On The Norms Of Circulant Matrices With The Complex Fibonacci And Lucas Numbers

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ABSTRACT

In this paper, we compute the norms of circulant matrices with the complex Fibonacci and Lucas numbers. Moreover, we give golden ratio in complex Fibonacci numbers.

Keywords: _Circulant matrix, Complex Fibonacci numbers, Matrix norm._

1. INTRODUCTION

In some scientific areas such as signal processing, coding theory and image processing, we often encounter circulant matrices. An $n \times n$ matrix C is called a circulant matrix if it is of the form

$$C = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & \cdots & c_{n-3} \\ \vdots & \vdots & & \vdots \\ c_1 & c_2 & \cdots & c_0 \end{bmatrix}$$

or an $n \times n$ matrix C is circulant if there exist $c_0, c_1, \ldots, c_{n-1}$ such that the i, j entry of C is $c_{(j-i) \bmod n}$, where the rows and columns are numbered from 0 to $n-1$ and $k \bmod n$ means the number between 0 to $n-1$ that is congruent to $k \bmod n$. Thus, we denote the circulant matrix C as $C = Circ(c_0, c_1, \ldots, c_{n-1})$. Any circulant matrix has many elegant properties. Some of them are [6,12]:

1. Let A be an $n \times n$ matrix. Then A is a circulant if and only if

   $$A\pi = \pi A$$

   where the matrix $\pi = Circ(0,1,\ldots,0)$.

2. $Circ(c_0, c_1, \ldots, c_{n-1}) = c_0 I + c_1 \pi + \cdots + c_{n-1}\pi^{n-1}$.

3. All circulants of the same order commute. If C is a circulant so is C^*. Hence C and C^* commute and therefore all circulants are normal matrices, where C^* is conjugate transpose of C.

4. If C is an invertible circulant matrix so is C^{-1}.

5. If C is a circulant matrix, then the eigenvalues of C are

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$$\lambda_m = \sum_{k=0}^{n-1} c_k w^{-mk} \qquad (1)$$
$$_{0 \le m \le n-1}$$

where $w = e^{\frac{2\pi i}{n}}$ and $i = \sqrt{-1}$, and the corresponding eigenvectors are

$$x_m = \left(1, w^m, w^{2m}, \ldots, w^{(n-1)m}\right)^T .$$
$$_{0 \le m \le n-1}$$

In the past decades, circulant matrices have been an important research area and many authors have dealt with circulant matrices. Hladnik [7] has given a formula for Schur norm of a block circulant matrix with circulant blocks. Karner et al. [12] have studied on spectral decompositions and singular value decompositions of four types of real circulant matrices. Bose and Mitra [4] have derived the limiting spectral distribution of a particular variant of a circulant random matrix. Atkin et al. [2] have studied the powers of a circulant. Zhang et al. [21] have worked on the minimal polynomials and inverses of a block circulant matrices over a field. Also, there have been several papers on the circulant matrices with famous numbers. Solak [16,17] has given some bounds for the spectral and Euclidean norms of the matrices with Fibonacci and Lucas numbers. Civciv and Türkmen [5] have defined the circulant matrix with the Lucas number and computed lower and upper bounds for the Euclidean and spectral norms of this matrix. Bahsi and Solak [3] have calculated eigenvalues, determinant, spectral norm, Euclidean norm of circulant matrix with arithmetic progression. Tuğlu and Kızılateş [19] have studied norms of some circulant matrices and some special matrices, which entries consist of harmonic Fibonacci numbers. Kocer and et al. [13] have studied circulant and semicirculant matrices with Horadam numbers. Shen and Cen [14] have given upper and lower bounds for the spectral norms of r-circulant matrices in the forms

$$A = Circ\left(F_0, F_1, \ldots, F_{n-1}\right), \ \ B = Circ\left(L_0, L_1, \ldots, L_{n-1}\right).$$

Solak and Bozkurt [18] have established upper bounds for the l_p norms of the matrix almost circulant matrix

$$C_n = Circ\left(a, 1, \frac{1}{2}, \ldots, \frac{1}{n-1}\right), \text{ where } a \in \mathbb{R} \ (\mathbb{R} \text{ denotes the}$$

set of real numbers) and $a \ne 0$. Ipek [11] has obtained the equality for the Solak's work in [16]. Shen and et al. [15] have given the determinant formulae for circulant matrices with Fibonacci and Lucas numbers. Jiang, Xin and Lu [10] have examined some types of circulant matrices whose entries are Gaussian Fibonacci numbers. Resently, Altınışık et al. [1] have dealt with determinant and inverse of circulant matrices associated with complex Fibonacci numbers.

