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Unique Common Fixed Points For Maps With (ψ,α,β)- Contractive Condition In W^*-Spaces

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ABSTRACT

In this paper, we introduce W^* -spaces which generalizes W -spaces introduced by Piao and Jin [10] and prove three unique common fixed point theorems in it. Some illustrative examples to highlight the results are furnished.

***Key Words**: W^* -spaces, Converse commuting maps, Common fixed points.*

1. INTRODUCTION

In recent years many researchers have done much work in metric spaces, symmetric spaces[9, 3, 4], D-metric spaces[1, 2], D^*- metric spaces[6, 7], G-metric spaces[12, 13], Partial metric spaces[5, 8] and so on. In this direction Piao and Jin [10] introduced the concept of W-spaces in 2012, which is weaker than the notions of metric and symmetric spaces and proved some fixed point theorems.

In this paper, we introduce W^* -spaces to generalize W -spaces and proved three unique common fixed point theorems in it. We also give examples to illustrate our theorems.

First we state the following known definitions.

Definition 1.1 *Let X be a non-empty set. If a function $d : X \times X \to [0,\infty)$ satisfies the property that $d(x, y) = 0$ implies $x = y$, then (X,d) is called a W -space.*

Definition 1.2 *Let f and g be two self mappings on a non-empty set X.*

(i) [11]. If $fgx = gfx$ for some $x \in X$ then x is called a commuting point of the pair (f,g).

(ii) [11]. If $fgx = gfx$ implies $fx = gx$ for all $x \in X$ then the pair (f,g) is said to be converse commuting.

Now we give the following definition.

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Definition 1.3 *Let* f *and* g *be two self mappings on a non-empty set* X *. We say that the pair* (f,g) *satisfy Property (K) if there exists* $u \in X$ *such that* $fgu = gfu$ *and* $fu = gu$ *.*

Remark 1.4 *Definition 1.2 (*(i) *and* (ii)*) imply the Property(K) but not the converse in view of the following example.*

Example 1.5 *Let* $X = \{0,1,2\}$, $f0 = 0$, $f1 = 2$, $f2 = 1$ *and* $g0 = g1 = g2 = 0$ *. Clearly the pair* (f,g) *satisfy Property(K). But* $fg1 = gf1$ *and* $f1 \neq g1$.

Piao and Jin [10] proved the following theorems.

Theorem 1.6 (*Theorem* 1, [10]) *Let* (X,d) *be a* W *-space and* f *and* g *be two converse commuting self maps which have a commuting point. Suppose that* $x, y \in X$ *with* $d(gx, gy) \neq 0$ *satisfy*
$d(gx, gy) \leq \varphi\left(d(fy, fx), d(gy, fx), d(gx, fy)\right)$ *where* $\varphi : R_+^3 \to R_+$ *is such that* $a \leq \varphi(b,b,a)$ *implies* $a < b$ *for all* $a > 0$, $b > 0$ *. Then* f *and* g *have a unique common fixed point.*

Theorem 1.7 (*Theorem* 2, [10]) *Let* (X,d) *be a* W *-space and* f *and* g *be two self maps which have a commuting point. Suppose that* $x, y \in X$ *with* $d(fx, gy) \neq 0$ *satisfy*
$d(fx, gy) \leq \psi\left(d(fx, gx), d(fy, gx), d(gx, gy)\right)$ *where* $\psi : R_+^3 \to R_+$ is such that

(i) ψ is monotone increasing for the first variable,

(ii) if $a > 0$, $b > 0$ then $a \leq \psi(a,b,a)$ implies $a < b$,

(iii) for any $a > 0$, there is $\psi(a,a,0) < a$.

Then f and g have a unique common fixed point.

