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A Short Note on The Relation $\mathcal{N}$ in Ordered Semihypergroups

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ABSTRACT
Let $(S,\bullet,\leq)$ be an ordered semihypergroup and $a\in S$. We denote by $N(a)$ the hyperfilter of S generated by a. We define an equivalence relation $\mathcal{N}:=\{(a,b)\in S\times S \mid N(a)=N(b)\}$ on S. In this note, we show that $\mathcal{N}$ is the intersection of the relations $\sigma_P:=\{(a,b)\in S\times S \mid a,b\in P$ or $a,b\notin P\}$, where P runs over the completely prime hyperideals of S. Moreover, we give some results on the ordered semihypergroups.
Keywords: *Ordered semihypergroup, completely prime hyperideal, hyperfilter, semilattice equivalence relation*

1. INTRODUCTION AND PREREQUISITES

The notion of a hyperstructure was introduced by Marty [10] in 1934, when he defined the hypergroups and began to investigate their properties with applications to groups, rational fractions and algebraic functions. The concept of a semihypergroup is a generalization of the concept of a semigroup. The concept of congruences play an important role in studying the structure of semihypergroups [4]. As a reference for more definitions on semihypergroups we refer to [3]. In [8], Heidari and Davvaz studied a semihypergroup $(S,\bullet)$ besides a binary relation $\leq$, where $\leq$ is a partial order relation such that satisfies the monotone condition. This structure is called an ordered semihypergroup. Davvaz et al. [6] introduced the notion of pseudoorders of ordered semihypergroups. Changphas and Davvaz [1, 2] investigated some properties of hyperideals and bi-hyperideals in ordered semihypergroups. In 2016, Davvaz and Omidi [5] introduced hyperideals and bi-hyperideals of ordered semihyperrings. Tang et al. [12] studied some properties of hyperfilters in ordered semihypergroups. The ordered regular equivalence relations on ordered semihypergroups were studied by Gu and Tang in [7]. In 2014, Kehayopulu [9] studied the Green's relations and the relation $\mathcal{N}$ in Γ-semigroups. The concept of a semilattice congruence of an ordered Γ-semigroup was introduced in [11].

Let S be a non-empty set. A mapping $\bullet: S\times S \longrightarrow P^*(S)$, where $P^*(S)$ denotes the family of all non-empty subsets of S, is called a *hyperoperation* on S. By a *hypergroupoid* we mean a non-empty set S endowed with a hyperoperation $\bullet$. In the above definition, if A and B are two non-empty

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subsets of S and $x \in S$, then we denote

$$A \bullet B = \bigcup_{\substack{a \in A \\ b \in B}} a \bullet b, \quad A \bullet x = A \bullet \{x\}, \quad x \bullet B = \{x\} \bullet B.$$

A hypergroupoid $(S,\bullet)$ is called a *semihypergroup* if for all x, y, z of S, we have

$$x \bullet (y \bullet z) = (x \bullet y) \bullet z,$$

which means that

$$\bigcup_{u \in y \bullet z} x \bullet u = \bigcup_{v \in x \bullet y} v \bullet z.$$

A non-empty subset K of a semihypergroup $(S,\bullet)$ is called a *subsemihypergroup* of S if $K \bullet K \subseteq K$. Let $(S,\bullet)$ be a semihypergroup. Then, S is called a *hypergroup* if it satisfies the reproduction axiom, for all $x \in S$, $x \bullet S = S = S \bullet x$. A non-empty subset K of S is a *subhypergroup* of S if $a \bullet K = K = K \bullet a$, for every $a \in K$.

Let σ be an equivalence relation on a semihypergroup $(S,\bullet)$ and $A, B \subseteq S$. Then $A\overline{\sigma}B$ means that for all $a \in A$, there exists $b \in B$ such that $a\sigma b$ and for all $b' \in B$, there exists $a' \in A$ such that $a'\sigma b'$. Also, $A\overline{\overline{\sigma}}B$ means that for all $a \in A$ and for all $b \in B$, we have $a\sigma b$. The equivalence relation σ is called (1) *regular* if for all $x \in S$, from $a\sigma b$, it follows that $a \bullet x\overline{\sigma}\, b \bullet x$ and $x \bullet a\overline{\sigma}\, x \bullet b$; (2) *strongly regular* if for all $x \in S$, from $a\sigma b$, it follows that $a \bullet x\overline{\overline{\sigma}}\, b \bullet x$ and $x \bullet a\overline{\overline{\sigma}}\, x \bullet b$.