In this paper, we give some relations between golden ratio and complex Fibonacci numbers and we compute Euclidean and spectral norms of circulant matrices with the complex Fibonacci and Lucas numbers in section 3. For these, we give some preliminaries, definitions and lemmas related to our study in Section 2.

2. PRELIMINARIES

Fibonacci numbers defined by the recurrence relation

$$F_{n+1} = F_n + F_{n-1} \ \left(n \ge 1\right), \ F_0 = 0 \text{ and } F_1 = 1$$

have many applications to different fields such as mathematics, statistics and physics. The Lucas numbers are defined by $L_{n+1} = L_n + L_{n-1} \ \left(n \ge 1\right), \ L_0 = 2$ and $L_1 = 1$. There are some elementary identities for these famous numbers. Some of them are [20]:

$$\sum_{s=0}^{n-1} F_s = F_{n+1} - 1, \ F_s^2 + F_{s+1}^2 = F_{2s+1}, \ \sum_{s=0}^{n-1} F_{2s+1} = F_{2n} \quad (2)$$

$$\sum_{s=0}^{n-1} L_s = L_{n+1} - 1, \ L_s^2 + L_{s+1}^2 = 5F_{2s+1}, \ L_{s-1} + L_{s+1} = 5F_s . \ (3)$$

The Binet's formula for Fibonacci and Lucas numbers are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } L_n = \alpha^n + \beta^n$$

where

$$\alpha = \frac{1 + \sqrt{5}}{2}, \ \ \beta = \frac{1 - \sqrt{5}}{2}$$

that is, α and β are the roots of

$$x^2 - x - 1 = 0.$$

The root α is called golden ratio. Over the past five centuries, golden ratio has been very attractive for researchers because it occurrences ubiquitous such as nature, art, architecture, and anatomy. Well-known some relations for golden ratio, Fibonacci and Lucas numbers are [20]:

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \alpha \ \ \text{ and } \ \ \lim_{n \to \infty} \frac{F_{n+s}}{F_n} = \alpha^s \qquad (4)$$

$$\lim_{n \to \infty} \frac{L_{n+1}}{L_n} = \alpha \ \ \text{ and } \ \ \lim_{n \to \infty} \frac{L_{n+s}}{L_n} = \alpha^s. \qquad (5)$$

Fibonacci and Lucas numbers have many generalizations. One of them is called complex Fibonacci number defined by

$$F_n^* = F_n + iF_{n+1}, \qquad (6)$$

where $i = \sqrt{-1}$ and F_n is nth Fibonacci number [8]. Also, complex Fibonacci numbers have the recurrence relation

$$F_{n+1}^* = F_n^* + F_{n-1}^* \ \left(n \ge 1\right), \ F_0^* = i \text{ and } F_1^* = 1 + i.$$

Similarly, complex Lucas number defined by

$$L_n^* = L_n + iL_{n+1}, \qquad (7)$$

and

$$L_{n+1}^* = L_n^* + L_{n-1}^* \ (n \geq 1), \ L_0^* = 2+i \text{ and } L_1^* = 1+3i,$$

where $i = \sqrt{-1}$ and L_n is nth Lucas number [8].

Definition 1. Let $A = (a_{ij})$ be any $m \times n$ matrix. The spectral norm of A is

$$\|A\|_E = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}.$$

Definition 2. Let $A = (a_{ij})$ be any $m \times n$ matrix. The Euclidean norm of A is

$$\|A\|_2 = \sqrt{\max_{1 \leq i \leq n} \lambda_i(A^H A)},$$

where $\lambda_i(A^H A)$ are eigenvalues of $A^H A$ and A^H is the conjugate transpose of A.

3. MAIN RESULTS

We start with golden ratio in complex Fibonacci numbers to our main results.

