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A Lattice-theoretic Generalization of the Lehmer Matrix

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ABSTRACT

In this paper, we present a lattice-theoretic generalization of the Lehmer matrix. We obtain some certain formulae for the determinant and the entries of the inverse of this new generalization by using lattice-theoretic tools. These formulae are the generalization of formulae for the determinant, the inverse of the classical Lehmer matrix and most of its generalizations presented in the literature.

Keywords: *the Lehmer matrix, lattice, meet matrix, determinant, inverse, Möbius inversion.*

1. INTRODUCTION

In 1946, Lehmer [7] proposed a question about finding the inverse of an $n \times n$ matrix $A = (a_{ij})$, where

$$a_{ij} = \frac{min\{i, j\}}{max\{i, j\}} \qquad (1)$$

or equivalently

$$a_{ij} = \begin{cases} i/j & \text{if } j \geq i, \\ j/i & \text{otherwise.} \end{cases}$$

The problem was solved by D. M. Smiley and M. F. Smiley, and J. Williamson [13]. Then Newman and Todd [11], and Marcus [8] used the matrix A to evaluate the accuracy of matrix inversion programs and Shampine [12] obtained P-condition number of the matrix A. In the same paper, Shampine called A Lehmer's matrix.

Recently, some authors [5, 2] study the Lehmer matrix associated with Fibonacci and Lucas numbers and their relatives. In [5], Kılıç and Stanica obtained the LU-factorization and the Cholesky factorization of the Lehmer matrix. Then, by the LU-factorization, they present an explicit formula for the inverse of the Lehmer matrix. Moreover, in the same paper, they consider a recursive analogue of the Lehmer matrix associated with the numbers u_n, where u_n satisfies the second order recurrence relation

$$u_n = pu_{n-1} - qu_{n-2} \qquad (2)$$

with the initial conditions $u_0 = 0$ and $u_1 = 1$ under the condition $q = -1$. Furthermore, by a similar method, they obtain the LU-factorization and the Cholesky factorization to calculate the entries of the inverse of this analogue. Let $\lambda \geq 1$, $r \geq 0$ and $k \geq 1$ be integer parameters and u_n be as in (2). In [2], Akkuş presents another recursive analogue of the Lehmer matrix associated with the numbers X_i, where

$$X_i := \prod_{s=1}^{k} u_{\lambda(i+s-1)+r} \qquad (3)$$

for all $i \geq 2$. The author obtains the LU-factorization, the Cholesky factorization, and hence the entries of the inverse of this new generalization.

On the other hand, more recently, Matilla and Haukkanen [10] study different properties MIN and MAX matrices of a finite multiset $T = \{t_1, t_2, t_3, \dots, t_n\}$ of real numbers $(t_1 \leq t_2 \leq t_3 \leq \cdots \leq t_n)$, that are $n \times n$ matrices whose ij-entry is $\min\{t_i, t_j\}$ and $\max\{t_i, t_j\}$, respectively (see [4] for the MIN matrix). In [10], by interpreting these matrices as meet and join matrices, and by applying some known results for meet and join matrices, present factorizations of MIN and MAX matrices, formulae for their determinants and explicit

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formulae for their inverses (see [3] and [6] for meet and join matrices).

In this paper, we present a lattice theoretic generalization of the Lehmer matrix as a particular combined meet and join matrix. We utilize lattice-theoretical tools of meet matrices to study the properties of the generalized Lehmer matrix associated with incidence functions on lattices. Without using the methods in [2] and [5], namely the LU-factorization or the Cholesky factorization, we obtain formulae for the determinants and the inverses of the classical Lehmer matrix and most of its generalizations given in the literature by our lattice-theoretic approach.

2. PRELIMINARIES

Firstly, we collect lattice-theoretic tools we use in this paper. Let $(P,\leq)$ be a locally finite poset and $\text{int}\,(P)$ denote the set of intervals of P. Let K be a field. Throughout this paper, we take K as $\mathbb{R}$. If $f:\text{int}(P)\to K$ is a function then we write $f(x,y)$ for $f([x,y])$. f is an incidence function on P whenever $f(x,y)=0$ unless $x\leq y$. The incidence algebra $I(P,K)$ of P over K is the K-algebra of all functions $f:\text{int}(P)\to K$, where multiplication or convolution is defined by

$$(f*g)(x,y)=\sum_{x\leq z\leq y} f(x,z)g(z,y).$$

The identity δ of $I(P,K)$ is defined by

$$\delta(x,y)=\begin{cases}1 & \text{if } x=y,\\ 0 & \text{otherwise,}\end{cases}$$

and the zeta function ζ of $I(P,K)$ is defined by

$$\zeta(x,y)=\begin{cases}1 & \text{if } x\leq y,\\ 0 & \text{otherwise.}\end{cases}$$