Theorem 1.8 (*Theorem* 3, [10]) *Let* (X,d) *be a* W *-space and* f_1, f_2 *and* g_1, g_2 *be four self maps. Also let* (f_1, f_2) *and* (g_1, g_2) *be pairs of converse commuting self mappings which have a commuting point respectively. Suppose that* $x, y \in X$ *with* $d(f_2 x, g_2 y) \neq 0$ *satisfy*

$$d(f_2x, g_2y) \leq \varphi\begin{pmatrix} d(g_1y, f_1x), d(g_2y, f_1x), d(g_1y, g_2y), \\ d(f_1x, f_2x), d(g_1y, f_2x) \end{pmatrix}$$

and suppose that $x, y \in X$ with $d(g_1x, f_1y) \neq 0$ satisfy

$$d(g_1x, f_1y) \leq \varphi'\begin{pmatrix} d(f_2y, g_2x), d(f_2y, g_1x), d(g_1x, g_2x), \\ d(f_1y, f_2y), d(f_1y, g_2x) \end{pmatrix}$$

where $\varphi, \varphi' : R_+^5 \to R_+$ satisfy $\varphi(a,a,0,0,a) < a$ for any $a > 0$ and $\varphi'(a,a,0,0,a) < a$ for any $a > 0$. Then f_1, f_2, g_1 and g_2 have a unique common fixed point.

Now we define W^* - spaces as follows.

Definition 1.9 *Let* X *be a non-empty set. If a function* $d : X \times X \times X \to [0,\infty)$ *satisfies the property that* $d(x,y,z) = 0$ *implies* $x = y = z$ *, then* (X,d) *is called a* W^**-space.*

Example 1.10 *Let* $X = [0,\infty)$ *and* $d(x,y,z) = \max\{x,y,z\}$ *or* $x + y + z$ *. Then* (X,d) *is a* W^**-space.*

Throughout this paper, let $\psi, \alpha, \beta : [0,\infty) \to [0,\infty)$ be such that $\psi(t) - \alpha(t) + \beta(t) > 0$ for all $t > 0$.

Immediately it follows that $\psi(t) - \alpha(t) + \beta(t) \leq 0$ implies $t = 0$.

Now we prove our main results which are different from Theorems 1.6,1.7 and 1.8.

2. MAIN RESULT

Theorem 2.1 . *Let* (X,d) *be a* W^**-space and* $f, g : X \to X$ *be satisfying*

$$(2.1.1)\ \psi\left(d(gx,gy,gz)\right) \leq \alpha\left(\max\begin{Bmatrix} d(fx,fy,fz), d(fx,gy,gz), \\ d(fx,gy,fz), d(fx,fy,gz), \\ d(gx,fy,fz), d(gx,fy,gz), \\ d(gx,gy,fz) \end{Bmatrix}\right) - \beta\left(\max\begin{Bmatrix} d(fx,fy,fz), d(fx,gy,gz), \\ d(fx,gy,fz), d(fx,fy,gz), \\ d(gx,fy,fz), d(gx,fy,gz), \\ d(gx,gy,fz) \end{Bmatrix}\right)$$

for all $x, y, z \in X$ with $d(gx, gy, gz) \neq 0$ and

(2.1.2) the pair (f,g) satisfies Property (K).

Then f and g have a unique common fixed point.

Proof. From (2.1.2), there exists $u \in X$ such that $fgu = gfu$ and $fu = gu$.

Hence

$$ffu = fgu = gfu = ggu. \tag{1}$$

Suppose $d(gu, gu, ggu) \neq 0$.

Putting $x = u$, $y = u$ and $z = gu$ in (2.1.1) and using (1), we obtain

$$\psi(d(gu, gu, ggu)) \le \alpha(d(gu, gu, ggu)) - \beta(d(gu, gu, ggu))$$

which in turn yields that $d(gu, gu, ggu) = 0$. It is a contradiction. Hence $ggu = gu$.

From (1), it follows that gu is a common fixed point of f and g.

Suppose x and y are two common fixed points of f and g. Then $d(x, y, y) = d(gx, gy, gy) \neq 0$.

Then using (2.1.1) with $x = x$, $y = y$ and $z = y$ we obtain

$$\psi(d(x, y, y)) \le \alpha(d(x, y, y)) - \beta(d(x, y, y))$$

which in turn yields that $d(x, y, y) = 0$. It is a contradiction. Hence $x = y$. Thus f and g have a unique common fixed point.

