Gazi University Journal of Science
GU J Sci
29(2):479-486 (2016)

An Upper Estimate of Complex *q*-Balázs-Szabados-Kantorovich Operators on Compact Disks

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Received:22/02/2016 *Accepted: 17/04/2016*

ABSTRACT

In this paper, the complex q-Balázs-Szabados-Kantorovich operators are defined, and a convergence result and an upper quantitative estimate of these operators are given.

Key Words: *convergence, order of convergence, q-Balázs-Szabados operators*

1. INTRODUCTION

The applications of q-calculus in the approximation theory have become one of the main area of research. Firstly, we recall some basic definitions used in q-calculus. Details can be found in [1, 10, 2]. For any non-negative integer r, the q-integer of the number r is defined by

$$[r]_q = \begin{cases} \dfrac{1-q^r}{1-q} & if \quad q \neq 1 \\ r & if \quad q = 1, \end{cases}$$

where q is a fixed positive real number. The q-factorial is defined by

$$[r]_q! = \begin{cases} [1]_q [2]_q \dots [r]_q & if \quad r = 1,2,\dots \\ 1 & if \quad r = 0, \end{cases}$$

For integers n, r with $0 \leq r \leq n$, the q-binomial coeficients are defined by

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q! \, [n-r]_q!}.$$

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The q-derivative operator is defined by

$$D_q[f(z)] = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & if \quad z \neq 0 \\ f'(0) & if \quad z = 0. \end{cases}$$

It is not very difficult to see that $lim_{q\to 1} D_q[f(z)] = f'(z)$ if the function f is differentiable at z. Suppose that $0 < a < b$. Further we have

$D_q[f(z)g(z)] = f(qz)D_q[g(z)] + g(z)D_q[f(z)]$,

$D_q[f(z)g(z)] = f(z)D_q[g(z)] + g(qz)D_q[f(z)]$,

which is often referred to as the q-product rule. The definite q-integral is defined by

$$\int_0^b f(t)d_qt = (1-q)\, b \sum_{j=0}^{\infty} f(q^j b) q^j$$

and

$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt$,

for $0 < q < 1$.

Bernstein type rational functions are defined by K. Balázs [3]. K. Balázs and J. Szabados modified and studied the approximation properties of these operators [4]. The q-analogue of Balázs -Szabados operators is defined by O. Dogru [6].

The rational complex Balázs-Szabados operators were defined by Gal in [8]. He studied the approximation properties of these operators on compact disks. In [9], the complex q- Balázs -Szabados operators were defined as follows

$$R_n(f;q,z) = \frac{1}{\prod_{s=0}^{n-1}(1+q^s a_n z)} \sum_{j=0}^{n} q^{j(j-1)/2} f\left(\frac{[j]_q}{b_n}\right) \begin{bmatrix} n \\ j \end{bmatrix}_q (a_n z)^j \tag{1.1}$$

and the approximation properties of these operators were studied on compact disks. In [13] and in [14], complex bivariate Balázs-Szabados operators and q- Balázs-Szabados operators of tensor product kind were studied on compact polydisks, respectively.

2. CONSTRUCT OF THE OPERATORS AND AUXILIARY RESULTS

In this part, we define the reel and complex q- Balázs-Szabados-Kantorovich operators, and we give some results for these operators.

Definition 1. We define the reel q- Balázs-Szabados-Kantorovich operators as follows

$$\tilde{R}_n(f;q,x) = \frac{b_n}{\prod_{s=0}^{n-1}(1+q^s a_n x)} \sum_{j=0}^{n} q^{-j} q^{\frac{j(j-1)}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q (a_n x)^j \int_{\frac{[j]_q}{b_n}}^{\frac{[j+1]_q}{b_n}} f(t)d_qt\,,$$

where $f: [0,\infty) \to \mathbb{R}$ is a continuous function, $x \in [0,\infty)$, $a_n = [n]_q^{\beta-1}$, $b_n = [n]_q^{\beta}$ for $q \in (0,1]$, $0 < \beta \leq \frac{2}{3}$ and $n \in \mathbb{N}$.