The inverse of ζ with respect to the convolution is the Möbius function μ, where $\mu(x,x)=1$ for all $x\in P$, and if $x<y$ then

$$\mu(x,y)=-\sum_{x\leq z<y}\mu(x,z).$$

When P is a chain, the Möbius function takes values as the following. If $x_i\leq x_j$, then

$$\mu(x_i,x_j)=\begin{cases}1 & \text{if } i=j,\\ -1 & \text{if } i+1=j,\\ 0 & \text{otherwise.}\end{cases}$$

Let A be a finite set and $P=\mathcal{P}(A)$, the power set of A, and let $\leq$ be the order of inclusion. Then, for all $C\subset B\subset A$,

$$\mu(C,B)=(-1)^{|B|-|C|},$$

where $|B|$ ve $|C|$ are the cardinalities of B and C, respectively. Let $P=\mathbb{Z}^+$ and let $\leq$ be the divisibility relation of integers. Then, we have the classical Möbius function of number theory, that is

$$\mu(n)=\begin{cases}1 & \text{if } n=1,\\ (-1)^r & \text{if } n=p_1\ldots p_r \text{ for distinct primes } p_i,\\ 0 & \text{if } p^2|n \text{ for a prime } p,\end{cases}$$

for all $n\in\mathbb{Z}^+$. For all $z\in P$, we associate each $f(z)$ with incidence function value $f(0,z)$, where 0 is the bottom of P. Suppose all principal ideals of P are finite. Then, for all $x\in P$,

$$g(x)=\sum_{y\leq x} f(y) \quad \text{if and only if} \quad f(x)=\sum_{y\leq x} g(y)\mu(y,x).$$

This result is so-called the Möbius inversion for posets. In particular, if $P=\mathbb{Z}^+$ and $\leq$ is the divisibility relation of integers, then we have the Möbius inversion formula of number theory

$$g(n)=\sum_{d|n} f(d) \quad \text{if and only if} \quad f(n)=\sum_{d|n} g(d)\,\mu\left(\frac{n}{d}\right).$$

for all $n\in\mathbb{Z}^+$. The reader can consult the text of M. Aigner [1] for the above lattice-theoretical tools and undefined terms in the present paper.

Let $S=\{x_1,x_2,\ldots,x_n\}$ be a subset of P and $f:P\to\mathbb{C}$ be a function. The $n\times n$ matrix $(S)_f$ whose the ij −entry is $f(x_i\wedge x_j)$ is called the meet matrix on S with respect to f. The join matrix $[S]_f$ is defined similarly. Briefly, $[S]_f=\left(f(x_i\vee x_j)\right)$. The meet matrix was firstly introduced by Bhat [3]. On the other hand, the join matrix was defined by Korkee and Haukkanen [6]. Recently, Mattila [9] introduces a new matrix related to meet and join matrix and he calls it the combined meet and join matrix associated with a semimultiplicative function. Let f be a semimultiplicative function and α,β,γ and λ be real parameters. The combined meet and join matrix on $S=\{x_1,x_2,\ldots,x_n\}$ associated with f is the $n\times n$ matrix $M_{S,f}^{\alpha,\beta,\gamma,\lambda}=(m_{ij})$, where

$$m_{ij}=\frac{f(x_i\wedge x_j)^{\alpha}.f(x_i\vee x_j)^{\beta}}{f(x_i)^{\gamma}f(x_j)^{\lambda}}.$$

It is clear that $M_{S,f}^{1,0,0,0}=(S)_f$ and $M_{S,f}^{0,1,0,0}=[S]_f$. Finally, the matrix $M_{S,f}^{\alpha,\beta,\gamma,\lambda}$ is the origin of our generalization of the Lehmer matrix.

