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# On product of Fuzzy Semiprime ideals in $\Gamma$-LA-Semigroups

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**ABSTRACT**

The purpose of this paper is to introduce the notion of a weakly fuzzy quasi-semiprime ideals in $\Gamma$-LA-semigroups, we study direct product of fuzzy semiprime, fuzzy weakly completely semiprime, weakly fuzzy semiprime and weakly fuzzy quasi-semiprime ideals in $\Gamma$-LA-semigroups. Some characterizations of weakly fuzzy semiprime and weakly fuzzy quasi-semiprime ideals are obtained. Moreover, we investigate relationships between fuzzy weakly completely semiprime and weakly fuzzy quasi-semiprime ideals in $\Gamma$-LA-semigroups

***Key words:****fuzzy semiprime, fuzzy quasi-semiprime, fuzzy weakly completely semiprime, weakly fuzzy semiprime, weakly fuzzy quasi-semiprime*

## 1. INTRODUCTION

A left almost semigroup (LA-semigroup) is a generalization of semigroup theory with wide range of usages in theory of flocks [23]. The fundamentals of this non-associative algebraic structure were first discovered by Kazim and Naseeruddin (1972). A groupoid $S$ is called an LA-semigroup if it satisfies the left invertive law:

$$(ab)c = (cb)a$$

for all $a,b,c \in S.$ It is interesting to note that an LA-semigroup with right identity becomes a commutative monoid [21]. This structure is closely related to a commutative semigroup. Because of containing a right identity, an LA-semigroup becomes a commutative monoid [21]. A left identity in an LA-semigroup is unique [21]. It lies between a groupoid and a commutative semigroup with wide range of applications in theory of flocks [23]. Ideals in LA-semigroups have been discussed in [22]. Now we define the concepts that we will used. Let $S$ be an LA-semigroup. By an LA-subsemigroup of [20], we means a non-empty subset $A$ of $S$ such that $A^2 \subseteq A.$ A non-empty subset $A$ of an LA-semigroup $S$ is called a left (right) ideal of [18] if

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$SA \subseteq A(AS \subseteq A).$ By two-sided ideal or simply ideal, we mean a non-empty subset of an LA-semigroup $S$ which is both a left and a right ideal of $S.$ In 1981, the notion of $\Gamma$-semigroups was introduced by M. K. Sen. A groupoid is called an $\Gamma$-LA-semigroup if it satisfies the left invertive law:

$$(a\gamma b)\alpha c = (c\gamma b)\alpha a$$

for all $a,b,c \in S$ and $\gamma, \alpha \in \Gamma$ [26]. This structure is also known as an $\Gamma$-Abel-Grassmann's groupoid ($\Gamma$-AG-groupoid). In this paper, we are going to investigate some interesting properties of recently discovered classes, namely $\Gamma$-LA-semigroup always satisfies the $\Gamma$-medial law:

$$(a\gamma b)\alpha(c\beta d) = (a\gamma c)\alpha(b\beta d)$$

for all $a,b,c,d \in S$ and $\gamma, \alpha, \beta \in \Gamma$ [26], while an $\Gamma$-LA-semigroup with left identity always satisfies $\Gamma$-paramedial law:

$$(a\gamma b)\alpha(c\beta d) = (d\beta c)\alpha(b\gamma a)$$

for all $a,b,c,d \in S$ and $\gamma, \alpha, \beta \in \Gamma$ [26]. Recently T. Shah and I. Rehman have discussed $\Gamma$-Ideals and $\Gamma$-Bi-Ideals in $\Gamma$-LA-semigroups. An ideal $P$ of an $\Gamma$-LA-semigroup $S$ is called semiprime if $A^2 \subseteq P$ implies that either $A \subseteq P,$ for all ideal $A$ in $S.$ Q. Mushtaq and M. Khan defined the direct product of left (resp, right) ideals, prime ideals, maximal ideals and investigate the properties of such ideals [19].

The fundamental concept of fuzzy sets was first introduced by Zadeh [28] in 1965. Given a set $S,$ a fuzzy subset of $S$ is, by definition an arbitrary mapping $f: S \to [0,1],$ where $[0,1]$ is the unit interval. Kuroki initiated the theory of fuzzy bi ideals in semigroups [15]. The thought of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset was defined by Murali [17]. Recently, M. Khan et al. introduced the concept of fuzzy ideals and anti fuzzy ideals of LA-semigroups in this papers [27]. There are many mathematicians who added several results to the theory fuzzy $\Gamma$-LA-semigroups, see [2, 3, 26]. In this paper we characterize the fuzzy subset in $\Gamma$-LA-semigroup. We investigate the relationships between fuzzy weakly completely semiprime and weakly fuzzy quasi-semiprime $\Gamma$-ideals in $\Gamma$-LA-semigroups.

## 2. PRELIMINARIES

Let $S$ be an $\Gamma$-LA-semigroup. A nonempty subset $A$ of $S$ is called a left $\Gamma$-ideal of $S$ if $S\Gamma A \subseteq A.$ $A$ is called a right $\Gamma$-ideal of $S$ if $A\Gamma S \subseteq A$ and $A$ is called an $\Gamma$-ideal of $S$ if $A$ is both a left and a right $\Gamma$-ideal of $S.$ A function $f$ from $S$ to the unit interval $[0,1]$ is a fuzzy subset of $S.$ The $\Gamma$-LA-semigroup $S$ itself is a fuzzy subset of $S$ such that $S(x) = 1$ for all $x \in S,$ denoted also by $S.$ Let $f$ and $g$ be two fuzzy subsets of $S.$ Then the inclusion relation $f \subseteq g$ is defined $f(x) \le g(x),$ for all $x \in S.$ $f \cap g$ and $f \cup g$ are fuzzy subsets of $S$ defined by

$$(f \cap g)(x) = min\{f(x), g(x)\},$$

$$(f \cup g)(x) = max\{f(x), g(x)\}$$ for all $x \in S.$

The product $f\Gamma g$ is defined as follows;