The operators $\tilde{R}_n$ are lineer and positive.

Lemma 1. The following equalities hold for the operators $\tilde{R}_n$

$$\tilde{R}_n(1;q,x) = 1, \tag{2.1}$$

$$\tilde{R}_n(t;q,x) = \frac{x}{1+a_n x} + \frac{1}{[2]_q b_n}, \tag{2.2}$$

$$\tilde{R}_n(t^2;q,x) = \frac{\left(1-\frac{a_n}{b_n}\right)qx^2}{(1+a_n x)(1+a_n qx)} + \frac{(q+[2]_q+[3]_q)x}{[3]_q b_n(1+a_n x)}. \tag{2.3}$$

Proof. Using the results of following integrals

$$\int_{\frac{[j]_q}{b_n}}^{\frac{[j+1]_q}{b_n}} d_q t = \frac{q^j}{b_n},$$

$$\int_{\frac{[j]_q}{b_n}}^{\frac{[j+1]_q}{b_n}} t d_q t = \frac{q^j}{[2]_q b_n^2}\left(1+[2]_q[j]_q\right),$$

$$\int_{\frac{[j]_q}{b_n}}^{\frac{[j+1]_q}{b_n}} t^2 d_q t = \frac{q^j}{[3]_q b_n^3}\left\{1+\left(q+[2]_q\right)[j]_q+[3]_q[j]_q^2\right\},$$

after simple calculation, the desired equalities obtained.

Let $q=(q_n)$ be a sequence satisfying the following conditions:

$\lim_{n\to\infty} q_n = 1$ and $\lim_{n\to\infty} q_n^n = c$ for $0 \le c < 1$. (2.4)

Lemma 2. Let q_n be a sequence satisfying the conditions 2.4 with $q_n \in (0,1]$ for all $n \in \mathbb{N}$. If $f:[0,\infty)\to\mathbb{R}$ is a continuous function, then the sequence of the operators $\left(\tilde{R}_n(f;q_n,x)\right)_{n\ge n_0}$ converges f on [0, r] uniformly for $n_0 \ge 2$ and $\frac{1}{2} < r < \frac{[n_0]^{1-\beta}}{2}$.

Proof. Using 2.1, 2.2 and 2.3, the lemma can be simply proved from Korovkin theorem (see [12]).

Definition 2. We define the complex q-Balázs-Szabados-Kantorovich operators as follows

$$\tilde{R}_n(f;q,z) = \frac{b_n}{\prod_{s=0}^{n-1}(1+q^s a_n z)}\sum_{j=0}^{n} q^{-j} q^{j(j-1)/2} \begin{bmatrix} n \\ j \end{bmatrix}_q (a_n z)^j \int_{\frac{[j]_q}{b_n}}^{\frac{[j+1]_q}{b_n}} f(t) d_q t\,,$$

where $f: D_R \cup [R,\infty) \to \mathbb{C}$ is uniformly continuous and bounded on $[0,\infty)$, $D_R = \{z \in \mathbb{C}: |z| < R\}$ with $R > 0$, $z \in \mathbb{C}$ with $z \ne -\frac{1}{q^s a_n}$ for $s = 0,1,\dots,n-1$, $a_n = [n]_q^{\beta-1}$, $b_n = [n]_q^{\beta}$ for $q \in (0,1]$, $0 < \beta \le \frac{2}{3}$ and $n \in \mathbb{N}$.

The complex q-Balázs-Szabados-Kantorovich operators $\tilde{R}_n(f;q,z)$ are well defined, linear, and these operators are analytic for all $n \ge n_0$ and $|z| \le r < [n_0]_q^{1-\beta}$.

Let us denote with $\|f\|_r = \max\{|f(z)| \in \mathbb{R}: z \in \overline{D}_r\}$ the norm of f in the space of continuous functions on $\overline{D}_r$ and with $\|f\|_{B[0,\infty)} = sup\{|f(x)| \in \mathbb{R}: x \in [0,\infty)\}$ the norm of f in the space of bounded functions on $[0,\infty)$.

