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Two Parameter Deformed Non-Extensive Entropy from the Two-Parameter Quantum Number

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Abstract

In this paper, two parameter (p, r) deformed non-extensive entropy is constructed from the view of the two-parameter quantum number. The time evolution of two parameter deformed entropy is obtained to give the condition that the entropy increases. For special example, $q = 1 + \varepsilon$, $r = 1 - \varepsilon$ for a sufficiently small ε is considered and up to a first order in ε, the MaxEnt probability distribution and the deformed internal energy is computed.

Keywords: *Tsallis entropy; maximum entropy probability; non-extensive entropy; deformed entropy; deformed internal energy.*

1. Introduction

There is a growing interest in generalizing the Boltzmann-Gibbs statistical mechanics. Because the entropy plays a fundamental role in the statistical physics, the entropy should be deformed so as to construct a new (deformed) theory. The first attempt has been accomplished by Tsallis [1,2]. Based on the fact that Boltzmann-Gibbs theory is not adequate for various complex, natural, artificial and social system, he introduced the non-extensive entropy is given by

$$S_q = k\left(\textstyle\sum_i^W p_i^q - 1\right)/(1-q) \tag{1}$$

where k is a Boltzmann constant. The non-extensive Boltzmann-Gibbs entropy has attracted much interest among physicists, chemists and mathematicians who study thermodynamics of complex system [3-6]. When the deformation parameter q goes to 1, the non-extensive entropy reduces to the ordinary one.

The entropy (1) can be obtained from the quantum derivative. Abe [7] rewrote the Tsallis entropy as follows:

$$S_q = -k \frac{d}{d_q\alpha} \sum_{i=1}^W p_i^\alpha |_{\alpha=1} \tag{2}$$

where $\frac{d}{d_q\alpha}$ is Jackson q-derivative defined by

$$\frac{d}{d_q\alpha} f(\alpha) = \frac{f(q\alpha) - f(x)}{(q-1)\alpha} \tag{3}$$

This derivative is related to the following q-number:

$$[x]_q = \frac{q^x - 1}{q-1} \tag{4}$$

Considering the system composed of two independent subsystems having the physical quantities A_1, A_2, respectively and assuming that the physical quantity is extensive $(\mathrm{A} = A_1 + A_2)$, we have

$$[A_1 + A_2]_q = [A_1]_q + [A_2]_q + (q-1)[A_1]_q[A_2]_q \tag{5}$$

which gives the psuedo-additivity relation:

$$S_q[p_1 \otimes p_2] = S_q[p_1] + S_q[p_2] + (1-q)S_q[p_1]S_q[p_2] \tag{6}$$

In this paper, we consider the two parameter deformed non-extensive entropy from the view of the two-parameter quantum number, which emerges in the quantum group theory. We consider the time evolution of two parameter deformed entropy to obtain the condition that the entropy increases with time. For special example, we dealt with the case of $r = -q + 2$, $b = q$ with $q = 1 + \varepsilon$, $r = 1 - \varepsilon$ for a sufficiently small ε. In our case we consider that ε is less than 10^{-2}. Up to a first order in ε, we computed the MaxEnt probability distribution and the deformed internal energy.

2. Two Parameter Deformed Entropy

Now let us consider two parameter deformation of the Eq. (1):

$$S_{q,r} = -k \frac{d}{d_{q,r}\alpha} \sum_{i=1}^W p_i^\alpha |_{\alpha=1} \tag{7}$$

where two parameter deformed derivative $\frac{d}{d_{q,r}\alpha}$ is defined by

$$\frac{d}{d_{q,r}\alpha} f(\alpha) = \frac{f(q\alpha) - f(r\alpha)}{(q-r)\alpha} \tag{8}$$

which is related to the following deformed number :

$$[x]_{q,r} = \frac{q^x - r^x}{q - r} \tag{9}$$

We know that $S_{q,r}$ reduces to a Tsallis entropy when r approaches 1. Considering the system composed of two independent subsystems having the physical quantities A_1, A_2, respectively and assuming that the physical quantity is extensive $(A = A_1 + A_2)$, we have

