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# Some properties of Konhauser matrix polynomials

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**ABSTRACT**

The main aim of this paper is to derive new summation formulas and integral representations for Konhauser matrix polynomials. In addition, we obtain a raising operator and a Rodrigues formula for these matrix polynomials.
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**Key words**: Laguerre matrix polynomials, Konhauser matrix polynomials, Raising operator, Rodrigues formula.

## 1. INTRODUCTION

Konhauser polynomials introduced by Joseph D. E. Konhauser [1] have the following explicit expressions

$$Z_n^{\alpha}(x;k)=\frac{\Gamma(\alpha+kn+1)}{n!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\frac{x^{kr}}{\Gamma(\alpha+kr+1)}\quad(1.1)$$

$$Y_n^{\alpha}(x;k)=\frac{1}{n!}\sum_{r=0}^{n}\frac{x^r}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}\left(\frac{s+\alpha+1}{k}\right)_n\quad(1.2)$$

and satisfy the biorthogonality condition with respect to weight function $\omega(x)=x^{\alpha}e^{-x}$ over the interval $(0,\infty)$

$$\int_0^{\infty}x^{\alpha}e^{-x}Z_i^{\alpha}(x;k)Y_j^{\alpha}(x;k)dx=\begin{cases}0 & ;\ i\neq j\\ \neq 0 & ;\ i=j\end{cases}\quad(1.3)$$

where $\alpha>-1$ and $k\in\mathbb{Z}^{+}$. These polynomials are the first examples of the notion of biorthogonal polynomials and generalize the Laguerre polynomials in a natural way. For the special case $k=2$, Konhauser polynomials were used in the paper dealing with the penetration of matter by gamma rays per Spencer and Fano [2].

**Remark 1.**For the special case $k=1$, we note that respectively Konhauser polynomials (1.1)-(1.2)and the biorthogonality condition (1.3) reduce to the well-known Laguerre polynomials and its orthogonality condition.

Lately, the notion of orthogonal matrix polynomials studied intensively in the literature. Some results in the theory of classical orthogonal polynomials have been extended to orthogonal matrix polynomials [3-10]. Correspond to these developments, some biorthogonal matrix polynomials, Konhauser matrix polynomials and biorthogonal matrix polynomials suggested by the Jacobi matrix polynomials, were defined in [11-12] during the recent years.
Konhauser matrix polynomials [11],

$$Z_n^{(A,\lambda)}(x;k)=\frac{\Gamma(A+(kn+1)I)}{n!}\sum_{r=0}^{n}(-1)^r\binom{n}{r}\Gamma^{-1}(A+(kr+1)I)(\lambda x)^{kr}\quad(1.4)$$

and

$$Y_n^{(A,\lambda)}(x;k)=\frac{1}{n!}\sum_{r=0}^{n}\frac{(\lambda x)^r}{r!}\sum_{s=0}^{r}(-1)^s\binom{r}{s}\left(\frac{1}{k}(A+(s+1)I)\right)_n\quad(1.5)$$

are biorthogonal with respect to matrix weight function

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$W(x,A) = x^A e^{-\lambda x}$ over the interval $(0,\infty)$. In other words, we have

$$\int_0^\infty x^A e^{-\lambda x} Z_n^{(A,\lambda)}(x;k) Y_m^{(A,\lambda)}(x;k)dx = \begin{cases} \mathbf{0} & ; \quad m \neq n \\ \neq \mathbf{0} & ; \quad m = n \end{cases} \quad (1.6)$$

where $A \in \mathbb{C}^{r\times r}$ is a matrix satisfying the spectral condition

$$\mathrm{Re}(\mu) > -1 \text{ for } \forall\mu \in \sigma(A) \ .$$

Here $\sigma(A)$ denotes the set of all eigenvalues of $A$ and $\lambda$ is a complex number with $\mathrm{Re}(\lambda) > 0$. Moreover, $Z_n^{(A,\lambda)}(x;k)$ matrix polynomials satisfy the following relation

$$\int_0^\infty x^A e^{-\lambda x} Z_n^{(A,\lambda)}(x;k)(\lambda x)^i dx = \begin{cases} \mathbf{0} & ; \quad i = 0,1,2,\ldots,n-1 \\ \neq \mathbf{0} & ; \quad i = n \end{cases} . \quad (1.7)$$

**Remark 2.** In a similar way, respectively put $k = 1$ in the Konhauser matrix polynomials (1.4)-(1.5), biorthogonality condition (1.6) and the relation (1.7), they reduce to Laguerre matrix polynomials and its orthogonality condition [7].