Also, the many results in this study are obtained under the condition that $f: D_R \cup [R,\infty) \to \mathbb{C}$ is analytic in D_R for $r < R$, which assures the representation $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $z \in \mathbb{C}$.

Lemma 3. Let be $n_0 \ge 2$, $0 < \beta \le \frac{2}{3}$. If f is uniformly continous on $D_R \cup [R,\infty)$, bounded on $[0,\infty)$ and analytic in D_R, then $\tilde{R}_n(f;q,z) = \sum_{k=0}^{\infty} c_k \tilde{R}_n(e_k;q,z)$ for all $z \in \overline{D}_r$, where $e_k(z) = z^k$.

Proof. For any $m \in \mathbb{N}$, we define

$f_m(z) = \sum_{k=0}^{\infty} c_k e_k(z)$ if $|z| \le r$ and $f_m(z) = f(z)$ if $z \in (r,\infty)$.

From hypothesis on f, it is clear that each f_m is bounded on $[0,\infty)$, that is, there exist $M_{f_m} > 0$ with $|f_m(z)| \le M_{f_m}$, which implies that

$$\left|\tilde{R}_n(f_m;q,z)\right| \le \frac{b_n}{\prod_{s=0}^{n-1}(1-q^s a_n|z|)}\sum_{j=0}^{n} q^{-j} q^{\frac{(j-1)j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q (a_n|z|)^j \left|\int_{\frac{[j]_q}{b_n}}^{\frac{[j+1]_q}{b_n}} f_m(t) d_q t\right|$$

$$\leq \frac{b_n}{\prod_{s=0}^{n-1}(1-q^s a_n r)} \sum_{j=0}^{n} q^{-j} q^{\frac{(j-1)j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q (a_n r)^j$$

$$\times (1-q)\, lim_{m\to\infty} \sum_{k=0}^{m} q^k \left| \frac{[j+1]_q}{b_n} f_m\left(\frac{[j+1]_q}{b_n}\right) - \frac{[j]_q}{b_n} f_m\left(\frac{[j]_q}{b_n}\right) \right|$$

$$\leq \frac{M_{f_m}}{\prod_{s=0}^{n-1}(1-q^s a_n r)} \sum_{j=0}^{n} q^{-j} q^{\frac{(j-1)j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q (a_n r)^j \left([j]_q + [j+1]_q\right)$$

$$= M_{f_m}\, \widetilde{M}_{r,n,q} < \infty,$$

for all $|z| \leq r$. That is all $\tilde{R}_n(f_m; q, z)$ with $n \geq n_0$, $r < \frac{[n_0]_q^{1-\beta}}{2}$, $m \in \mathbb{N}$ are well defined for all $z \in \overline{D}_r$. Defining

$f_{m,k}(z) = \sum_{k=0}^{\infty} c_k\, e_k(z)$ if $|z| \leq r$ and $f_{m,k}(z) = \frac{f(z)}{m+1}$ if $z \in (r, \infty)$,

it is clear that each $f_{m,k}$ is bounded on $[0, \infty)$ and that $f_m(z) = \sum_{k=0}^{m} f_{m,k}(z)$. From the linearity of $\tilde{R}_n$, we have

$\tilde{R}_n(f_m; q, z) = \sum_{k=0}^{m} c_k \tilde{R}_n(e_k; q, z)$ for all $|z| \leq r$.

It is suffices to prove that $\lim_{m\to\infty} \tilde{R}_n(f_m; q, z) = \tilde{R}_n(f; q, z)$. For any fixed $n \in \mathbb{N}$, $n \geq n_0$ and $|z| \leq r$. We have the following inequality for all $|z| \leq r$

$$\left|\tilde{R}_n(f_m; q, z) - \tilde{R}_n(f; q, z)\right| \leq \|f_m - f\|_r \widetilde{M}_{r,n,q}, \tag{2.5}$$

where $\widetilde{M}_{r,n,q} = \frac{1}{\prod_{s=0}^{n-1}(1-q^s a_n r)} \sum_{j=0}^{n} q^{-j} q^{\frac{(j-1)j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q (a_n r)^j \left([j]_q + [j+1]_q\right) < \infty$.