$$(f\Gamma g)(y) = \begin{cases} sup[min\{f(y), g(z)\}] & ;\text{if there exist } y,z \in S, \text{such that } x = yz \\ 0 & ;\text{otherwise.} \end{cases}$$

for all $x, y \in S, \gamma \in \Gamma,$ if $f$ is both fuzzy left and right $\Gamma$-ideal of $S,$ then $f$ is called a fuzzy $\Gamma$-ideal of $S$ [24]. It is easy that $f$ is a fuzzy $\Gamma$-ideal of $S$ if and only if $f(x\gamma y) \ge max\{f(x), f(y)\}$ for all $x, y \in S, \gamma \in \Gamma$ and any fuzzy left (right) $\Gamma$-ideal of $S$ is a fuzzy sub$\Gamma$-LA-semigroup of $S.$ Equivalently, We can prove easily that $A$ is a (left, right) $\Gamma$-ideal of $S$ if and only if the characteristic function $f_A$ of $A$ is a

A fuzzy subset $f$ of $S$ is called a fuzzy sub$\Gamma$-LA-semigroup of $S$ if

$$f(x\gamma y) \ge min\{f(x), f(y)\}$$

for all $x, y \in S, \gamma \in \Gamma,$ and is called a fuzzy left (right) $\Gamma$-ideal of $S$ if

$$f(x\gamma y) \ge f(y) (f(x\gamma y) \ge f(x))$$

fuzzy (left, right) $\Gamma$ -ideal of $S$ [6].

**Lemma 2.1.** [6, 24] If $S$ is an $\Gamma$ -LA-semigroup and $f, g, h$ are fuzzy subsets of $S,$ then $(f\Gamma g)\Gamma h = (h\Gamma g)\Gamma f.$

**Proof.** The proof is available in [6, 24].

**Lemma 2.2.** [6, 24] If $S$ is an $\Gamma$ -LA-semigroup with left identity and $f, g, h, k$ are fuzzy subsets of $S,$ then

1. $f\Gamma(g\Gamma h) = g\Gamma(f\Gamma h);$
2. $(f\Gamma g)\Gamma(h\Gamma k) = (k\Gamma h)\Gamma(g\Gamma f).$

**Proof.** The proof is available in [6, 24].

**Lemma 2.3.** [6, 24] Let $f$ be a fuzzy subset of an $\Gamma$ -LA-semigroup $S.$ Then the following properties hold.

1. $f$ is a fuzzy sub $\Gamma$ -LA-semigroup of $S$ if and only if $f\Gamma f \subseteq f.$
2. $f$ is a fuzzy left $\Gamma$ -ideal of $S$ if and only if $S\Gamma f \subseteq f.$
3. $f$ is a fuzzy right $\Gamma$ -ideal of $S$ if and only if $f\Gamma S \subseteq f.$
4. $f$ is a fuzzy $\Gamma$ -ideal of $S$ if and only if $S\Gamma f \subseteq f$ and $f\Gamma S \subseteq f.$

**Proof.** The proof is available in [6, 24].

**Lemma 2.4.** [6] Let $f$ be a fuzzy left ideal of an $\Gamma$ -LA-semigroup $S.$ Then

1. $S\Gamma S = S.$
2. $S\Gamma f = f.$

**Proof.** The proof is available in [6].

**Definition 2.5.** A fuzzy subset $f$ of an $\Gamma$ -LA-semigroup $S$ is called fuzzy quasi-semiprime if for any fuzzy left $\Gamma$ -ideal $g$ of $S$ such that $g\Gamma g \subseteq f$ implies $g \subseteq f.$

**Definition 2.6.** A fuzzy subset $f$ of an $\Gamma$ -LA-semigroup $S$ is called fuzzy semiprime of $S$ if for any fuzzy $\Gamma$ -ideal $g$ of $S$ such that $g\Gamma g \subseteq f$ implies $g \subseteq f.$

It is easy to see that every fuzzy semiprime $\Gamma$ -ideal is fuzzy quasi-semiprime.

**Definition 2.7.** A fuzzy subset $f$ of an $\Gamma$ -LA-semigroup of $S$ is called fuzzy weakly completely semiprime if $f(x) \geq f(x^2),$ for all $x \in S.$

**Lemma 2.8.** A fuzzy $\Gamma$ -ideal $f$ of an $\Gamma$ -LA-semigroup of $S$ is fuzzy weakly completely semiprime if and only if $f(x) = f(x^2),$ for all $x \in S.$

**Proof.** It is straightforward by Definition 2.7.

**Theorem 2.9.** Let $S$ be an $\Gamma$ -LA-semigroup. Then $f$ is fuzzy sub $\Gamma$ -LA-semigroup of $S$ if and only if $1-f$ is fuzzy weakly completely semiprime.

**Proof.** $(\Longrightarrow)$ Assume that $f$ is a fuzzy sub $\Gamma$ -LA-semigroup of $S.$ Since $f(x^2) \geq f(x),$ we have $1-f(x^2) \leq 1-f(x),$ for all $x \in S.$ Then $1-f$ is fuzzy weakly completely semiprime.

$(\Longleftarrow)$ Suppose that $1-f$ is fuzzy weakly completely semiprime of $S.$ Since

$$1-f(x) \geq 1-f(x^2),$$

we have $f(x^2) \geq f(x),$ for all $x \in S.$ Hence $f$ is a fuzzy sub $\Gamma$ -LA-semigroup of $S.$

**Theorem 2.10.** Let $S$ be an $\Gamma$ -LA-semigroup. If $P_i, i \in I$ are fuzzy weakly completely semiprime subsets of $S,$ then $\bigcup_{i\in I} P_i$ is fuzzy weakly completely semiprime subset of $S.$

**Proof.** Suppose that $P_i, i \in I$ are fuzzy weakly completely semiprime subset of $S.$ Then $P_i(x^2) \leq P_i(x),$ for all $x \in S,$ and for $i \in I.$ Since $\bigcup_{i\in I} P_i(x) \geq P_i(x^2),$ for all $i \in I,$ we get

$\bigcup_{i\in I} P_i(x) \geq \bigcup_{i\in I} P_i(x^2)$. Hence $\bigcup_{i\in I} P_i$ is a fuzzy weakly completely semiprime subset of $S$.