3. THE VALUE OF DETERMINANT AND THE INVERSE OF THE GENERALIZED LEHMER MATRIX

In the rest of the paper, let P be a lattice, $S=\{x_1,x_2,\ldots,x_n\}\subset P$ and let $f:P\to\mathbb{C}$ be a semi-multiplicative function such that $f(x_i)\neq 0$ for all $x_i\in S$. We say that f is a semi-multiplicative function on P if f satisfies $f(x)f(y)=f(x\wedge y)f(x\vee y)$ for all $x,y\in P$. In this paper, we study the matrix

$$M_{S,f}^{1,-1,0,0}=\left(\frac{f(x_i\wedge x_j)}{f(x_i\vee x_j)}\right)$$

on S. Indeed, it is an abstract generalization of the Lehmer matrix. Let $P = \mathbb{R}$ with the ordinary order on real numbers, $S = \{x_1, x_2, \dots, x_n\}$ a finite chain and $f = I$, the identity function. Then, the matrix $M_{S,f}^{1,-1,0,0}$ becomes the classical Lehmer matrix. For the sake of simplicity, we denote the generalized Lehmer matrix $M_{S,f}^{1,-1,0,0}$ by $M = (m_{ij})$. Since f is a semi-multiplicative function, we have

$$m_{ij} = \frac{f^2(x_i \wedge x_j)}{f(x_i)f(x_j)}.$$

On the other hand, by the Möbius inversion formula,

$$f^2(x_i) = \sum_{x_k \le x_i} G(x_k)$$

if and only if

$$G(x_i) = \sum_{x_k \le x_i} f^2(x_k)\, \mu_S(x_k, x_i).$$

The incidence matrix $= (e_{ij})$ on S is defined as follows

$$e_{ij} = \begin{cases} 1 & \text{if } x_j \le x_i, \\ 0 & \text{otherwise,} \end{cases}$$

or equivalently $e_{ij} = \zeta_s(x_j, x_i)$. Thus, we have

$$\begin{aligned} m_{ij} &= \frac{1}{f(x_i)} \sum_{x_k \le x_i \wedge x_j} G(x_k) \frac{1}{f(x_j)} \\ &= \frac{1}{f(x_i)f(x_j)} \sum_{\substack{x_k \le x_i \\ x_k \le x_j}} G(x_k) \\ &= \frac{1}{f(x_i)f(x_j)} \sum_{k=1}^{n} e_{ik} G(x_k) e_{jk}. \end{aligned}$$

Hence, the matrix M can be written as

$$M = \Lambda_{1/f} E \Lambda_{\mathrm{G}} E^T \Lambda_{1/f},$$

where $\Lambda_{1/f} = \operatorname{diag}\left(\frac{1}{f(x_1)}, \dots, \frac{1}{f(x_n)}\right)$ and $\Lambda_{\mathrm{G}} = \operatorname{diag}\left(G(x_1), \dots, G(x_n)\right)$ and we have obtained the following lemma.

Lemma 1. Let M be the generalized Lehmer matrix. Then

$$M = \Lambda_{1/f} E \Lambda_{\mathrm{G}} E^T \Lambda_{1/f}.$$

Theorem 2. Let M be the generalized Lehmer matrix. Then

$$\det M = \prod_{k=1}^{n} \frac{G(x_k)}{f(x_k)^2}.$$

Proof: Since $\det \Lambda_{1/f} = \prod_{k=1}^{n} \frac{1}{f(x_k)^2}$, $\det E = 1$, and $\det \Lambda_{\mathrm{G}} = \prod_{k=1}^{n} G(x_k)$, the proof is immediate.

Now, we specialize Theorem 2 to particular lattices and hence we obtain the following results.

Corollary 3. Let A be a finite set and $P = \mathcal{P}(A)$, the power set of A, and let $\le$ be the order of inclusion. If $S = P$ then we have

$$\det M = \prod_{B \in \mathcal{P}(A)} \sum_{\substack{C \subset B \\ C \in \mathcal{P}(A)}} \frac{f(C)^2}{f(B)^2} (-1)^{|B|-|C|}.$$

Proof. If $C \subset B$ in $\mathcal{P}(A)$ then $\mu_S(C, B) = (-1)^{|B|-|C|}$ and hence

$$G(B) = \sum_{C \subset B} f^2(C)(-1)^{|B|-|C|}.$$

Thus, by Theorem 2, the proof is obvious.

Corollary 4. Let $P = \mathbb{Z}^+$ and let $\le$ be the divisibility relation of integers. If $S = \{x_1, x_2, \dots, x_n\}$ is the set of positive divisors of x_n, then

$$\det M = \prod_{k=1}^{n} \sum_{x_t | x_k} \frac{f(x_t)^2}{f(x_k)^2} \mu\left(\frac{x_k}{x_t}\right),$$

where μ is the classical Möbius function.