Using 2.5, $\lim_{m\to\infty} \|f_m - f\|_r = 0$ and the $\|f_m - f\|_{B[0,\infty)} \leq \|f_m - f\|_r$, the proof of the lemma is completed.

Lemma 4. We have the following recurrence formula for the complex q-Balázs-Szabados-Kantorovich operators

$$\tilde{R}_n(e_{k+1}; q, z) = \frac{[k+1]_q}{[k+2]_q} \frac{(1+q^n a_n z) q z}{b_n(1+a_n z)} D_q[\tilde{R}_n(e_k; q, z)]$$

$$+ \frac{[k+1]_q}{[k+2]_q} \left\{ \frac{qz}{1+a_n z} + \frac{q}{b_n} \right\} \tilde{R}_n(e_k; q, z) + \frac{1}{[k+2]_q} R_n(e_{k+1}; q, z)$$

where R_n is q-Balázs-Szabados operators given in 1.1, $n \in \mathbb{N}$, $z \in \mathbb{C}$ and $k = 0,1,2, \ldots$.

Proof. Firstly, we calculate $D_q[\tilde{R}_n(e_k; q, z)]$

$$D_q[\tilde{R}_n(e_k; q, z)] = D_q\left[\frac{1}{\prod_{s=0}^{n-1}(1+q^s a_n z)}\right] b_n \sum_{j=0}^{n} q^{-j} q^{j(j-1)/2} \begin{bmatrix} n \\ j \end{bmatrix}_q (a_n z)^j \int_{\frac{[j]_q}{b_n}}^{\frac{[j+1]_q}{b_n}} t^k d_q t$$

$$+ \frac{b_n}{\prod_{s=0}^{n-1}(1+q^{s+1} a_n z)} \sum_{j=0}^{n} q^{-j} q^{j(j-1)/2} \begin{bmatrix} n \\ j \end{bmatrix}_q (a_n)^j z^{j-1} [j]_q \int_{\frac{[j]_q}{b_n}}^{\frac{[j+1]_q}{b_n}} t^k d_q t \tag{2.6}$$

From the fundamental theorem of calculus, we calculate

$$[j]_q \int_{\frac{[j]_q}{b_n}}^{\frac{[j+1]_q}{b_n}} t^k d_q t = \frac{[j]_q\left([j+1]_q^{k+1} - [j]_q^{k+1}\right)}{b_n^{k+1}[k+1]_q}$$

$$= \frac{[j]_q [j+1]_q^{k+1} - [j]_q^{k+2}}{b_n^{k+1}[k+1]_q}$$

$$= \frac{\frac{[j+1]_q - 1}{q}[j+1]_q^{k+1} - [j]_q^{k+2}}{b_n^{k+1}[k+1]_q}$$

$$= \frac{\frac{[j+1]_q^{k+2}}{q} - \frac{[j+1]_q^{k+1}}{q} - [j]_q^{k+2}}{b_n^{k+1}[k+1]_q}$$

$$= \frac{\frac{[j+1]_q^{k+2}}{q} - \frac{[j]_q^{k+2}}{q} - \frac{[j+1]_q^{k+1}}{q} + \frac{[j]_q^{k+1}}{q} + \frac{[j]_q^{k+2}}{q} - [j]_q^{k+2} - \frac{[j]_q^{k+1}}{q}}{b_n^{k+1}[k+1]_q}$$

$$= \frac{[k+2]_q}{[k+1]_q}\frac{b_n}{q}\frac{[j+1]_q^{k+2} - [j]_q^{k+2}}{b_n^{k+2}[k+2]_q} - \frac{1}{q}\frac{[j+1]_q^{k+1} - [j]_q^{k+1}}{b_n^{k+1}[k+1]_q} - \frac{\left(\frac{1}{q}-1\right)[j]_q^{k+2} - \frac{1}{q}[j]_q^{k+1}}{b_n^{k+1}[k+1]_q}$$