**Theorem 2.11.** [24] Let $I$ be a non-empty subset of an $\Gamma$-LA-semigroup $S$ and $f_I : S \to [0,1]$ be a fuzzy subset of $S$ such that

$$f_I(x) = \begin{cases} 1; x \in I \\ 0; x \notin I. \end{cases}$$

Then $I$ is a left $\Gamma$-ideal (right $\Gamma$-ideal, $\Gamma$-ideal) of $S$ if and only if $f_I$ is a fuzzy left $\Gamma$-ideal (resp. fuzzy right $\Gamma$-ideal, fuzzy $\Gamma$-ideal) of $S$.

**Proof.** The proof is available in [24].

**Theorem 2.12.** Let $I$ be an $\Gamma$-ideal (left, right $\Gamma$-ideal) of an $\Gamma$-LA-semigroup $S, m \in (0,1]$. If $f_I$ is fuzzy set of $S$ such that

$$f_I(x) = \begin{cases} m; x \in I \\ 0; x \notin I, \end{cases}$$

then $f_I$ is a fuzzy $\Gamma$-ideal (fuzzy left, fuzzy right $\Gamma$-ideal) $S$.

**Proof.** It is straightforward by Theorem 2.11.

**Definition 2.13.** [24] Let $S$ be an $\Gamma$-LA-semigroup, $x \in S$ and $t \in [0,1]$. A fuzzy point $x_t$ of $S$ is defined by the rule that

$$x_t(y) = \begin{cases} t; x = y \\ 0; x \neq y. \end{cases}$$

It is accepted that $x_t$ is a mapping from $S$ into $[0,1]$, then a fuzzy point of $S$ is a fuzzy subset of $S$. For any fuzzy subset $f$ of $S$, we also denote $x_t \subseteq f$ by $x_t \in f$ in sequel. Let $tf_A$ be a fuzzy subset of $S$ defined as follows:

$$tf_A(x) = \begin{cases} t \in (0,1]; x \in A \\ 0 \qquad ; x \notin A. \end{cases}$$

**Lemma 2.14.** Let $A$ be a subset of an $\Gamma$-LA-semigroup $S$ and $f$ be a fuzzy set of $S$. Then the following statements are equivalent

1. $tg_A \subseteq f, t \in [0,1]$
2. $A \subseteq f_t, t \in [0,1]$.

**Proof.** It is straightforward by Definition 2.13.

**Definition 2.15.** A fuzzy subset $f$ of $S$ is said to be a weakly fuzzy semiprime if $tg_A \Gamma tg_A \subseteq f$ implies $tg_A \subseteq f$, for the $\Gamma$-ideal $A$ in $S$ and for all $t \in (0,1]$.

**Definition 2.16.** A fuzzy subset $f$ of $S$ is said to be a weakly fuzzy quasi-semiprime if $tg_A \Gamma tg_A \subseteq f$ implies $tg_A \subseteq f$, for the left $\Gamma$-ideal $A$ in $S$ and for all $t \in (0,1]$.

It is easy to see that every weakly fuzzy semiprime is weakly fuzzy quasi-semiprime.

## 3. FUZZY QUASI-SEMIPRIME $\Gamma$-IDEALS OF $\Gamma$-SEMIGROUPS

The results of the following lemmas seem to play an important role to study fuzzy semiprime $\Gamma$-ideals in $\Gamma$-LA-semigroups; these facts will be used frequently and normally we shall make no reference to this lemma.

**Lemma 3.1.** Let $A, B$ be any non-empty subset of an $\Gamma$-LA-semigroup $S$. Then for any $t \in (0,1]$ the following statements are true.

1. $tf_A \Gamma tf_B = tf_{A\Gamma B}$.
2. $tf_A \cap tf_B = tf_{A\cap B}$.
3. $tf_A \cup tf_B = tf_{A\cup B}$.
4. $tf_A = \bigcup_{a\in A} a_t$.
5. $S\Gamma tf_A = tf_{S\Gamma A}$, $tf_A \Gamma S = tf_{A\Gamma S}$ and $S\Gamma(tf_A \Gamma S) = tf_{S\Gamma(A\Gamma S)}$.
6. If $A$ is a left $\Gamma$-ideal (right, $\Gamma$-ideal) of

$S$, then $tf_A$ is a fuzzy left $\Gamma$-ideal (fuzzy left, fuzzy $\Gamma$-ideal) of $S$.

**Proof.** 1. If $x \in A\Gamma B$, then $tf_{A\Gamma B}(x) = t$, and $x = a\gamma b$, for some $a \in A, b \in B$ and $\gamma \in \Gamma$. Thus $tf_A \Gamma tf_B(x) = sup(min\{tf_A(a), tf_B\}) = sup(min\{t,t\}) = t$. If $x \notin A\Gamma B$, then $f_{A\Gamma B}(x) = 0$. We now prove that $(tf_A \Gamma tf_B)(x) = 0$. If $x \neq y\gamma z$, then

$$(tf_A \Gamma tf_B)(x) = 0,$$

and $(tf_A \Gamma tf_B)(x) = tf_{A\Gamma B}(x)$. If $x = y\gamma z$ and $y \in A$ and $z \in B$, then $y\gamma z \in A\Gamma B$, so $x \in A\Gamma B$, which is impossible. Thus $y \notin A$ or $z \notin B$. If $y \notin A$, then $tf_A(y) = 0$. Since $tf_B(z) \geq 0$, we have $min\{tf_A(y), tf_B(z)\} = 0$. If $z \notin B$ then, as in the previous case, we also have $min\{tf_A(y), tf_B(z)\} = 0$. Therefore,