$$[A_1 + A_2]_{q,r} = [A_1]_{q,r} \times \frac{1}{2}\{q^{A_2} + r^{A_2}\} + [A_2]_{q,r} \times \frac{1}{2}\{q^{A_1} + r^{A_1}\} \tag{10}$$

or

$$\begin{aligned}[A_1 + A_2]_{q,r} &= [A_1]_{q,r} + [A_2]_{q,r} \\ &+ \frac{q-1}{2}\{[A_1]_{q,r}[A_2]_q + [A_2]_{q,r}[A_1]_q\} \\ &+ \frac{r-1}{2}\{[A_1]_{q,r}[A_2]_r + [A_2]_{q,r}[A_1]_r\}\end{aligned} \tag{11}$$

which gives the pseudo-additivity relation:

$$\begin{aligned}S_q[p_1 \otimes p_2] &= S_q[p_1] + S_q[p_2] \\ &+ \frac{1-q}{2}\{S_{q,r}(p_1)S_q(p_2) + S_{q,r}(p_2)S_q(p_1)\} \\ &+ \frac{1-r}{2}\{S_{q,r}(p_1)S_r(p_2) + S_{q,r}(p_2)S_r(p_1)\}\end{aligned} \tag{12}$$

Clearly, we know that $S_{q,r}$ is nonextensive unless $q = r = 1$. Two parameter deformed entropy can be written in terms of two parameter deformed logarithmic function:

$$S_{q,r} = -\frac{k}{q-r}\sum_{i=1}^{W}(p_i^q - p_i^r) \tag{13}$$

$$S_{q,r} = -k\sum_{i=1}^{W} p_i^\alpha \ln_{q,r} p_i \tag{14}$$

$$\ln_{q,r} x = \frac{1 - x^{r-q}}{q - r} \tag{15}$$

$$e_{q,r}(x) = [1 + (r-q)x]_+^{\frac{1}{r-q}} \tag{16}$$

The two parameter deformed exponential function obeys:

$$e_{q,r}(x)e_{q,r}(y) = e_{q,r}(x \otimes y) \tag{17}$$

where the deformed addition is defined by:

$$x \otimes y = x + y + (r-q)xy \tag{18}$$

3. MaxEnt Probability Distribution

Now let us start with two parameter deformed entropy (13) with the following two constraints

$$\sum_{i=1}^{W} p_i = 1 \tag{19}$$

$$\sum_{i=1}^{W} p_i^b E_i = U_b = const \tag{20}$$

Then, the maximum entropy principle reduces to optimize the following quantity

$$\tilde{S} = \frac{S}{k} - \alpha\sum_{i=1}^{W} p_i - \beta\sum_{i=1}^{W} p_i^b E_i \tag{21}$$

where α, β are Lagrange multipliers. Differentiating $\tilde{S}$ with respect to p_i yields

$$\frac{1}{r-q}\left(qp_i^{q-1} - rp_i^{r-1}\right) - \alpha - \beta b E_i p_i^{b-1} = 0 \tag{22}$$

which does not give a concrete form of p_i for arbitrary q, r and b. However, for special choices of these values we can solve the Eq. (22).

Now let us consider the time evolution of the deformed entropy. Differentiating the deformed entropy with respect to time, we get

$$\frac{dS_{q,r}}{dt} = \frac{k}{r-q}\sum_{i=1}^{W}\left(qp_i^{q-1} - rp_i^{r-1}\right)\frac{dp_i}{dt} \tag{23}$$

From $\sum_{i-1}^{W} p_i = 1$, the time evolution of the probability is

$$\frac{dp_i}{dt} = \sum_{j=1}^{W}\left(A_{ji}p_j - A_{ij}p_i\right) \tag{24}$$

where A_{ji} is the probability of transition per unit time from the j-th microstate to the i-th micro state. Assuming the detailed balance, we can set

$$A_{ij} = A_{ji} \tag{25}$$

Then, we have

$$\frac{dS_{q,r}}{dt} = \frac{k}{r-q}\sum_{i,j=1}^{W}\left(qp_i^{q-1} - rp_i^{r-1}\right)A_{ij}\left(p_j - p_i\right) \tag{26}$$