$$= \frac{[k+2]_q}{[k+1]_q}\frac{b_n}{q}\frac{[j+1]_q^{k+2} - [j]_q^{k+2}}{b_n^{k+2}[k+2]_q} - \frac{1}{q}\frac{[j+1]_q^{k+1} - [j]_q^{k+1}}{b_n^{k+1}[k+1]_q} - \frac{q^j[j]_q^{k+1}}{qb_n^{k+1}[k+1]_q}$$

$$= \frac{[k+2]_q}{[k+1]_q}\frac{b_n}{q}\int_{\frac{[j]_q}{b_n}}^{\frac{[j+1]_q}{b_n}} t^{k+1} d_q t - \frac{1}{q}\int_{\frac{[j]_q}{b_n}}^{\frac{[j+1]_q}{b_n}} t^k d_q t - \frac{q^j[j]_q^{k+1}}{qb_n^{k+1}[k+1]_q} \tag{2.7}$$

Applying 2.7 in 2.6, we obtain

$$D_q[\tilde{R}_n(e_k;q,z)] = -\frac{b_n}{1+q^n a_n z}\tilde{R}_n(e_k;q,z) + \frac{[k+2]_q}{[k+1]_q}\frac{b_n(1+a_n z)}{qz(1+q^n a_n z)}\tilde{R}_n(e_{k+1};q,z)$$
$$-\frac{1+a_n z}{qz(1+q^n a_n z)}\tilde{R}_n(e_k;q,z) - \frac{1}{[k+1]_q}\frac{b_n(1+a_n z)}{qz(1+q^n a_n z)}R_n(e_{k+1};q,z)$$

(2.8)

Arranging 2.8, the desired recurrence formula is obtained.

Lemma 5 ([9]). Let $n_0 \geq 2$, $0 < \beta \leq \frac{2}{3}$ and $\frac{1}{2} < r < R \leq [n_0]_q^{1-\beta}$. For all $n \geq n_0$, $|z| \leq r$ and $k = 0,1,2,...$, we have $|R_n(e_k;q,z)| \leq k!\,(20r)^k$.

Considering Corallary 1.10.4 in [5] (or Corallary 1 in [9]) and by the mean value theorem [7] in complex analysis, we have the following corollary.

Corollary 1. Let $f(z) = \frac{p_k(z)}{\prod_{j=1}^{k}(z-a_j)}$ where $p_k(z)$ is a polynomial of degree $\leq k$, and we suppose that $|a_j| \geq R > 1$ for all $j = 1,2,...,k$. If $1 \leq r < R$, then for all $|z| \leq r$ we have

$$|D_q[f(z)]| \leq \frac{R+r}{R-r}\frac{k}{r}\|f\|_r.$$

Lemma 6. Let $n_0 \geq 2$, $0 < \beta \leq \frac{2}{3}$ and $\frac{1}{2} < r < R \leq [n_0]_q^{1-\beta}$. For all $n \geq n_0$, $|z| \leq r$ and $k = 0,1,2,...$, we have $|\tilde{R}_n(e_k;q,z)| \leq kk!\,(20r)^k$.

Proof. Taking the absolute value of the recurrence formula in Lemma 4 and using the triangle inequality, we get

$$|\tilde{R}_n(e_{k+1};q,z)| \leq \frac{[k+1]_q}{[k+2]_q}\frac{(1+q^n a_n r)qr}{b_n(1-a_n r)}|D_q[\tilde{R}_n(e_k;q,z)]|$$
$$+\frac{[k+1]_q}{[k+2]_q}\left(\frac{qr}{1-a_n r}+\frac{q}{b_n}\right)|\tilde{R}_n(e_k;q,z)| + \frac{1}{[k+2]_q}|R_n(e_{k+1};q,z)|$$