$$(tf_A \Gamma tf_B)(x) = min\{tf_A(y), tf_B(z)\} = 0.$$

2. We will show that

$$(tf_A \cap tf_B)(x) = tf_{A\cap B}(x),$$

for all $x \in S$. If $x \in A \cap B$, then $tf_{A\cap B}(x) = t$. Since $x \in A$ and $x \in B$, we have

$$tf_A(x) = tf_B(x) = t,$$

so that $(tf_A \cap tf_B)(x) = tf_A(x) \wedge tf_B(x) = t$. If $x \notin A \cap B$, then $tf_{A\cap B}(x) = 0$. Suppose that $x \notin A$. Then $(tf_A \cap tf_B)(x) \leq tf_A(x) = 0$. Thus we obtain that $(tf_A \cap tf_B)(x) = tf_{A\cap B}(x)$, for all $x \in S$.

3. The proof is similar to the proof of 1 with suitable modification by using the definition.

4. If $x \in A$, then

$$\bigcup_{a\in A} a_t(x) = sup_{a\in A} a_t(x) = t = tf_A(x).$$

If $x \notin A$, then $tf_A(x) = 0$. Since $x \notin A$, we have $x \neq a$, for all $a \in A$, and so $a_t(x) = 0$. It implies that

$$\bigcup_{a\in A} a_t(x) = sup_{a\in A} a_t(x) = 0 = tf_A(x).$$

5. The proof is similar to the proof of 1 with a slight modification.

6. Suppose that $A$ is a left $\Gamma$-ideal of $S$. Then $tf_A(x\gamma y) \geq tf_A(y)$, for all $x, y \in S, \gamma \in \Gamma$. If $y \notin A$, then $tf_A(y) = 0$. Since $tf_A$ is a fuzzy subset of $S$, we have $tf_A(x\gamma y) \geq 0 = tf_A(y)$. If $y \in A$, then $tf_A(y) = t$. Since $A$ is a left $\Gamma$-ideal of $S$ and $x \in S, y \in A, \gamma \in \Gamma$, we then have $x\gamma y \in A$. Thus, $tf_A(x\gamma y) = t = tf_A(y)$.

**Theorem 3.2.** Let $P$ be a fuzzy left $\Gamma$-ideal of an $\Gamma$-LA-semigroup with left identity $S$. Then the following statements are equivalent:

1. $P$ is a weakly fuzzy quasi-semiprime of $S$.

2. For any $x \in S$ and $t \in (0,1]$, if $x_t \Gamma (S \Gamma x_t) \subseteq P$, then $x_t \in P$.

3. For any $x \in S$ and $t \in (0,1]$, if $tf_x \Gamma tf_x \subseteq P$, then $x_t \in P$.

4. If $A$ is a left $\Gamma$-ideal of $S$ such that $tf_A \Gamma tf_A \subseteq P$, then $tf_A \subseteq P$.

**Proof.** $(1 \Rightarrow 2)$ Let $P$ be a weakly fuzzy quasi-semiprime of $S$. For any $x \in S$ and $t \in (0,1]$, if $x_t \Gamma (S \Gamma x_t) \subseteq P$, then $tf_{S\Gamma(x\Gamma S)} \Gamma tf_{S\Gamma(x\Gamma S)}$

$$\begin{aligned}
&= (S\Gamma(x_t\Gamma S))\Gamma(S\Gamma(x_t\Gamma S)) \\
&= (S\Gamma S)\Gamma((x_t\Gamma S)\Gamma(x_t\Gamma S)) \\
&= (S\Gamma S)\Gamma((x_t\Gamma x_t)\Gamma(S\Gamma S)) \\
&= (S\Gamma S)\Gamma((S\Gamma S)\Gamma(x_t\Gamma x_t)) \\
&\subseteq S\Gamma(S\Gamma(x_t\Gamma x_t)) \\
&= S\Gamma(x_t\Gamma(S\Gamma x_t)) \\
&\subseteq S\Gamma P \\
&\subseteq P.
\end{aligned}$$

Since $P$ is a weakly fuzzy quasi-semiprime, we get $tf_{x\gamma x^2} \subseteq tf_{S\Gamma(x\Gamma S)} \subseteq P.$ Hence $x_t \in tf_x \subseteq P.$

$(2 \Rightarrow 3)$ Let $x \in S,\ t \in (0,1]$ and $tf_x \Gamma tf_x \subseteq P.$ Then

$$\begin{aligned} x_t\Gamma(S\Gamma x_t) &\subseteq tf_x\Gamma(S\Gamma tf_x) \\ &= S\Gamma(tf_x\Gamma tf_x) \\ &\subseteq S\Gamma P \\ &\subseteq P. \end{aligned}$$

Thus, by hypothesis $x_t \in P.$

$(3 \Rightarrow 4)$ Let $A$ be a left $\Gamma$-ideal of $S.$ Then, by Lemma 3.1, we get $tf_A$ is a fuzzy left $\Gamma$-ideal of $S.$ Suppose that $tf_A\Gamma tf_A \subseteq P$ and $tf_A \not\subset P,$ then there exists $x \in A$ such that $x_t \notin P.$ By Lemma 3.1 and hypothesis, we have

$$\begin{aligned} tf_x\Gamma tf_x &= tf_{x^2} \\ &\subseteq tf_{A\Gamma A} \\ &= tf_A\Gamma tf_A \\ &\subseteq P. \end{aligned}$$

Since $x_t \notin P,$ which implies $tf_x \not\subset P.$ But this leads to a contradiction.

$(4 \Rightarrow 1)$ By Definition 2.16, the following corollary is obvious.