Example 2.2 *Let* $X = \{0,1,2\}$ *and* $d(x, y, z) = x + y + z$. *Let* $g0 = g1 = 0, g2 = 1$ *and* $f0 = 0,\ f1 = 1,\ f2 = 2$. *Let* $\psi, \alpha, \beta : [0,\infty) \to [0,\infty)$ *be defined by* $\psi(t) = t$, $\alpha(t) = \frac{3t}{4}$ *and* $\beta(t) = \frac{t}{4}$.

Then clearly $(2.1.1)$ and $(2.1.2)$ are satisfied and 0 is the unique common fixed point of f and g.

Next we give the following theorem without using the converse commuting condition.

Theorem 2.3 . *Let* (X, d) *be a* W^**-space and* $f, g : X \to X$ *be satisfying*

$$(2.3.1)\ \psi\left(\max\begin{Bmatrix} d(fx, gy, gz), \\ d(gx, fy, fz) \end{Bmatrix} \right) \le \alpha\left(\max\begin{Bmatrix} d(gz, fx, fy), d(fz, gx, gy), \\ d(gy, fz, fx), d(fy, gz, gx) \end{Bmatrix} \right) - \beta\left(\max\begin{Bmatrix} d(gz, fx, fy), d(fz, gx, gy), \\ d(gy, fz, fx), d(fy, gz, gx) \end{Bmatrix} \right)$$

for all $x, y, z \in X$ with $\max\{d(fx, gy, gz), d(gx, fy, fz)\} \neq 0$

$(2.3.2)$ the pair (f, g) has a commuting point in X.

In addition to these, assume that α is monotonically increasing and β is monotonically decreasing.

Then f and g have a unique common fixed point in *X*.

Proof. Let u be a commuting point of f and g.

i.e $fgu = gfu$ for some $u \in X$.

Suppose that $fu \neq gu$.

From $(2.3.1)$, we have

$$\psi\left(\max\begin{Bmatrix} d(fu, gu, gu), \\ d(gu, fu, fu) \end{Bmatrix} \right) \le \alpha\left(\max\begin{Bmatrix} d(gu, fu, fu), \\ d(fu, gu, gu) \end{Bmatrix} \right) - \beta\left(\max\begin{Bmatrix} d(gu, fu, fu), \\ d(fu, gu, gu) \end{Bmatrix} \right)$$

which in turn yields that $\max\begin{Bmatrix} d(fu, gu, gu), \\ d(gu, fu, fu) \end{Bmatrix} = 0$.

It is a contradiction. Hence $fu = gu$. (2)

Hence from (2), we have

$$ffu = fgu = gfu = ggu. \quad (3)$$

Suppose that $ffu \neq fu$.

Putting $x = fu$, $y = u$, $z = u$ in $(2.3.1)$, we have

$$\psi\left(\max\begin{Bmatrix} d(ffu, gu, gu), \\ d(gfu, fu, fu) \end{Bmatrix} \right) \le \alpha\left(\max\begin{Bmatrix} d(gu, ffu, fu), d(fu, gfu, gu), \\ d(gu, fu, ffu), d(fu, gu, gfu) \end{Bmatrix} \right) - \beta\left(\max\begin{Bmatrix} d(gu, ffu, fu), d(fu, gfu, gu), \\ d(gu, fu, ffu), d(fu, gu, gfu) \end{Bmatrix} \right)$$

Hence

$$\psi\big(d(ffu, fu, fu)\big) \le \alpha\left(\max\begin{Bmatrix} d(fu, ffu, fu), \\ d(fu, fu, ffu) \end{Bmatrix} \right) - \beta\left(\max\begin{Bmatrix} d(fu, ffu, fu), \\ d(fu, fu, ffu) \end{Bmatrix} \right) \quad (4)$$