Theorem 1. Let α, F_n^*, L_n^* be golden ratio, nth complex Fibonacci number and nth complex Lucas number, respectively. Then

$$a) \lim_{n \to \infty} \frac{F_{n+1}^*}{F_n^*} = \alpha$$

$$b) \lim_{n \to \infty} \frac{L_{n+1}^*}{L_n^*} = \alpha.$$

Proof. *a)* From the equalities in (4) and the properties of limit, we have

$$\lim_{n \to \infty} \frac{F_{n+1}^*}{F_n^*} = \lim_{n \to \infty} \frac{F_{n+1} + iF_{n+2}}{F_n + iF_{n+1}}$$

$$= \lim_{n \to \infty} \frac{F_{n+1}}{F_n + iF_{n+1}} + \lim_{n \to \infty} \frac{iF_{n+2}}{F_n + iF_{n+1}}$$

$$= \lim_{n \to \infty} \frac{1}{\dfrac{F_n}{F_{n+1}} + i} + \lim_{n \to \infty} \frac{i}{\dfrac{F_n}{F_{n+2}} + \dfrac{iF_{n+1}}{F_{n+2}}}$$

$$= \frac{1}{\dfrac{1}{\alpha} + i} + \frac{i}{\dfrac{1}{\alpha^2} + \dfrac{i}{\alpha}}$$

$$= \frac{\alpha}{1 + \alpha i} + \frac{\alpha^2 i}{1 + \alpha i} = \alpha.$$

b) The proof is similar to proof of *a)*.

Now we give norms of circulant matrices with the complex Fibonacci and Lucas numbers. Our theorems

have two parts. We prove part *a)* of theorems because one can see easily that the part *b)* of theorems is true by using the method of the proof of part a) and equalities in (3).

Theorem 2. The Euclidean norms of the $n \times n$ circulant matrices $F = Circ\left(F_0^*, F_1^*, \ldots, F_{n-1}^*\right)$ and $L = Circ\left(L_0^*, L_1^*, \ldots, L_{n-1}^*\right)$ are

$$a) \|F\|_E = \sqrt{nF_{2n}}$$

$$b) \|L\|_E = \sqrt{5nF_{2n}}.$$

Proof. *a)* From the definition of Euclidean norm and the equalities in (2), we have

$$\|F\|_E^2 = n\sum_{s=0}^{n-1} |F_s^*|^2 = n\sum_{s=0}^{n-1} |F_s + iF_{s+1}|^2 = n\sum_{s=0}^{n-1} \left(F_s^2 + F_{s+1}^2\right)$$

$$= n\sum_{s=0}^{n-1} F_{2s+1}$$

$$= nF_{2n}.$$

Theorem 3. The spectral norms of the $n \times n$ circulant matrices $F = Circ\left(F_0^*, F_1^*, \ldots, F_{n-1}^*\right)$ and $L = Circ\left(L_0^*, L_1^*, \ldots, L_{n-1}^*\right)$ are

$$a) \|F\|_2 = \sqrt{F_{2n+3} - 2F_{n+3} + 2}$$

$$b) \|L\|_2 = \sqrt{5\left(F_{2n+3} - 2F_{n+3} + 2\right)}.$$

Proof. *a)* Since F is a circulant matrix, from (1) its eigenvalues are

$$\lambda_m = \sum_{s=0}^{n-1} F_s^* e^{\frac{-2\pi ims}{n}}.$$
$$\scriptstyle 0 \leq m \leq n-1$$

Then,

$$\lambda_0 = \sum_{s=0}^{n-1} F_s^* = \sum_{s=0}^{n-1} \left(F_s + iF_{s+1}\right) = \left(F_{n+1} - 1\right) + i\left(F_{n+2} - 1\right),$$

by using second equality in (2),

$$|\lambda_0| = \sqrt{\left(F_{n+1} - 1\right)^2 + \left(F_{n+2} - 1\right)^2}$$

$$= \sqrt{F_{n+1}^2 + F_{n+2}^2 - 2F_{n+1} - 2F_{n+2} + 2}$$

$$= \sqrt{F_{2n+3} - 2F_{n+3} + 2}. \qquad (8)$$

Also,

$$|\lambda_m| = \left|\sum_{s=0}^{n-1} F_s^* e^{\frac{-2\pi ims}{n}}\right| \leq \sum_{s=0}^{n-1} \left|F_s^*\right| \left|e^{\frac{-2\pi ims}{n}}\right| \leq \left|\sum_{s=0}^{n-1} F_s^*\right| = |\lambda_0|. \quad (9)$$
$$\scriptstyle 1 \leq m \leq n-1$$

Since the matrix F is a normal matrix, we have

$$\|F\|_2 = \max_{0 \leq m \leq n-1} |\lambda_m| = \max\left(|\lambda_0|, \max_{1 \leq m \leq n-1} |\lambda_m|\right). \qquad (10)$$

From (8), (9) and (10), we have

$$\|F\|_2 = \sqrt{F_{2n+3} - 2F_{n+3} + 2}.$$

Thus the proof is completed.