**Corollary 3.3.** Let $P$ be a fuzzy $\Gamma$-ideal of an $\Gamma$-LA-semigroup with left identity $S.$ Then the following statements are equivalent:

1. $P$ is a weakly fuzzy semiprime $\Gamma$-ideal of $S.$
2. For any $x \in S$ and $t \in (0,1],$ if $x_t\Gamma(S\Gamma x_t) \subseteq P,$ then $x_t \in P.$
3. For any $x \in S$ and $t \in (0,1],$ if $tf_x\Gamma tf_x \subseteq P,$ then $x_t \in P.$
4. If $A$ is an $\Gamma$-ideal of $S$ such that $tf_A\Gamma tf_A \subseteq P,$ then $tf_A \subseteq P.$

**Proof.** This follows from Theorem 3.2.

**Theorem 3.4.** Let $S$ be an $\Gamma$-LA-semigroup with left identity. If $supf(a\Gamma(S\Gamma a)) = f(a),$ for all $a \in S,$ then $f$ is a fuzzy quasi-semiprime of $S.$

**Proof.** Let $g$ be a fuzzy left $\Gamma$-ideal of $S$ such that $g\Gamma g \subseteq f.$ If $g \not\subset f,$ then there exist $a \in S$ such that $f(a) < g(a).$ Since

$$f(a) = supf(a\Gamma(S\Gamma a)),$$

there exists $s \in S, \gamma, \alpha \in \Gamma$ such that

$$f(a\gamma(s\alpha a)) \le f(a).$$

Then $f(a\gamma(s\alpha a)) < g(a)$ so that

$$\begin{aligned} g(a) &> f(a\gamma(s\alpha a)) \\ &\ge g\Gamma g(a\gamma(s\alpha a)) \\ &\ge sup[min\{g(a), g(s\alpha a)\}] \\ &\ge min\{g(a), g(s\alpha a)\} \\ &= g(a) \end{aligned}$$

since $g$ is fuzzy left $\Gamma$-ideal of $S.$ But this leads to a contradiction.

**Theorem 3.5.** Let $S$ be an $\Gamma$-LA-semigroup with left identity. If $f$ is a fuzzy quasi-semiprime of $S,$ then $inf(f(a\Gamma(S\Gamma a))) = f(a),$ for all $a \in S.$

**Proof.** Suppose that $inf\left(f(a\Gamma(S\Gamma a))\right) \neq f(a),$ for some $a \in S.$ Since $f$ is fuzzy left $\Gamma$-ideal of $S,$ we get $f(a\gamma(s\alpha a)) \ge f(s\alpha a) \ge f(a),$ for all $s \in S, \gamma, \alpha \in \Gamma.$ Then

$$f(a) < inf(f(a\Gamma(S\Gamma a))).$$

Let $inf(f(a\Gamma(S\Gamma a))) = m$ and $g_{a\Gamma S}$ be fuzzy subset of $S$ such that

$$g_{a\Gamma S}(x) = \begin{cases} m; x \in a\Gamma S \\ 0; x \notin a\Gamma S. \end{cases}$$

Then by above Theorem 2.13, $g_{a\Gamma S}$ is a fuzzy left $\Gamma$-ideal of $S$. If $g_{a\Gamma S}\Gamma g_{a\Gamma S}(x)=m$, then

$$m=sup_{x=yz}[min\{g_{a\Gamma S}(y),g_{a\Gamma S}(z)\}].$$

This means there exist some $u,v\in a\Gamma S$ such that $u\gamma v=x$. Put $u=a\alpha t, v=a\beta k$. Then

$$\begin{aligned} f(x) &= f(u\gamma v)\\ &= f((a\alpha t)\gamma(a\beta k))\\ &= f((a\alpha a)\gamma(t\beta k))\\ &= f((k\beta t)\gamma(a\alpha a))\\ &\geq f(a\alpha a)\\ &= f(a\alpha(e\delta a))\\ &\geq inf(f(a\Gamma(a\Gamma S)))\\ &= m \end{aligned}$$

so that $g_{a\Gamma S}\Gamma g_{a\Gamma S}\subseteq f$ and hence $g_{a\Gamma S}\subseteq f$. Thus $g_{a\Gamma S}(a)=g_{a\Gamma S}(a\gamma e)=m$. But from

$$m=g_{a\Gamma S}(a)\leq f(a)<inf(f(a\Gamma(S\Gamma a))=m,$$

we have a contradiction.

**Corollary 3.6.** Let $S$ be an $\Gamma$-LA-semigroup with left identity. If $f$ is a fuzzy semiprime of $S$, then $inf(f(a\Gamma(S\Gamma a)))=f(a)$, for all $a\in S$.

**Proof.** This follows from Theorem 3.5.

**Theorem 3.7.** Let $S$ be an $\Gamma$-LA-semigroup with left identity. A fuzzy $\Gamma$-ideal $P$ of an $\Gamma$-LA-semigroup $S$ is weakly fuzzy quasi-semiprime $\Gamma$-ideal if and only if $P(x^2)=P(x)$, for all $x\in S$.

**Proof.** $(\Rightarrow)$ Suppose that $P$ is a fuzzy $\Gamma$-ideal of $S$. Then $P(x^2)\geq f(x)$, for all $x\in S$. On the other hand, if $P(x^2)>P(x)$, then there exists $t\in(0,1)$ such that $P(x^2)>t>P(x)$. Thus

$$x_t\Gamma(S\Gamma x_t)=S\Gamma(x_t\Gamma x_t)\subseteq S\Gamma(x^2)_t\in S\Gamma P\subseteq P,$$

for all $x\in S$. Since $P$ is a weakly fuzzy quasi-semiprime $\Gamma$-ideal of $S$, we get $x_t\in P$, but $x_t\notin P$, which is impossible. Therefore, $P(x^2)=P(x)$, for all $x\in S$.

$(\Leftarrow)$ Suppose that $x_t(t\in(0,1])$ are the fuzzy point of $S$ such that $x_t\Gamma(S\Gamma x_t)\subseteq P$. Since

$$S\Gamma(x^2)_t=S\Gamma(x_t\Gamma x_t)=x_t\Gamma(S\Gamma x_t)\subseteq P$$

and $P(x^2)=P(x)$, we have $P(x^2)\geq t$, which implies that $P(x)\geq t$. Then $x_t\in P$.