Proof. Let $(P, \le) = (\mathbb{Z}^+, |)$, where $|$ is the divisibility relation of integers. Then, we have $G(x_k) = \sum_{x_t | x_k} f^2(x_t) \mu(\frac{x_k}{x_t})$ and hence the proof is clear from Theorem 2.

Corollary 5. Let $(P, \le)$ be a lattice and let $S = \{x_1, x_2, \dots, x_n\}$ be a finite chain with respect to $\le$ such that $x_1 \le x_2 \le \dots \le x_n$. Then

$$\det M = \prod_{k=2}^{n} \frac{f(x_k)^2 - f(x_{k-1})^2}{f(x_k)^2}.$$

Proof. Let $S = \{x_1, x_2, \dots, x_n\}$ be a finite chain with respect to $\le$. By the value of the Möbius function for chains, $G(x_1) = f(x_1)^2$ and $G(x_k) = f(x_k)^2 - f(x_{k-1})^2$ for each $2 \le k \le n$. Then, by Theorem 2, the proof is immediate.

As an application of Corollary 5, we can obtain some certain results concerning the determinants of the classical Lehmer matrix and its recursive analogues presented in the literature.

Corollary 6. (Corollary 5 in [2]) Let u_n and X_i be as in (2) and (3), respectively. Define an $n \times n$ matrix $\mathcal{F}_n = \left(\frac{\min\{X_{i+1}, X_{j+1}\}}{\max\{X_{i+1}, X_{j+1}\}}\right)$ on the set $S = \{X_1, X_2, \dots, X_n\}$. Then

$$\det \mathcal{F}_n = \prod_{i=2}^{n} \frac{X_{i+1}^2 - X_i^2}{X_{i+1}^2}.$$

Proof. Taking $f(x_i) = X_{i+1}$ in Corollary 5, we obtain the claim.

Corollary 7. (Corollary 2 in [5]) Let u_n be as in (2). Define an $n \times n$ recursive generalized Lehmer matrix $\mathcal{U} = \left(\frac{\min\{u_{i+1}, u_{j+1}\}}{\max\{u_{i+1}, u_{j+1}\}}\right)$ on the set $S = \{u_1, u_2, \dots, u_n\}$. Then

$$\det \mathcal{U} = \prod_{i=2}^{n} \frac{u_{i+1}^2 - u_i^2}{u_{i+1}^2}.$$

Proof. Taking $f(x_i) = u_{i+1}$ in Corollary 5, we obtain the claim.

Here we should note that we obtain the claim of Corollary 7 without any restriction on u_n although Kılıç and Stanica only proved Corollary 7 under the condition $q = -1$. Moreover, we will prove the claim of Corollary 14 without any restriction on u_n.

Corollary 8. (Corollary 1 in [5]) Let A be the classical Lehmer matrix defined by (1). Then $\det A = \frac{(2n)!}{2^n (n!)^3}$.

Proof. If we take $f(x_i) = i$ in Corollary 5, then we obtain the claim.

Theorem 9. Let $M = (m_{ij})$, where $m_{ij} = \frac{f(x_i \wedge x_j)}{f(x_i \vee x_j)}$, be the generalized Lehmer matrix on the set $S = \{x_1, x_2, \dots, x_n\}$. If $G(x_i) \neq 0$ for all $x_i \in S$ then M is invertible and its inverse is the $n \times n$ matrix $Z = (z_{ij})$, where

$$z_{ij} = f(x_i) f(x_j) \sum_{k=1}^{n} \frac{1}{G(x_k)} \mu_S(x_i, x_k) \mu_S(x_j, x_k).$$

Proof. Let $M = (m_{ij})$, where $m_{ij} = \frac{f(x_i \wedge x_j)}{f(x_i \vee x_j)}$, and $G(x_i) \neq 0$ for all $x_i \in S$. By Theorem 2, M is invertible. Let $Z = (z_{ij})$ is the inverse of M. Moreover, by Theorem 1, we have

$$Z = \operatorname{diag}(f(x_1), \dots, f(x_n))\, (E^T)^{-1}$$

$$\times \operatorname{diag}\left(\frac{1}{G(x_1)}, \dots, \frac{1}{G(x_n)}\right) E^{-1}\ \operatorname{diag}(f(x_1), \dots, f(x_n)).$$