Put $x = u$, $y = fu$, $z = u$ and $x = u$, $y = u$, $z = fu$ in $(2.3.1)$, we have

$$\psi\big(d(fu, ffu, fu)\big) \le \alpha\left(\max\begin{Bmatrix} d(fu, fu, ffu), \\ d(ffu, fu, fu) \end{Bmatrix} \right) - \beta\left(\max\begin{Bmatrix} d(fu, fu, ffu), \\ d(ffu, fu, fu) \end{Bmatrix} \right) \quad (5)$$

$$\psi\big(d(fu, fu, ffu)\big) \le \alpha\left(\max\begin{Bmatrix} d(ffu, fu, fu), \\ d(fu, ffu, fu) \end{Bmatrix} \right) - \beta\left(\max\begin{Bmatrix} d(ffu, fu, fu), \\ d(fu, ffu, fu) \end{Bmatrix} \right) \quad (6)$$

From (4), (5) and (6), using monotonically increasing and decreasing properties of α and β respectively, we get

$$\psi\left(\max\left\{\begin{array}{l} d(ffu, fu, fu), \\ d(fu, ffu, fu), \\ d(fu, fu, ffu) \end{array}\right\}\right) = \max\left\{\begin{array}{l} \psi(d(ffu, fu, fu)), \\ \psi(d(fu, ffu, fu)), \\ \psi(d(fu, fu, ffu)) \end{array}\right\}$$

$$\leq \alpha\left(\max\left\{\begin{array}{l} d(ffu, fu, fu), \\ d(fu, ffu, fu), \\ d(fu, fu, ffu) \end{array}\right\}\right) - \beta\left(\max\left\{\begin{array}{l} d(ffu, fu, fu), \\ d(fu, ffu, fu), \\ d(fu, fu, ffu) \end{array}\right\}\right)$$

which in turn yields that $ffu = fu$.

Thus fu is a common fixed point of f and g.

Suppose v and v' are common fixed points of f and g. Taking $x = v'$, $y = v$, $z = v$; $x = v$, $y = v'$, $z = v$ and $x = v$, $y = v$, $z = v'$ in $(2.3.1)$ and using monotonically increasing of α and decreasing of β we can show that $v = v'$.

Thus f and g have a unique common fixed point.

Finally we give a unique common fixed point theorem for two pairs of mappings satisfying Property (K).

Theorem 2.4 *Let* (X, d) *be a* W^**-space and* $f, g, S, T : X \to X$ *be* satisfying

$$(2.4.1)\ \psi(d(fx, gy, Sz)) \leq \alpha\left(\max\left\{\begin{array}{l} d(gx, fy, Sz), \\ d(gx, fy, Tz), \\ d(fx, gy, Tz) \end{array}\right\}\right) - \beta\left(\max\left\{\begin{array}{l} d(gx, fy, Sz), \\ d(gx, fy, Tz), \\ d(fx, gy, Tz) \end{array}\right\}\right)$$

for all $x, y, z \in X$ with $d(fx, gy, Sz) \neq 0$ and

$(2.4.2)$ the pairs (f, g) and (S, T) satisfy the Property (K).

Then f, g, S and T have a unique common fixed point in X.

Proof. From $(2.4.2)$, there exist u and v in X such that $fu = gu$, (7)

$fgu = gfu$ (8)

and $Sv = Tv$, (9)

$STv = TSv$. (10)

Hence $ffu = fgu = gfu = ggu$ (11)

and

$SSv = STv = TSv = TTv$. (12)

Now suppose that $fu \neq Sv$. Then $d(fu, fu, Sv) \neq 0$.

From $(2.4.1)$, we have

$$\psi(d(fu, fu, Sv)) = \psi(d(fu, gu, Sv))$$

$$\leq \alpha\left(\max\left\{\begin{array}{l} d(gu, fu, Sv), \\ d(gu, fu, Tv), \\ d(fu, gu, Tv) \end{array}\right\}\right) - \beta\left(\max\left\{\begin{array}{l} d(gu, fu, Sv), \\ d(gu, fu, Tv), \\ d(fu, gu, Tv) \end{array}\right\}\right)$$

$$= \alpha(d(fu, fu, Sv)) - \beta(d(fu, fu, Sv)) \text{ from } (7),\ (9)$$

which in turn yields that $d(fu, fu, Sv) = 0$. Hence $fu = Sv$.