**Corollary 3.8.** Let $S$ be an $\Gamma$-LA-semigroup with left identity. If $P$ is a fuzzy weakly completely semiprime, then $P$ is weakly fuzzy quasi-semiprime of $S$.

Proof. One can easily show by induction method.

## 4. PRODUCT OF FUZZY $\Gamma$-IDEALS OF $\Gamma$-SEMIGROUPS

We start with the following theorem that gives a relation between product of fuzzy $\Gamma$-ideal and fuzzy $\Gamma$-ideal in $\Gamma$-LA-semigroup. Our starting points are the following definitions:

Let $S_1$ and $S_2$ be two $\Gamma$-LA-semigroups. Then

$$S_1\times S_2:=\{(x,y)\in S_1\times S_2 \mid x\in S_1, y\in S_2\}$$

and for any $(a,b),(c,d)\in S_1\times S_2, \gamma\in\Gamma$ we define $(a,b)\gamma(c,d):=(a\gamma c, b\gamma d)$, then $S_1\times S_2$ is an $\Gamma$-LA-semigroup as well. Let $f:S_1\to[0,1]$ and $g:S_2\to[0,1]$ be two fuzzy subsets of $\Gamma$-LA-semigroups $S_1$ and $S_2$ respectively. Then the product of fuzzy subsets is denoted by $f\times g$ and defined as $f\times g:S_1\times S_2\to[0,1]$, where

$$(f\times g)(x,y)=min\{f(x),g(y)\}.$$

**Lemma 4.1.** If $f$ and $g$ are fuzzy sub $\Gamma$-LA-semigroups of $S_1$ and $S_2$ respectively, then $f\times g$ is a fuzzy sub $\Gamma$-LA-semigroup of $S_1\times S_2$.

**Proof.** Let $(x_1,y_1),(x_2,y_2)\in S_1\times S_2$ and $\gamma\in\Gamma$. Then $(f\times g)((x_1,y_1)\gamma(x_2,y_2))$

$$= (f \times g)(x_1 \gamma x_2, y_1 \gamma y_2)$$

$$= min\{ f(x_1 \gamma x_2), g(y_1 \gamma y_2)\}$$

$$\geq min\{ f(x_1), f(x_2), g(y_1), g(y_2)\}$$

$$\geq min\{min\{ f(x_1), g(y_1))\}, min\{ f(x_2), g(y_2)\}\}$$

$$= min\{(f \times g)(x_1, y_1), (f \times g)(x_2, y_2)\}.$$

Therefore $f \times g$ is a fuzzy sub $\Gamma$-LA-semigroup of $S_1 \times S_2$.

**Lemma 4.2.** If $f$ and $g$ are fuzzy left $\Gamma$-ideals (fuzzy right $\Gamma$-ideals, fuzzy $\Gamma$-ideals) of $S_1$ and $S_2$ respectively, then $f \times g$ is a fuzzy left $\Gamma$-ideal (fuzzy right $\Gamma$-ideal, fuzzy $\Gamma$-ideal) of $S_1 \times S_2$.

**Proof.** Let $(x_1, y_1), (x_2, y_2) \in S_1 \times S_2$ and $\gamma \in \Gamma$. Then $(f \times g)((x_1, y_1)\gamma(x_2, y_2))$

$$\begin{aligned} &= (f \times g)(x_1 \gamma x_2, y_1 \gamma y_2) \\ &= min\{ f(x_1 \gamma x_2), g(y_1 \gamma y_2)\} \\ &\geq min\{ f(x_2), g(y_2)\} \\ &= (f \times g)(x_2, y_2). \end{aligned}$$

Therefore $f \times g$ is a fuzzy left $\Gamma$-ideal of $S_1 \times S_2$.

**Corollary 4.3.** Let $f_1, f_2, f_3, \ldots, f_n$ be a fuzzy subsets of $\Gamma$-LA-semigroups $S_1, S_2, S_3, \ldots, S_n$ respectively.

1. If $f_1, f_2, \ldots, f_n$ are fuzzy sub $\Gamma$-LA-semigroups of $S_1, S_2, \ldots, S_n$ respectively, then $\prod_{i=1}^{n} f_i$ is fuzzy sub $\Gamma$-LA-semigroup of $\prod_{i=1}^{n} S_i$.

2. If $f_1, f_2, f_3, \ldots, f_n$ are fuzzy left $\Gamma$-ideals (fuzzy right $\Gamma$-ideals, fuzzy $\Gamma$-ideals) of $S_1, S_2, S_3, \ldots, S_n$ respectively, then $\prod_{i=1}^{n} f_i$ is fuzzy left $\Gamma$-ideal (fuzzy right $\Gamma$-ideal, fuzzy $\Gamma$-ideal) of $\prod_{i=1}^{n} S_i$.

**Proof.** This follows from Lemma 4.1 and Lemma 4.2.

**Lemma 4.4.** Let $f, g$ be fuzzy subsets of $\Gamma$-LA-semigroup with left identity $S_1, S_2$ respectively such that $f \times g$ is a fuzzy sub $\Gamma$-LA-semigroup of $S_1 \times S_2$. Then $f$ or $g$ is fuzzy sub $\Gamma$-LA-semigroup of $S_1$ or $S_2$ respectively.