Corollary 1. The norms of the $n \times n$ circulant matrices
$F = Circ\left(F_0^*, F_1^*, \ldots, F_{n-1}^*\right)$ and

$L = Circ\left(L_0^*, L_1^*, \ldots, L_{n-1}^*\right)$ hold

$a)\, \|L\|_E = \sqrt{5}\,\|F\|_E$

$b)\, \|L\|_2 = \sqrt{5}\,\|F\|_2$.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] Altınışık, E., Yalçın, N.F. and Büyükköse, Ş., "Determinants and inverses of circulant matrices with complex Fibonacci numbers", Special Matrices, 3:82-90 (2015).

[2] Atkin A.O.L., Boros,E., Cechlarova, K.U. and N. Peled, "Powers of Circulants in Bottlenack Algebra", Linear Algebra and Its Appl., 258: 137-148 (1997).

[3] Bahsi, M. and Solak, S., "On the circulant matrices with arithmetic sequence", Int. J. Cont. Math. Sciences, 5(25): 1213 – 1222 (2010).

[4] Bose, A. and Mitra, J., "Limiting Spectral Distribution of a Special Circulant", Statistics & Prob. Let., 60: 111-120 (2002).

[5] Civciv, H. and Türkmen, R., "Notes on norms of circulant matrices with Lucas number", Int. Journal for Inf. and System Sci., 4(1): 142-147 (2008).

[6] Davis, P.J., Circulant Matrices, Wiley, New York, Chichester, Brisbane, 1979.

[7] Hladnik, M., "Schur Norms of Bicirculant Matrices", Linear Algebra and Its Appl., 286: 261-272 (1999).

[8] Horadam, A.F., "Complex Fibonacci numbers and Fibonacci quaternions", Amer. Math. Monthly 70: 289-291 (1963).

[9] Horn, R.A., Johnson, C.R., Matrix Analysis, Cambridge University Press, Cambridge, 1985.
.

[10] Jiang, Z., Xin, H. and Lu, F., "Gaussian Fibonacci circulant type matrices", Abstr. Appl. Anal., Art. ID 592782, 10 pp, (2014).

[11] Ipek, A., "On the spectral norms of circulant matrices with classical Fibonacci and Lucas numbers entries", Appl. Math. Comp. 217: 6011-6012 (2011).

[12] Karner, H., Schneid, J., and Ueberhuber, C.W., "Spectral Decomposition of Real Circulant Matrices", Linear Algebra and Its Appl., 367: 301-311 (2003).

[13] Kocer, E.G., Mansour T. and Tuglu N., "Norms of Circulant and Semicirculant Matrices with Horadam's Numbers", Ars Combinatoria 85:353-359 (2007).

[14] Shen, S. and Cen, J., "On the bounds for the norms of r- Circulant Matrices with the Fibonacci and Lucas Numbers", Appl. Math. Comput., 216: 2891-2897 (2010).

[15] Shen, S.Q., Cen, J.M. and Hao, Y., "On the determinants and inverses of circulant matrices with and Lucas numbers", Appl. Math. Comput., 217: 9790-9797 (2011).

[16] Solak, S., "On the norms of circulant matrices with the Fibonacci and Lucas numbers", Appl. Math. Comp. 160: 125-132 (2005).

[17] Solak, S., Erratum to "On the Norms of Circulant Matrices with the Fibonacci and Lucas Numbers" [Appl. Math. Comp., 160, (2005), 125-132], Appl. Math. Comp., 190: 1855-1856 (2007).

[18] Solak, S. and Bozkurt, D., "Some bounds on l_p matrix and l_p operator norms of almost circulant, Cauchy-Toeplitz and Cauchy-Hankel matrices", Math. Comp. Appl. Int. J., 7(3): 211-218 (2002).

[19] Tuğlu, N. and Kızılateş C., "On the Norms of Some Special Matrices with the Harmonic Fibonacci Numbers", Gazi University Journal of Science, 28(3): 497-501 (2015).

[20] Vajda, S., Fibonacci & Lucas Numbers, and the Golden Section. West Sussex, England: Ellis Horwood Limited, 1989.

[21] Zhang, S., Jiang, Z. and Liu, S., "An Application of the Gröbner Basis in Computation for the Minimal Polynomials and Inverses of Block Circulant Matrices", Linear Algebra and Its Appl., 347: 101-114 (2002).