An *ordered semihypergroup* $(S,\bullet,\leq)$ *is a semihypergroup* $(S,\bullet)$ *together with a partial order* $\leq$ *that is compatible with the hyperoperation* $\bullet$, *meaning that for any* $x, y, z \in S$,

$$x \leq y \Rightarrow z \bullet x \leq z \bullet y \text{ and } x \bullet z \leq y \bullet z.$$

Here, $z \bullet x \leq z \bullet y$ *means for any* $a \in z \bullet x$ *there exists* $b \in z \bullet y$ *such that* $a \leq b$. *The case* $x \bullet z \leq y \bullet z$ *is defined similarly.* An ordered semihypergroup $(S,\bullet,\leq)$ is called an *ordered hypergroup* if $(S,\bullet)$ is a hypergroup. A non-empty subset A of an ordered semihypergroup $(S,\bullet,\leq)$ is called a *subsemihypergroup* of S if $(A,\bullet,\leq)$ is an ordered semihypergroup. For a non-empty subset K of an ordered semihypergroup S, we denote

$$(K] = \{x \in S \mid x \leq a \text{ for some } a \in K\}.$$

Definition 1.1 *A non-empty subset* I *of an ordered semihypergroup* $(S,\bullet,\leq)$ *is called a hyperideal of* S *if*

(1) $S \bullet I \subseteq I$ and $I \bullet S \subseteq I$;

(2) When $x \in I$ and $y \in S$ such that $y \leq x$, imply that $y \in I$.

Note that the condition (2) in Definition 1.1 is equivalent to $(I] \subseteq I$.

A hyperideal I of an ordered semihypergroup S is called *completely prime* if for any a, b of S such that $a \bullet b \cap I \neq \emptyset$, then $a \in I$ or $b \in I$. A subsemihypergroup F of S is called a *hyperfilter* [12] of S if (1) $(\forall a, b \in S)$ $a \bullet b \cap F \neq \emptyset$ implies $a, b \in F$; (2) For any $a \in F$ and $c \in S$, $a \leq c$ implies $c \in F$. The intersection of all hyperfilters of S containing a non-empty subset A of S is the hyperfilter of S generated by A. For an element $a \in S$, we denote by $N(a)$ the hyperfilter of S generated by a, and by $\mathcal{N}$ the relation on S defined by

$$\mathcal{N} \coloneqq \{(a,b) \in S \times S \mid N(a) = N(b)\}.$$

A semilattice equivalence relation on S is a strongly regular relation σ on S such that $a \bullet a\overline{\overline{\sigma}}\, a$ and $a \bullet b\overline{\overline{\sigma}}\, b \bullet a$ for each $a, b \in S$.

Theorem 1.2 *[7] Let* $(S,\bullet,\leq)$ *be an ordered semihypergroup. Then* $\mathcal{N}$ *is an ordered semilattice equivalence relation on* S.

Theorem 1.3 *[12] Let* $(S,\bullet,\leq)$ *be an ordered semihypergroup and* F *a non-empty subset of* S. *Then, the following statements are equivalent:*

(1) F is a hyperfilter of S.

(2) $S \backslash F = \emptyset$ or $S \backslash F$ is a completely prime hyperideal of S.

2 . MAIN RESULTS

Let a be an element of an ordered semihypergroup $(S,\bullet,\leq)$. We denote by $I_S(a)$ the hyperideal of S generated by a. We have

$$I_S(a) = (a \cup S \bullet a \cup a \bullet S \cup S \bullet a \bullet S].$$

Let $(S,\bullet,\leq)$ be an ordered semihypergroup. An equivalence relation $\mathcal{I}$ is defined on S by $a\mathcal{I}b$ if and only if $I_S(a) = I_S(b)$ for any $a, b \in S$. This equivalence relation is called Green's relation on an ordered semihypergroup S. For a subset A of S we denote by σ_A the relation on S defined by

$$\sigma_A \coloneqq \{(a,b) \in S \times S \mid a, b \in A \text{ or } a, b \notin A\}.$$

Theorem 2.1 *Let* P *be a completely prime hyperideal of an ordered semihypergroup* $(S,\bullet,\leq)$. *Then the set* σ_P *is a semilattice equivalence relation on* S.

Proof. Clearly, σ_P is a relation on S which is reflexive and symmetric. So, we show that it is transitive. Suppose that $(a,b) \in \sigma_P$ and $(b,c) \in \sigma_P$ .Then $a, b \in P$ or $a, b \notin P$ and $b, c \in P$ or $b, c \notin P$. If $a, b \in P$ and $b, c \in P$, then $a, c \in P$. Hence, $(a,c) \in \sigma_P$ .If $a, b \notin P$ and $b, c \notin P$, then $a, c \notin P$. So, $(a,c) \in \sigma_P$. The case $a, b \in P$ and $b, c \notin P$ is impossible and so is the case $a, b \notin P$ and $b, c \in P$. Thus, σ_P is an equivalence relation on S.