**Proof.** We know that

$$\begin{aligned} min\{ f(e_1), g(e_2)\} &= (f \times g)(e_1, e_2) \\ &\geq (f \times g)(x, y) \\ &= min\{ f(x), g(y)\}, \end{aligned}$$

for all $(x, y) \in S_1 \times S_2$. Then $f(x) \leq f(e_1)$ or $g(y) \leq g(e_2)$. If $f(x) \leq f(e_1)$, then

$$f(x) \leq g(e_2) \text{ or } g(y) \leq g(e_2).$$

Let $f(x) \leq g(e_2)$. Then $(f \times g)(x, e_2) = f(x)$ so that

$$\begin{aligned} f(x \gamma y) &= (f \times g)(x \gamma y, e_2) \\ &= (f \times g)((x, e_2)\gamma(y, e_2)) \\ &\geq min\{(f \times g)(x, e_2), (f \times g)(y, e_2)\} \\ &= min\{ f(x), f(y)\}. \end{aligned}$$

Therefore $f$ is a fuzzy sub $\Gamma$-LA-semigroup of $S_1$. Now suppose that $f(x) \leq g(e_2)$ is not true for all $x \in S_1$. If $f(x) > g(e_2)$ for some $x \in S_1$, then $g(y) \leq g(e_2)$, for all $y \in S_2$. Therefore $(f \times g)(e_1, y) = g(y)$, for all $y \in S_2$. Similarly

$$\begin{aligned} g(x \gamma y) &= (f \times g)(e_1, x \gamma y) \\ &= (f \times g)((e_1, x)\gamma(e_1, y)) \\ &\geq min\{(f \times g)(e_1, x), (f \times g)(e_1, y)\} \\ &= min\{g(x), g(y)\}. \end{aligned}$$

Hence $g$ is a fuzzy sub $\Gamma$-LA-semigroup of $S_2$.

**Lemma 4.5.** Let $f,g$ be fuzzy subsets of $\Gamma$-LA-semigroups with left identity $S_1,S_2$ respectively such that $f\times g$ be a fuzzy left $\Gamma$-ideal (fuzzy right $\Gamma$-ideal, fuzzy $\Gamma$-ideal) of $S_1\times S_2$. Then $f$ or $g$ is fuzzy left $\Gamma$-ideal (fuzzy right $\Gamma$-ideal, fuzzy $\Gamma$-ideal) of $S_1$ or $S_2$ respectively.

**Proof.** We know that

$$\begin{aligned} min\{f(e_1),g(e_2)\} &= (f\times g)(e_1,e_2)\\ &\geq (f\times g)(x,y)\\ &= min\{f(x),g(y)\}, \end{aligned}$$

for all $(x,y)\in S_1\times S_2$. Then $f(x)\leq f(e_1)$ or $g(y)\leq g(e_2)$. If $f(x)\leq f(e_1)$, then

$$f(x)\leq g(e_2) \text{ or } g(y)\leq g(e_2).$$

Let $f(x)\leq g(e_2)$. Then $(f\times g)(x,e_2)=f(x)$ so that

$$\begin{aligned} f(x\gamma y) &= (f\times g)(x\gamma y,e_2)\\ &= (f\times g)((x,e_2)\gamma(y,e_2))\\ &\geq (f\times g)(y,e_2)\\ &= f(y). \end{aligned}$$

Therefore $f$ is a fuzzy left $\Gamma$-ideal of $S_1$. Now suppose that $f(x)\leq g(e_2)$ is not true for all $x\in S_1$. If $f(x)>g(e_2)$ for some $x\in S_1$, then $g(y)\leq g(e_2)$, for all $y\in S_2$. Therefore $(f\times g)(e_1,y)=g(y)$, for all $y\in S_2$. Similarly

$$\begin{aligned} g(x\gamma y) &= (f\times g)(e_1,x\gamma y)\\ &= (f\times g)((e_1,x)(e_1,y))\\ &\geq (f\times g)(e_1,y)\\ &= g(y). \end{aligned}$$

Hence $g$ is fuzzy left $\Gamma$-ideal of $S_2$.

**Corollary 4.6.** Let $f_1, f_2, f_3,\ldots, f_n$ be a fuzzy subsets of $\Gamma$-LA-semigroups $S_1,S_2,S_3,\ldots,S_n$ respectively.

1. If $\prod_{i=1}^{n} f_i$ is a fuzzy sub $\Gamma$-LA-semigroup of $\prod_{i=1}^{n} S_i$, then $f_1$ or $f_2$ or $f_3$ or $\ldots$ or $f_n$ is a fuzzy sub $\Gamma$-LA-semigroup of $S_1,S_2,S_3,\ldots,S_n$ respectively.

2. If $\prod_{i=1}^{n} f_i$ is a fuzzy left $\Gamma$-ideal (fuzzy right $\Gamma$-ideal, fuzzy $\Gamma$-ideal) of $\prod_{i=1}^{n} S_i$, then $f_1$ or $f_2$ or $f_3$ or $\ldots$ or $f_n$ is a fuzzy left $\Gamma$-ideal (fuzzy right $\Gamma$-ideal, fuzzy $\Gamma$-ideal) of $S_1,S_2,S_3,\ldots,S_n$ respectively.

**Proof.** This follows from Lemma 4.5.

**Lemma 4.7.** Let $f,g$ be fuzzy subsets of $\Gamma$-LA-semigroups $S_1,S_2$ respectively and $t\in[0,1]$. Then $(f\times g)_t = f_t\times f_t$.

**Proof.** Let $f,g$ be fuzzy subsets of $\Gamma$-LA-semigroup $S_1,S_2$ respectively and $t\in[0,1]$. Then

$$\begin{aligned} (x,y)\in f_t\times g_t &\Leftrightarrow x\in f_t \text{ and } y\in g_t\\ &\Leftrightarrow f(x)\geq t \text{ and } g(y)\geq t\\ &\Leftrightarrow min\{f(x),g(y)\}\geq t\\ &\Leftrightarrow (f\times g)(x,y)\geq t\\ &\Leftrightarrow (x,y)\in(f\times g)_t \end{aligned}$$

for all $x\in S_1, y\in S_2$. Hence $(f\times g)_t = f_t\times f_t$.

**Corollary 4.8.** Let $f_1, f_2, f_3,\ldots, f_n$ be a fuzzy subsets of $\Gamma$-LA-semigroups $S_1,S_2,S_3,\ldots,S_n$ respectively and $t\in[0,1]$. Then $(\prod_{i=1}^{n} f_i)_t = \prod_{i=1}^{n}(f_i)_t$.

