambda\leq\mu=C_{\tau,\tau^\star}(I_{\tau,\tau^\star}(\mu,r),r)\}=\lambda$. Thus λ is intuitionistic r-fuzzy δ-closed. □

Theorem 5.19. *For any intuitionistic fuzzy set λ, $\delta C_{\tau,\tau^\star}(\lambda,r)$ is an intuitionistic r-fuzzy δ-closed set.*

Proof. By Theorems 5.14 and 5.18. □

Theorem 5.20. *Intuitionistic fuzzy δ-closure satisfies the Kuratowski closure axioms.*

From the results of intuitionistic fuzzy δ-closure which are obtained above, we have following properties of intuitionistic fuzzy δ-interior.

Theorem 5.21. *Let λ and μ be two intuitionistic fuzzy sets in an intuitionistic smooth fuzzy topological space $(X,\tau,\tau^\star)$. Then we have the following properties:*

1. $\delta I_{\tau,\tau^\star}(\overline{0},r)=\overline{0}$,
2. $\lambda\geq\delta I_{\tau,\tau^\star}(\lambda,r)$,
3. $\lambda\leq V\Rightarrow\delta I_{\tau,\tau^\star}(\lambda,r)\leq\delta I_{\tau,\tau^\star}(\mu,r)$,
4. $\delta I_{\tau,\tau^\star}(\lambda,r)\wedge\delta I_{\tau,\tau^\star}(\mu,r)=\delta I_{\tau,\tau^\star}(\lambda\wedge\mu,r)$,
5. $\delta I_{\tau,\tau^\star}(\lambda\vee\mu,r)\leq\delta I_{\tau,\tau^\star}(\lambda,r)\vee\delta I_{\tau,\tau^\star}(\mu,r)$.

Theorem 5.22. *Let $(X,\tau,\tau^\star)$ be an intuitionistic smooth fuzzy topological space. Then the following hold:*

1. *Finite intersection of intuitionistic r-fuzzy δ-open sets in X is an intuitionistic r-fuzzy θ-open set in X.*
2. *Arbitrary union of intuitionistic r-fuzzy δ-open sets in X is an intuitionistic r-fuzzy δ-open set in X.*

Theorem 5.23. *Let λ be an intuitionistic fuzzy set in an intuitionistic smooth fuzzy topological space $(X,\tau,\tau^\star)$. Then $\delta I_{\tau,\tau^\star}(\lambda,r) = \bigvee\{\gamma : I_{\tau,\tau^\star}(C_{\tau,\tau^\star}(\gamma,r),r) = \gamma \leq \lambda\}$. As a result, $\delta I_{\tau,\tau^\star}(\lambda,r)$ is an intuitionistic r-fuzzy open set.*

Corollary 5.24.
1. *If λ is an intuitionistic r-fuzzy δ-open set in an intuitionistic smooth fuzzy topological space $(X,\tau,\tau^\star)$, then λ is an intuitionistic r-fuzzy open set.*
2. *If λ is an intuitionistic r-fuzzy θ-open set in an intuitionistic smooth fuzzy topological space $(X,\tau,\tau^\star)$, then λ is an intuitionistic r-fuzzy δ-open set.*

Theorem 5.25. *For any intuitionistic fuzzy set λ in an intuitionistic smooth fuzzy topological space $(X,\tau,\tau^\star)$, $\delta I_{\tau,\tau^\star}(\lambda,r) \leq \delta I_{\tau,\tau^\star}(\lambda,r) \leq I_{\tau,\tau^\star}(\lambda,r)$. In particular, for any intuitionistic r-fuzzy closed set λ, $\delta I_{\tau,\tau^\star}(\lambda,r) = \delta I_{\tau,\tau^\star}(\lambda,r) = I_{\tau,\tau^\star}(\lambda,r)$.*

Corollary 5.26. *If λ is an intuitionistic r-fuzzy regular open set, then λ is an intuitionistic r-fuzzy δ-open set.*

Corollary 5.27. *For any intuitionistic fuzzy set λ, $\delta I_{\tau,\tau^\star}(\lambda,r)$ is an intuitionistic r-fuzzy δ-open set.*

6. Conclusion

In this paper, we have studied an application of intuitionistic fuzzy set theory to a smooth Topological space which is a well known topological structure. Based on these results, we plan to study applications of intuitionistic fuzzy set theory in other topological structures. We also plan to use of the results of this paper to solve decision making problems and study (Pythagorean) fuzzy trends. We hope that researchers can apply these to applications in the real world.

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