In this paper, our aim is to derive some new summation formulas and integral representations for the $Y_n^{(A,\lambda)}(x;k)$ matrix polynomials by the help of following matrix generating function from [11]

$$g(x,w,A) = (1-w)^{-\frac{1}{k}(A+I)} \exp\left[-\lambda x\left\{(1-w)^{-\frac{1}{k}} - 1\right\}\right]$$

$$= \sum_{n=0}^\infty Y_n^{(A,\lambda)}(x;k)w^n \ , \ |w| < 1. \quad (1.8)$$

Furthermore, we obtain raising operator and Rodrigues formula for the matrix polynomials $Z_n^{(A,\lambda)}(x;k)$.

## 2.SOME RESULTS FOR $Y_n^{(A,\lambda)}(x;k)$ MATRIX POLYNOMIALS

First of all, let us recall the following definitions and lemmas.

**Definition 1.** A matrix polynomial of degree n means an expression of the form

$$P_n(x) = A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0$$

where $x$ is a real variable and $A_j \in \mathbb{C}^{r\times r}$ for $0 \le j \le n$.

**Lemma 1. (Dunford and Schwartz [13])** If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable $z$, which are defined in an open set $\Omega$ of the plane, and if $A$ is a matrix in $\mathbb{C}^{r\times r}$ for $\sigma(A) \subset \Omega$, then

$$f(A)g(A) = g(A)f(A).$$

Hence, if $B \in \mathbb{C}^{r\times r}$ is a matrix for $\sigma(B) \subset \Omega$, such that $AB = BA$, then

$$f(A)g(B) = g(B)f(A).$$

**Definition 2.** Let $A$ be a matrix in $\mathbb{C}^{r\times r}$. We say that $A$ is positive stable if $\mathrm{Re}(\delta) > 0$ for all $\delta \in \sigma(A)$.

**Definition 3.** For $A \in \mathbb{C}^{r\times r}$, the matrix form of the Pochhammer symbol is defined by

$$(A)_k = A(A+I)\ldots(A+(k-1)I) \quad , \quad k \in \mathbb{N}$$

with $(A)_0 = I$ where $I$ is the identity matrix of $\mathbb{C}^{r\times r}$.

**Lemma 2. (Jódar and Cortés [14])** For any matrix $A \in \mathbb{C}^{r\times r}$, the following serial expansion holds

$$(1-x)^{-A} = \sum_{n=0}^\infty \frac{(A)_n}{n!}x^n \quad , \quad |x| < 1.$$

With the following theorems, we derive new summation formulas and integral representations for the $Y_n^{(A,\lambda)}(x;k)$ matrix polynomials.

**Theorem 1.** Let $n \in \mathbb{N}$, $s$ and $k$ be the positive integers with $s \ge 2$. The following equality holds for the $Y_n^{(A,\lambda)}(x;k)$ matrix polynomials

$$\sum_{m=0}^n \frac{\left(\frac{(s-1)}{k}(A+I)\right)_m}{m!} Y_{n-m}^{(A,\lambda)}(sx;k) = \sum_{n_1+\ldots+n_s=n} Y_{n_1}^{(A,\lambda)}(x;k)\ldots Y_{n_s}^{(A,\lambda)}(x;k).$$

(2.1)

**Proof.** Applying $(s-1)$ times the partial derivativeoperator with respect to $x$ to the both sides of

$$g(sx,w,sA) = (1-w)^{-\frac{1}{k}(sA+I)} \exp\left[-\lambda sx\left\{(1-w)^{-\frac{1}{k}} - 1\right\}\right],$$

then by virtue of the matrix generating function and the equality $Ix = x^I$, we have

$$\frac{\partial^{(s-1)} g(sx,w,sA)}{\partial x^{(s-1)}}$$
$$= (-1)^{s-1}\left[\lambda s\left(1-(1-w)^{\frac{1}{k}}\right)\right]^{s-1}(1-w)^{-\frac{s}{k}(A+I)}\exp\left[-\lambda sx\left\{(1-w)^{-\frac{1}{k}}-1\right\}\right]$$
$$= (-1)^{s-1}\left[\lambda s\left(1-(1-w)^{\frac{1}{k}}\right)\right]^{s-1}(1-w)^{-\frac{(s-1)}{k}(A+I)}$$
$$\times (1-w)^{-\frac{1}{k}(A+I)}\exp\left[-\lambda sx\left\{(1-w)^{-\frac{1}{k}}-1\right\}\right]$$
$$= (-1)^{s-1}\left[\lambda s\left(1-(1-w)^{\frac{1}{k}}\right)\right]^{s-1}(1-w)^{-\frac{(s-1)}{k}(A+I)}\sum_{n=0}^\infty Y_n^{(A,\lambda)}(sx;k)w^n.$$