From the hypothesis of the lemma, we have $1 < 2r$, $\frac{1}{1-a_n r} < 2$, $1 + q^n a_n r < \frac{3}{2}$ and $\frac{1}{b_n} < 1$, which implies

$$|\tilde{R}_n(e_{k+1};q,z)| \leq 3r|D_q[\tilde{R}_n(e_k;q,z)]| + 4r|\tilde{R}_n(e_k;q,z)| + |R_n(e_{k+1};q,z)| \tag{2.9}$$

Considering Corollary 1, under the condition $r < [n_0]_q^{1-\beta}$, it holds $\frac{[n_0]_q^{1-\beta}+\mathrm{r}}{[n_0]_q^{1-\beta},-\mathrm{r}}$<3, which implies

$$\left|D_q[\tilde{R}_n(e_k;q,z)]\right| \le \frac{3k}{r}\left\|\tilde{R}_n(e_k;q,.)\right\|_r. \tag{2.10}$$

Applying 2.10 and Lemma 5 in 2.9, we get

$\left|\tilde{R}_n(e_{k+1};q,z)\right| \le 20r(k+1)\left\|\tilde{R}_n(e_k;q,.)\right\|_r+(k+1)!\,(20r)^{k+1}.$

Taking step by step $k = 0,1,2$ ..., we obtain

$\left|\tilde{R}_n(e_k;q,z)\right| \le kk!\,(20r)^k,$

which complete the proof.

3. CONVERGENCE RESULTS AND UPPER ESTIMATE

Now, we give the following convergence theorem and upper estimate for the complex q-Balázs-Szabados-Kantorovich operators.

Theorem 1. Let (q_n) be a sequence satisfying the conditions 2.4 with $q_n \in (0,1]$ for all, $n \in \mathbb{N}$, and let $n_0 \ge 2$, $0 < \beta \le \frac{2}{3}$ and $\frac{1}{2} < r < R \le [n_0]_{q_n}^{1-\beta}$. If $f: D_R \cup [R,\infty) \to \mathbb{C}$ is uniformly continuous, bounded on $[0,\infty)$ and analytic in D_R and there exist M > 0, $0 < A < \frac{1}{20r}$ with $|c_k| \le M\frac{A^k}{k!}$ (which implies $|f(z)| \le Me^{A|z|}$ for all $z \in D_R$), then the sequence $\left(\tilde{R}_n(e_k;q_n,.)\right)_{n\ge n_0}$ is uniformly convergent to f in D_R.

Proof. Using Lemma 3 and Lemma 6, we have for all $n \ge n_0$ and $|z| \le r$

$$\left|\tilde{R}_n(f;q_n,z)\right| \le \sum_{k=0}^{\infty}|c_k|\left|\tilde{R}_n(e_k;q_n,z)\right| \le M\sum_{k=0}^{\infty}k(20rA)^k,$$

where the series $\sum_{k=0}^{\infty}k(20rA)^k$ is convergent for $0 < A < \frac{1}{20r}$. From Lemma 2, since $\lim_{n\to\infty}\tilde{R}_n(f;q_n,x) = f(x)$ for all $x \in [0,r]$, by Vitali theorem (see [11], Theorem 3.2.10, p. 112), it follows that $\left(\tilde{R}_n(f;q_n,.)\right)$converges uniformly to f in $\overline{D}_r$ for $n \ge n_0$.

Theorem 2. Let (q_n) be a sequence satisfying the conditions 2.4 with $q_n \in (0,1]$ for all $n \in \mathbb{N}$, and let $n_0 \ge 2$, $0 < \beta \le \frac{2}{3}$ and $\frac{1}{2} < r < R \le [n_0]_{q_n}^{1-\beta}$. If $f: D_R \cup [R,\infty) \to \mathbb{C}$ is uniformly continuous, bounded on $[0,\infty)$ and analytic in D_R and there exist M > 0, $0 < A < \frac{1}{20r}$ with $|c_k| \le M\frac{A^k}{k!}$ (which implies $|f(z)| \le Me^{A|z|}$ for all $z \in D_R$), then we have

$$\left|\tilde{R}_n(f;q_n,z) - f(z)\right| \le C_r(f)\left(a_n + \frac{1}{b_n}\right).$$

where $C_r(f) = C\frac{M}{A}\sum_{k=1}^{\infty}(k-1)k(20rA)^{k+1}$ and $\sum_{k=1}^{\infty}(k-1)k(20rA)^{k+1} < \infty$.