**Proof.** This follows from Lemma 4.7.

**Theorem 4.9.** Let $f$ and $g$ be two fuzzy weakly completely semiprime (fuzzy semiprime, quasi-semiprime) $\Gamma$-ideals of an $\Gamma$-LA-semigroups $S_1, S_2$ respectively. Then $(f \times g)$ is a fuzzy weakly completely semiprime (fuzzy semiprime, quasi-semiprime) $\Gamma$-ideal of $S_1 \times S_2$.

**Proof.** Let $(a,b) \in S_1 \times S_2$. Since $f$ and $g$ are fuzzy weakly completely semiprime $\Gamma$-ideals of $S$, we get

$$\begin{aligned}(f \times g)(a,b)^2 &= (f \times g)(a^2, b^2)\\ &= min\left\{f(a^2), g(b^2)\right\}\\ &= min\left\{f(a), g(b)\right\}\\ &= (f \times g)(a,b).\end{aligned}$$

Hence $(f \times g)$ is a fuzzy weakly completely semiprime $\Gamma$-ideal of $S_1 \times S_2$.

**Theorem 4.10.** Let $f, g$ be fuzzy subsets of $\Gamma$-LA-semigroup with left identity $S_1, S_2$ respectively such that $f \times g$ is a fuzzy weakly completely semiprime (fuzzy semiprime $\Gamma$-ideal, quasi-semiprime $\Gamma$-ideal) of $S_1 \times S_2$. Then $f$ or $g$ is fuzzy weakly completely semiprime (fuzzy semiprime $\Gamma$-ideal, quasi-semiprime $\Gamma$-ideal) of $S_1$ or $S_2$ respectively.

**Proof.** We know that

$$\begin{aligned}min\left\{f(e_1), g(e_2)\right\} &= (f \times g)(e_1, e_2)\\ &\geq (f \times g)(x, y)\\ &= min\left\{f(x), g(y)\right\},\end{aligned}$$

for all $(x,y) \in S_1 \times S_2$. Then $f(x) \leq f(e_1)$ or $g(y) \leq g(e_2)$. If $f(x) \leq f(e_1)$, then

$$f(x) \leq g(e_2) \text{ or } g(y) \leq g(e_2).$$

Let $f(x) \leq g(e_2)$. Then $(f \times g)(x, e_2) = f(x)$ so that

$$\begin{aligned}f(x^2) &= (f \times g)(x^2, e_2)\\ &= (f \times g)(x, e_2)^2\\ &\leq (f \times g)(x, e_2)\\ &= f(x).\end{aligned}$$

Therefore $f$ is a fuzzy weakly completely semiprime of $S_1$. Now suppose that $f(x) \leq g(e_2)$ is not true for all $x \in S_1$. If $f(x) > g(e_2)$ for some $x \in S_1$, then $g(y) \leq g(e_2)$, for all $y \in S_2$. Therefore $(f \times g)(e_1, y) = g(y)$, for all $y \in S_2$. Similarly

$$\begin{aligned}g(y^2) &= (f \times g)(e_1, y^2)\\ &= (f \times g)(e_1, y)^2\\ &\leq (f \times g)(e_1, y)\\ &= g(y).\end{aligned}$$

Hence $g$ is fuzzy weakly completely semiprime of $S_2$.

**Theorem 4.11.** Let $f_1, f_2$ be a fuzzy subsets of $\Gamma$-LA-semigroups $S_1, S_2$ respectively. Then $f \times g$ is a fuzzy weakly completely semiprime $\Gamma$-ideal of $S_1 \times S_2$ if and only if the level subset $(f \times g)_t, t \in Im(f \times g)$ of $f \times g$ is a weakly completely semiprime $\Gamma$-ideal of $S_1 \times S_2$, for every $t \in [0,1]$.

**Proof.** $(\Rightarrow)$ Suppose that $f \times g$ is a fuzzy weakly completely semiprime $\Gamma$-ideal of $S_1 \times S_2$. Let $(x,y) \in S_1 \times S_2$ such that $(x,y)^2 \in (f \times g)_t$. Then $(f \times g)(x,y)^2 \geq t$ so that

$$(f \times g)(x^2, y^2) \geq t.$$

Since $f \times g$ is a fuzzy weakly completely semiprime $\Gamma$-ideal of $S_1 \times S_2$, we have

$$(f \times g)(x,y)^2 = (f \times g)(x,y).$$

Then $t \leq (f \times g)(x,y)$, so $(x,y) \in (f \times g)_t$.

$(\Leftarrow)$ Suppose that $(f \times g)_t$ is a weakly completely semiprime $\Gamma$-ideal of $S_1 \times S_2$, for every $t \in [0,1]$. Let $(x,y) \in S_1 \times S_2$. By Definition fuzzy subset, we get $(f\times g)(x,y)^2 \geq 0.$ Since

$$(x,y)^2 \in (f\times g)_{(f\times g)(x,y)^2}$$

by hypothesis, we have $(x,y)\in (f\times g)_{(f\times g)(x,y)^2}$. Thus $(f\times g)(x,y)\geq (f\times g)(x,y)^2.$

**Corollary 4.12.** Let $f_1, f_2, f_3,\ldots, f_n$ be a fuzzy subsets of $\Gamma$-LA-semigroups $S_1, S_2, S_3,\ldots, S_n$ respectively and and $t\in[0,1].$ Then $\prod_{i=1}^{n} f_i$ is a fuzzy weakly completely semiprime $\Gamma$-ideal of $\prod_{i=1}^{n} S_i$ ifand only if the level subset $(\prod_{i=1}^{n} f_i)_t, t\in Im(\prod_{i=1}^{n} S_i)$ is a weakly completely semiprime $\Gamma$-ideal of $\prod_{i=1}^{n} S_i$.

**Proof.** This follows from Theorem 4.11.

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**CONFLICT OF INTEREST**

No conflict of interest was declared by the authors.

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