Now, let $(a,b) \in \sigma_P$ and $c \in S$. Then $a, b \in P$ or $a, b \notin P$. Let $a, b \in P$. Since P is a hyperideal of S, we have $a \bullet c \subseteq P$ and $b \bullet c \subseteq P$. Hence, for all $u \in a \bullet c$ and for all $v \in b \bullet c$, we have $(u,v) \in \sigma_P$. This implies that $a \bullet c\overline{\overline{\sigma_P}}b \bullet c$. Let $a, b \notin P$ and $c \in P$. Since P is a hyperideal of S, we have $a \bullet c \subseteq P$ and $b \bullet c \subseteq P$. Thus for any $m \in a \bullet c$, $n \in b \bullet$

c, we have $(m,n) \in \sigma_P$. Hence, $a \bullet c \overline{\overline{\sigma_P}} b \bullet c$. Now, let $a, b \notin P$ and $c \notin P$. Since P is a completely prime hyperideal of S, we have $a \bullet c \cap P = \emptyset$ and $b \bullet c \cap P = \emptyset$. If $x \in a \bullet c$ and $y \in b \bullet c$, then $x \notin P$ and $y \notin P$. This implies that $(x,y) \in \sigma_P$, and hence $a \bullet c \overline{\overline{\sigma_P}} b \bullet c$. Similarly, $c \bullet a \overline{\overline{\sigma_P}} c \bullet b$. Therefore, σ_P is a strongly regular equivalence relation on S.

Finally, we prove that σ_P is a semilattice equivalence relation on S. Suppose that $a \in P$, where $a \in S$. Then, $a \bullet a \subseteq P$. Hence, $(x,a) \in \sigma_P$ for every $x \in a \bullet a$. Now, let $a \notin P$. Then, $a \bullet a \cap P = \emptyset$. Thus $a, y \notin P$ for every $y \in a \bullet a$, which means that $(y,a) \in \sigma_P$. Thus $a \bullet a \overline{\overline{\sigma_P}} a$. Let $a, b \in S$ and $x \in a \bullet b, y \in b \bullet a$. If $a \bullet b \cap P \neq \emptyset$, then $a \in P$ or $b \in P$. By condition (1) of Definition 1.1, we have $x \in a \bullet b \subseteq P$ and $y \in b \bullet a \subseteq P$. So, for all $x \in a \bullet b$ and $y \in b \bullet a$, we have $x, y \in P$. This implies that $(x,y) \in \sigma_P$. If $a \bullet b \cap P = \emptyset$, then $b \bullet a \cap P = \emptyset$. So, for all $x \in a \bullet b$ and $y \in b \bullet a$, we have $x, y \notin P$. This implies that $(x,y) \in \sigma_P$. Thus $a \bullet b \overline{\overline{\sigma_P}} b \bullet a$. Hence the proof is completed.

Theorem 2.2 *Let $\mathcal{CP}(S)$ be the set of completely prime hyperideals of an ordered semihypergroup $(S, \bullet, \leq)$. Then,*

$$\mathcal{N} = \bigcap_{P \in \mathcal{CP}(S)} \sigma_P.$$

Proof. Suppose that P is a completely prime hyperideal of S. First, we show that $\mathcal{N} \subseteq \sigma_P$. Let $(a,b) \in \mathcal{N}$. If $(a,b) \notin \sigma_P$, then one of two following cases happens:

Case 1. $a \in P$ and $b \notin P$. Since $b \in S \backslash P$, we have $S \backslash P \neq \emptyset$. By assumption, $S \backslash (S \backslash P) = P$ is a completely prime hyperideal of S. By Theorem 1.3, $S \backslash P$ is a hyperfilter of S. Since $b \in S \backslash P$, it follows that $N(b) \subseteq S \backslash P$. So, $a \in N(a) = N(b) \subseteq S \backslash P$, which is a contradiction. This leads to $(a,b) \in \sigma_P$.

Case 2. $a \notin P$ and $b \in P$. Since $a \in S \backslash P$, we have $S \backslash P \neq \emptyset$. By assumption, $S \backslash (S \backslash P) = P$ is a completely prime hyperideal of S. By Theorem 1.3, $S \backslash P$ is a hyperfilter of S. Since $a \in S \backslash P$, it follows that $N(a) \subseteq S \backslash P$. So, $b \in N(b) = N(a) \subseteq S \backslash P$, which is a contradiction. This leads to $(a,b) \in \sigma_P$.