Interchanging the dummy indices, we get

$$\frac{dS_{q,r}}{dt} = \frac{k}{r-q}\sum_{i,j=1}^{W}\left(qp_j^{q-1} - rp_i^{r-1}\right)A_{ji}\left(p_j - p_i\right) \tag{27}$$

Summing the Eq. (26) and the Eq. (27), side by side, we get

$$\frac{dS_{q,r}}{dt} = \frac{k}{2(q-r)}\sum_{i,j=1}^{W} A_{ij}\left(p_i - p_j\right)^2\left[q\sum_{l=0}^{q-2} p_i^l p_j^{q-2-l} - r\sum_{l=0}^{q-2} p_i^l p_j^{r-2-l}\right] \tag{28}$$

3.1 The Case of q $>$ r

In this case, we have:

$$\frac{dS_{q,r}}{dt} = \frac{k}{2(q-r)}\sum_{i,j=1}^{W} A_{ij}\left(p_i - p_j\right)^2\left[(q - r)\sum_{l=0}^{r-2} p_i^l p_j^{r-2-l} + q\sum_{l=0}^{q-2} p_i^l p_j^{q-2-l}\right] \tag{29}$$

Thus, we know that the deformed entropy increases with time for $q > 0$ while it may increase or decrease for $q < 0$.

3.2 The Case of $\mathbf{q = r}$

In this case, we have:

$$\frac{dS_{q,r}}{dt} = 0 \tag{30}$$

Thus, the deformed entropy remain invariant with time.

3.3 The Case of $q < r$

In this case, we have:

$$\frac{dS_{q,r}}{dt} = \frac{k}{2(q-r)} \sum_{i,j=1}^{W} A_{ij}\left(p_i - p_j\right)^2 \left[(q - r) \sum_{l=0}^{q-2} p_i^l p_j^{q-2-l} - r \sum_{l=0}^{r-2} p_i^l p_j^{r-2-l}\right] \tag{31}$$

Thus, we know that the deformed entropy increases with time for \$ $r > 0$ \$ while it may increase or decrease for $r < 0$.

3.4 Special Case

In this section we consider the special case with some approximation. When $r = -q + 2, b = q$, the Eq.(22) gives the following solution:

$$p_i = \left[\frac{\alpha(1-q) \pm \sqrt{(\alpha(1-q))^2 + q(2-q)[1-2(1-q)\beta E_i]}}{q-2}\right]^{\frac{1}{1-q}} \tag{32}$$

when $q \to 1$ we have $r \to 1$, so in this limit the Eq. (32) should reduce to the Boltzman-Gibbs case, which makes one determine p_i uniquely. Indeed, we have the following relation:

$$p_i = \left[\frac{\alpha(1-q) - \sqrt{(\alpha(1-q))^2 + q(2-q)[1-2(1-q)\beta E_i]}}{q-2}\right]^{\frac{1}{1-q}} \tag{33}$$

From the limit $q \to 1, r \to 1$, we know

$$\alpha = \ln Z_0 - 1, \ \beta = \frac{1}{kT} \tag{34}$$

where,

$$Z_0 = \sum_{l=0}^{W} e^{-\beta E_i} \tag{35}$$

From the probability distribution given in the Eq. (33), we cannot obtain some physical quantities such as internal energy, specific heat. Thus, we adopt the approximation method.

Let us consider the case that $q = 1 + \varepsilon, r = 1 - \varepsilon$ for a sufficiently small ε. Up to a first order in ε, we have

$$p_i = \frac{1}{Z} e^{-\beta E_i}\left[1 + \varepsilon\left(\frac{1}{2} + \ln Z_0 - 1\right)\beta E_i + \beta^2 {E_i}^2\right] \tag{36}$$

where

$$Z = \sum_{l=1}^{W} e^{-\beta E_i}\left[1 + \varepsilon\left(\frac{1}{2} + \ln Z_0 - 1\right)\beta E_i + \beta^2 {E_i}^2\right] \tag{37}$$