Thus

$gu = fu = Sv = Tv$. (13)

Suppose that $ffu \neq fu$. Then $d(ffu, fu, fu) \neq 0$.

From $(2.4.1)$ and (13), we have

$$\psi(d(ffu, fu, fu)) = \psi(d(ffu, gu, Sv))$$

$$\leq \alpha\left(\max\left\{\begin{array}{l} d(gfu, fu, Sv), \\ d(gfu, fu, Tv), \\ d(ffu, gu, Tv) \end{array}\right\}\right) - \beta\left(\max\left\{\begin{array}{l} d(gfu, fu, Sv), \\ d(gfu, fu, Tv), \\ d(ffu, gu, Tv) \end{array}\right\}\right)$$

$$= \alpha(d(ffu, fu, fu)) - \beta(d(ffu, fu, fu)) \text{ from } (7),\ (9)$$

which in turn yields that $d(ffu, fu, fu) = 0$. Hence

$ffu = fu$. (14)

Now from (11) and (14) we have

$gfu = fu$. (15)

Also from (13) and (10), we get

$Tfu = TSv = STv = Sfu$. (16)

Suppose that $fu \neq Sfu$.

Again from $(2.4.1)$, we have

$$\psi(d(fu, fu, Sfu)) = \psi(d(fu, gu, Sfu))$$

$$\leq \alpha\left(\max\left\{\begin{array}{l} d(gu, fu, Sfu), \\ d(gu, fu, Tfu), \\ d(fu, gu, Tfu) \end{array}\right\}\right) - \beta\left(\max\left\{\begin{array}{l} d(gu, fu, Sfu), \\ d(gu, fu, Tfu), \\ d(fu, gu, Tfu) \end{array}\right\}\right)$$

$$= \alpha(d(fu, fu, Sfu)) - \beta(d(fu, fu, Sfu)) \text{ from } (7),\ (9)$$

which gives that $d(fu, fu, Sfu) = 0$. Hence

$Sfu = fu$. (17)

Hence from (16) and (17)

$Tfu = fu$. (18)

Thus from (14), (15), (17) and (18) fu is a common fixed point of f, g, S and T.

If p and q are common fixed points of f, g, S and T, by $(2.4.1)$ one can easily prove that $p = q$.

Thus f, g, S and T have a unique common fixed point in X.

Example 2.5 *Let* $X = [0,1]$ *and* $d(x, y, z) = x + y + z$.

Define $fx = gx = 0$, $Sx = \frac{x}{8}$ *and* $Tx = \frac{x}{4}$, $\forall\ x \in X$.

Let $\psi(t) = t$, $\alpha(t) = \frac{3t}{4}$ *and* $\beta(t) = \frac{t}{4}$. *Then clearly the pairs* (f, g) *and* (S, T) *satisfy the Property (K) respectively.*

Now consider for $d(fx, gy, Sz) \neq 0$,

$$\psi\big(d(fx, gy, Sz)\big) = \frac{z}{8} = \frac{1}{2} d(fx, gy, Tz)$$

$$\leq \frac{1}{2} \max\begin{Bmatrix} d(gx, fy, Sz), \\ d(gx, fy, Tz), \\ d(fx, gy, Tz) \end{Bmatrix}$$

$$= \alpha\left(\max\begin{Bmatrix} d(gx, fy, Sz), \\ d(gx, fy, Tz), \\ d(fx, gy, Tz) \end{Bmatrix} \right)$$

$$- \beta\left(\max\begin{Bmatrix} d(gx, fy, Sz), \\ d(gx, fy, Tz), \\ d(fx, gy, Tz) \end{Bmatrix} \right)$$

Hence the condition $(2.4.1)$ is satisfied and 0 is the unique common fixed point of f, g, S and T .

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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