Proof. Using the recurrence formula in Lemma 4, we have

$$\begin{aligned}\tilde{R}_n(e_{k+1};q,z) - z^{k+1} &= \frac{[k+1]_q}{[k+2]_q}\frac{(1+q^n a_n z)qz}{b_n(1+a_n z)}D_q[\tilde{R}_n(e_k;q,z)]\\ &\quad+\frac{[k+1]_q}{[k+2]_q}\frac{qz}{1+a_n z}[\tilde{R}_n(e_k;q,z)-z^k]+\frac{q[k+1]_q}{b_n[k+2]_q}\tilde{R}_n(e_k;q,z)\\ &\quad+\frac{1}{[k+2]_q}[R_n(e_{k+1};q,z)-z^{k+1}]+S_{k,n,q}(z),\end{aligned}$$

where $S_{k,n,q}(z) := \left(\frac{1}{[k+2]_q}-1\right)\frac{a_n}{1+a_n z}z^{k+2}$. Taking absolute value for $|z| \le r$, we obtain

$$\left|\tilde{R}_n(e_{k+1};q,z)-z^{k+1}\right| \le \frac{(1+q^n a_n r)qr}{b_n(1-a_n r)}\left|D_q[\tilde{R}_n(e_k;q,z)]\right| + \frac{qr}{1-a_n r}\left|\tilde{R}_n(e_k;q,z)-\quad z^k\right|$$

$$+\frac{q}{b_n}\left|\tilde{R}_n(e_k;q,z)\right|+\left|R_n(e_{k+1};q,z)-z^{k+1}\right|+\frac{2a_n}{1-a_n r}r^{k+2}.$$

From the hypothesis of the theorem, we have $a_n r<\frac{1}{2}$, $\frac{1}{1-a_n r}<2$ and $1+q^n a_n r<\frac{3}{2}$, using 2.10, we can write

$$\left|R_n(e_{k+1};q,z)-z^{k+1}\right| \le \frac{9(k+1)}{b_n}\left\|\tilde{R}_n(e_k;q,.)\right\|_r + 2r\left|\tilde{R}_n(e_k;q,z)-\quad z^k\right|$$

$$+\left|R_n(e_{k+1};q,z)-z^{k+1}\right| + 4a_n r^{k+2}$$

Applying the following inequality given in [9] with (11)

$$\left|R_n(e_{k+1};q,z)-z^{k+1}\right| \le \frac{9}{b_n}k(k+1)!\,(20r)^k + 2a_n r^2(k+1)(2r)^k,$$

and Lemma 6 in 3.1, we get

$$\left|\tilde{R}_n(e_{k+1};q,z)-z^{k+1}\right| \le C\left(a_n+\frac{1}{b_n}\right)\,k(k+1)!\,(20r)^{k+2} + 2r\left|\tilde{R}_n(e_k;q,z)-\quad z^k\right|.$$

Taking step by step $k=0,1,2,\dots$, we arrive

$$\left|\tilde{R}_n(e_k;q,z)-\quad z^k\right| \le C\left(a_n+\frac{1}{b_n}\right)(k-1)k!\ (20r)^{k+1}.$$

Choosing $q=(q_n)$ and $C_r(f)=C\frac{M}{A}\sum_{k=1}^{\infty}(k-1)k(20rA)^{k+1}$, we obtain

$$\left|\tilde{R}_n(f;q_n,z)-f(z)\right| \le \sum_{k=0}^{\infty}|c_k|\left|\tilde{R}_n(e_k;q_n,z)-\quad z^k\right|$$

$$\le \left(a_n+\frac{1}{b_n}\right)C\frac{M}{A}\sum_{k=1}^{\infty}(k-1)k(20rA)^{k+1},$$

which is the desired result.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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