Up to a first order in ε, the deformed internal energy is given by

$$U_\varepsilon = \frac{1}{Z_0}\sum_{l=1}^{W} E_i e^{-\beta E_i} + \frac{\varepsilon}{Z_0}\sum_{l=1}^{W} e^{-\beta E_i} E_i \left(\frac{1}{2} - \ln Z_0 - \frac{Z_1}{Z_0} + (\ln Z_0 - 2)\beta E_i + \beta^2 {E_i}^2\right) \tag{38}$$

where

$$\mathrm{Z} = Z_0 + \varepsilon Z_1 \tag{39}$$

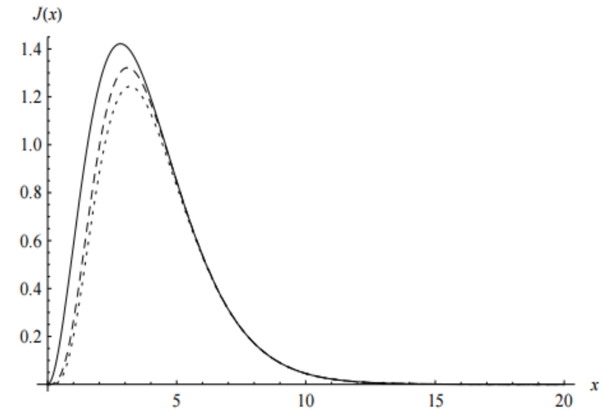

Figure 1. Plot of J(x) versus x for ε= 0 (solid line), ε= 0.01 (dashed line) and ε = −0.01 (dotted line)

As an example consider quantum harmonic oscillator governed by the Hamiltonian operator $\mathrm{H} = p^2/2 + m\omega^2 x^2/2$ where m is a particle mass and w is an angular frequency. It is known that the energy of the quantum harmonic oscillator is then given by $E_n = n\hbar\omega$, where $n = 0,1,2,\cdots$. The deformed internal energy is given by

$$\beta U_\varepsilon = \frac{xe^{-x}}{1e^{-x}} + x\varepsilon\left(\ln(1-e^{-x}) + \frac{xe^{-x}}{1-e^{-x}}\ln(1-e^{-x}) + \frac{x^2 e^{-x}(1+e^{-x})}{1-e^{-x}}\right)\frac{e^{-x}}{1-e^{-x}} - \varepsilon x^2\left(\ln(1-e^{-x}+1)\frac{xe^{-x}(1+e^{-x})}{(1-e^{-x})^2} + \varepsilon\frac{x^3 e^{-x}(1+4e^{-x}+e^{-2x})}{(1-e^{-x})^3}\right) \tag{40}$$

where we set $x = \beta\hbar\omega$. When we apply the formula (40) to the photon, we should compute the intensity which is proportional to the density of states times internal energy. If we denote the q-intensity by I(x), we get

$$\mathrm{I(x)} = \mathrm{cx}^2 U(x) \tag{41}$$

where c is some constant. The Figure 1 shows the plot of $J(x) = \beta\, I(x)/c$ versus x for $\varepsilon = 0$ (solid line), $\varepsilon = -0.01$ (dashed line) and $\varepsilon = 0.01$ (dotted line).

4. Conclusion

In this paper we considered the two parameter deformed non-extensive entropy from the view of the two parameter quantum number which emerges in the quantum group theory. For this deformed entropy we obtained the pseudo-additivity relation which reveals the non-extensive property. We also applied the maximum entropy principle to obtain the condition for the MaxEnt probability distribution. We considered the time evolution of two parameter deformed entropy to obtain the condition that the entropy increases with time. For special case, we dealt with the case of $r = -q + 2$, $b = q$ with $q = 1 + \varepsilon$, $r = 1 - \varepsilon$ for a sufficiently small ε. Up to a first order in ε, we have computed the MaxEnt probability distribution and the deformed internal energy. For $\varepsilon = \pm 0.01$, we computed the deformed intensity and plotted it in Figure 1 which shows that the peak of the intensity goes down for both case.

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