On the other hand, since $\zeta_S * \mu_S = \delta$ or more explicitly

$$\delta(x_i, x_j) = \sum_{x_i \le x_k \le x_j} \zeta_S(x_i, x_k)\, \mu_S(x_k, x_j)$$

whenever $x_i \le x_j$, the inverse of E^T is the $n \times n$ matrix $C = (c_{ij})$, where $c_{ij} = \mu_S(x_i, x_j)$. Thus, we have

$$z_{ij} = f(x_i) f(x_j) \sum_{k=1}^{n} \frac{1}{G(x_k)} \mu_S(x_i, x_k) \mu_S(x_j, x_k).$$

Now, we obtain formulae for the inverses of generalized Lehmer matrices for each particular lattices as a result of Theorem 9.

Corollary 10. Let A, $(P, \le)$ and S be as in Corollary 3. Label the rows and the columns of M with the subsets of A. If $\sum_{C \subset B} f(C)^2 (-1)^{|B|-|C|} \neq 0$ for all $B \in \mathcal{P}(A)$, then M defined on S is invertible and its inverse is the $n \times n$ matrix $Z = \left(z_{A_i A_j}\right)$, where

$$z_{A_i A_j} = f(A_i) f(A_j) \sum_{A_k \in \mathcal{P}(A)} \frac{(-1)^{|A_k|-|A_i|-|A_j|}}{\sum_{A_t \subset A_k} f(A_t)^2 (-1)^{|A_t|}}.$$

Proof. If $\sum_{C \subset B} \frac{f(C)^2}{f(B)^2} (-1)^{|B|-|C|} \neq 0$ for all $B \in \mathcal{P}(A)$ then, by Corollary 3, M defined on S is invertible. On the other hand, $G(A_k) = \sum_{A_t \subset A_k} f(A_t)^2 (-1)^{|A_k|-|A_t|}$. Thus, by Theorem 9, the proof is obvious.

Corollary 11. Let $P = \mathbb{Z}^+$ and let $\le$ be the divisibility relation of integers. If $S = \{x_1, x_2, \dots, x_n\}$ is the set of positive divisors of x_n and $\sum_{x_t | x_k} f(x_t)^2 \mu\left(\frac{x_k}{x_t}\right) \neq 0$ for all $x_k \in S$, then M defined on S is invertible and its inverse is the $n \times n$ matrix $Z = (z_{ij})$, where

$$z_{ij} = f(x_i) f(x_j) \sum_{k=1}^{n} \frac{\mu\left(\frac{x_k}{x_i}\right) \mu\left(\frac{x_k}{x_j}\right)}{\sum_{x_t | x_k} f(x_t)^2 \mu\left(\frac{x_k}{x_t}\right)}.$$

Proof. The proof is clear from Corollary 4 and Theorem 9.

Corollary 12. Let $(P, \le)$ be a lattice and let $S = \{x_1, x_2, \dots, x_n\}$ be a finite chain with respect to $\le$ such that $x_1 \le x_2 \le \dots \le x_n$. If $|f(x_k)| \neq |f(x_{k-1})|$ for each $k = 2, 3, \dots, n$, then M is invertible and its inverse is the $n \times n$ matrix $Z = (z_{ij})$, where $z_{11} = \frac{f^2(x_2)}{f^2(x_2) - f^2(x_1)}$, $z_{ii} = \frac{f^2(x_i)(f^2(x_{i+1}) - f^2(x_{i-1}))}{(f^2(x_i) - f^2(x_{i-1}))(f^2(x_{i+1}) - f^2(x_i))}$ for $2 \le i \le n-1$, $z_{nn} = \frac{f^2(x_n)}{f^2(x_n) - f^2(x_{n-1})}$, $z_{ij} = \frac{f(x_i) f(x_{i+1})}{f^2(x_i) - f^2(x_{i+1})}$ for $|i - j| = 1$ and $z_{ij} = 0$ otherwise.