This implies that $\mathcal{N} \subseteq \sigma_P$ for every $P \in \mathcal{CP}(S)$. Thus $\mathcal{N} \subseteq \bigcap_{P \in \mathcal{CP}(S)} \sigma_P$.

Now, let $(a,b) \in \sigma_P$ for every $P \in \mathcal{CP}(S)$. Let $(a,b) \notin \mathcal{N}$. Then $N(a) \neq N(b)$. So, $a \notin N(b)$ or $b \notin N(a)$. If, say, $a \notin N(b)$, then $a \in S \backslash N(b)$. Since $b \in N(b)$, we have $b \notin S \backslash N(b)$. This implies that $(a,b) \notin \sigma_{S \backslash N(b)}$. By Theorem 1.3, $S \backslash N(b)$ is a completely prime hyperideal of S, which is a contradiction. If, say, $b \notin N(a)$, then $b \in S \backslash N(a)$. Since $a \in N(a)$, we have $a \notin S \backslash N(a)$. This implies that $(a,b) \notin \sigma_{S \backslash N(a)}$. By Theorem 1.3, $S \backslash N(a)$ is a completely prime hyperideal of S, which is a contradiction. Then, $(a,b) \in \mathcal{N}$ which implies that $\sigma_P \subseteq \mathcal{N}$ for every $P \in \mathcal{CP}(S)$. Hence, $\bigcap_{P \in \mathcal{CP}(S)} \sigma_P \subseteq \mathcal{N}$.

Theorem 2.3 *If $(S, \bullet, \leq)$ is an ordered semihypergroup, then $\mathcal{J} \subseteq \mathcal{N}$.*

Proof. Let Ω be the set of hyperideals of S. First, we show that

$$\mathcal{J} = \bigcap_{A \in \Omega} \sigma_A.$$

Let $(a,b) \in \mathcal{J}$ and $A \in \Omega$. If $a \in A$, then we have

$$\begin{aligned} b \in I_S(b) = I_S(a) &= (a \cup S \bullet a \cup a \bullet S \cup S \bullet a \bullet S] \\ &\subseteq (A \cup S \bullet A \cup A \bullet S \cup S \bullet A \bullet S] \\ &\subseteq (A] \\ &= A. \end{aligned}$$

This means that $b \in A$, and so $(a,b) \in \sigma_A$. If $a \notin A$, then we have $b \notin A$. Since $a, b \notin A$, it follows that $(a,b) \in \sigma_A$. Hence, $\mathcal{J} \subseteq \sigma_A$ for every $A \in \Omega$. So, we have $\mathcal{J} \subseteq \bigcap_{A \in \Omega} \sigma_A$. Now, let $A \in \Omega$ and $(x,y) \in \sigma_A$. Then, $(x,y) \in \sigma_{I_S(x)}$. So, we have $x, y \in I_S(x)$ or $x, y \notin I_S(x)$. Since $x \in I_S(x)$, it follows that $y \in I_S(x)$. This implies that $I_S(y) \subseteq I_S(x)$. Similarly, we can prove that $I_S(x) \subseteq I_S(y)$. Then, $I_S(x) = I_S(y)$ and so $(x,y) \in \mathcal{J}$. Hence, $\bigcap_{A \in \Omega} \sigma_A \subseteq \mathcal{J}$. Thus we have $\mathcal{J} = \bigcap_{A \in \Omega} \sigma_A$. Now, by Theorem 2.2, we have

$$\mathcal{J} = \bigcap_{A \in \Omega} \sigma_A \subseteq \bigcap_{A \in \mathcal{CP}(S)} \sigma_A = \mathcal{N}.$$

Theorem 2.4 *Let $(S, \bullet, \leq)$ be an ordered semihypergroup. If $N(a) = \{b \in S \mid a \in (S \bullet b \bullet S]\}$ for all $a \in S$, then $\mathcal{J} = \mathcal{N}$.*

Proof. By Theorem 2.3, we have $\mathcal{J} \subseteq \mathcal{N}$. Let $(x,y) \in \mathcal{N}$. Then, $x \in N(x) = N(y)$. By hypothesis, we have

$$y \in (S \bullet x \bullet S] \subseteq (x \cup S \bullet x \cup x \bullet S \cup S \bullet x \bullet S] = I_S(x).$$

Hence, $I_S(y) \subseteq I_S(x)$. Similarly, we have $I_S(x) \subseteq I_S(y)$. Thus $I_S(x) = I_S(y)$ and so $(x,y) \in \mathcal{J}$. Therefore, $\mathcal{N} \subseteq \mathcal{J}$ and the proof is completed.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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