Proof. Let $S = \{x_1, x_2, \dots, x_n\}$ be a finite chain with respect to $\le$ such that $x_1 \le x_2 \le \dots \le x_n$. By the values of the Möbius function on a chain, $G(x_1) = f(x_1)^2$ and $G(x_k) = f(x_k)^2 - f(x_{k-1})^2$ for each $2 \le k \le n$. Then, by Theorem 9,

$$z_{11} = f(x_1)^2 \left(\frac{1}{f(x_1)^2} + \frac{1}{f(x_2)^2 - f(x_1)^2} \right)$$

$$= \frac{f^2(x_2)}{f^2(x_2) - f^2(x_1)}.$$

Let $2 \le i \le n - 1$. Then, by Theorem 9,

$$z_{ii} = f(x_i)^2 \left(\frac{1}{f(x_i)^2 - f(x_{i-1})^2} + \frac{1}{f(x_{i+1})^2 - f(x_i)^2} \right)$$

$$= \frac{f(x_i)^2 (f(x_{i+1})^2 - f(x_{i-1})^2)}{(f(x_i)^2 - f(x_{i-1})^2)(f(x_{i+1})^2 - f(x_i)^2)}.$$

Let $1 \le i \le n - 1$ and $j = i + 1$. Then, we have

$$z_{ij} = f(x_i) f(x_j) \frac{\mu(x_i, x_j) \mu(x_j, x_j)}{G(x_{i+1})}$$

$$= f(x_i) f(x_j) \frac{-1}{f(x_j)^2 - f(x_i)^2}.$$

Since M is symmetric, $z_{ij} = \frac{f(x_i) f(x_{i+1})}{f^2(x_i) - f^2(x_{i+1})}$ for $|i - j| = 1$. By the definition of the Möbius function for a chain, $z_{ij} = 0$ for $|i - j| \ge 2$.

As an application of Corollary 12, we can obtain some certain results concerning the inverses of the classical

Lehmer matrix and its recursive analogues presented in the literature.

Corollary 13. (Theorem 8 in [2]) Let the numbers X_i and the matrix $\mathcal{F}_n$ be as in Corollary 6. If $|X_{k+1}| \neq |X_k|$ for each $k = 2,3,\dots,n$, then the matrix $\mathcal{F}_n$ is invertible and its inverse is the $n \times n$ matrix $H = (h_{ij})$, where $h_{11} = \frac{X_3^2}{X_3^2 - X_2^2}$, $h_{ii} = \frac{X_{i+1}^2 (X_{i+2}^2 - X_i^2)}{(X_{i+1}^2 - X_i^2)(X_{i+2}^2 - X_{i+1}^2)}$ for $2 \leq i \leq n-1$, $h_{nn} = \frac{X_{n+1}^2}{X_{n+1}^2 - X_n^2}$, $h_{ij} = \frac{X_{i+1} X_{i+2}}{X_{i+1}^2 - X_{i+2}^2}$ for $|i-j| = 1$ and $h_{ij} = 0$ otherwise.

Proof. If we take $f(x_i) = X_{i+1}$ in Corollary 12, then we obtain the claim.

Corollary 14. (Theorem 6 in [5]) Let the numbers u_i and the matrix $\mathcal{U}$ be as in Corollary 7. If $|u_{k+1}| \neq |u_k|$ for each $k = 2,3,\dots,n$, then the matrix $\mathcal{U}$ is invertible and its inverse is the $n \times n$ matrix $Q = (q_{ij})$, where $q_{11} = \frac{u_3^2}{u_3^2 - u_2^2}$, $q_{ii} = \frac{u_{i+1}^2 (u_{i+2}^2 - u_i^2)}{(u_{i+1}^2 - u_i^2)(u_{i+2}^2 - u_{i+1}^2)}$ for $2 \leq i \leq n-1$, $q_{nn} = \frac{u_{n+1}^2}{u_{n+1}^2 - u_n^2}$, $q_{ij} = \frac{u_{i+1} u_{i+2}}{u_{i+1}^2 - u_{i+2}^2}$ for $|i-j| = 1$ and $q_{ij} = 0$ otherwise.

Proof. If we take $f(x_i) = u_{i+1}$ in Corollary 12, then we obtain the claim.

Corollary 15. (Theorem 3 in [5]) The classical Lehmer matrix A defined by (1) is invertible and its inverse is the $n \times n$ matrix $B = (b_{ij})$, where $b_{ii} = \frac{4i^3}{4i^2 - 1}$ for $1 \leq i \leq n-1$, $b_{nn} = \frac{n^2}{2n-1}$, $b_{ij} = -\frac{i(i+1)}{2i+1}$ for $|i-j| = 1$ and $b_{ij} = 0$ otherwise.

Proof. If we take $f(x_i) = i$ in Corollary 12, then we obtain the claim.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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