(2.2)

According to Lemma 2 and the Cauchy product, (2.2) becomes

$$\frac{\partial^{(s-1)} g(sx,w,sA)}{\partial x^{(s-1)}}$$
$$= (-1)^{s-1}\left[\lambda s\left(1-(1-w)^{\frac{1}{k}}\right)\right]^{s-1}\sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}\frac{\left(\frac{(s-1)}{k}(A+I)\right)_m}{m!}Y_{n-m}^{(A,\lambda)}(sx;k)\right]w^n. \tag{2.3}$$

On the other hand, in view of the matrix generating function (1.8), we get

$$\frac{\partial^{(s-1)} g(sx,w,sA)}{\partial x^{(s-1)}}$$
$$= (-1)^{s-1}\left[\lambda s\left(1-(1-w)^{\frac{1}{k}}\right)\right]^{s-1}\left[(1-w)^{-\frac{1}{k}(A+I)}\exp\left[-\lambda x\left\{(1-w)^{-\frac{1}{k}}-1\right\}\right]\right]^s$$
$$= (-1)^{s-1}\left[\lambda s\left(1-(1-w)^{\frac{1}{k}}\right)\right]^{s-1}\left[\sum_{n=0}^{\infty}Y_n^{(A,\lambda)}(x;k)w^n\right]^s$$
$$= (-1)^{s-1}\left[\lambda s\left(1-(1-w)^{\frac{1}{k}}\right)\right]^{s-1}\sum_{n=0}^{\infty}\left[\sum_{n_1+\ldots+n_s=n}Y_{n_1}^{(A,\lambda)}(x;k)\ldots Y_{n_s}^{(A,\lambda)}(x;k)\right]w^n. \tag{2.4}$$

Comparing (2.3) with (2.4), we obtain the equality (2.1).

**Theorem 2.** Let $n \in \mathbb{N}$, $s$ and $k$ be the positive integers with $s \geq 2$. $Y_n^{(A,\lambda)}(x;k)$ matrix polynomials satisfy the following equality

$$\sum_{n_1+\ldots+n_s=n}\int_0^{\infty}Y_{n_1}^{(A,\lambda)}(x;k)\ldots Y_{n_s}^{(A,\lambda)}(x;k)\exp(-\lambda s x)dx = \frac{1}{\lambda s}\frac{\left(\frac{1}{k}(sA+(s-1)I)\right)_n}{n!}. \tag{2.5}$$

**Proof.** Let us consider the following equality

$$(1-w)^{-\frac{s}{k}(A+I)}\exp\left[-\lambda sx\left\{(1-w)^{-\frac{1}{k}}-1\right\}\right] = \left[\sum_{n=0}^{\infty}Y_n^{(A,\lambda)}(x;k)w^n\right]^s$$
$$= \sum_{n=0}^{\infty}\left[\sum_{n_1+\ldots+n_s=n}Y_{n_1}^{(A,\lambda)}(x;k)\ldots Y_{n_s}^{(A,\lambda)}(x;k)\right]w^n. \tag{2.6}$$

Multiplying the both sides of the equality (2.6) by $\exp(-\lambda sx)$, then integrating with respect to $x$ over the interval $(0,\infty)$, in view of Lemma 2, (2.6) leads to

$$\sum_{n=0}^{\infty}\left[\sum_{n_1+\ldots+n_s=n}\int_0^{\infty}Y_{n_1}^{(A,\lambda)}(x;k)\ldots Y_{n_s}^{(A,\lambda)}(x;k)\exp(-\lambda sx)dx\right]w^n$$
$$= (1-w)^{-\frac{s}{k}(A+I)}\int_0^{\infty}\exp\left[-\lambda sx\left\{(1-w)^{-\frac{1}{k}}-1\right\}\right]\exp(-\lambda sx)dx$$
$$= \frac{1}{\lambda s}(1-w)^{-\frac{1}{k}(sA+(s-1)I)} = \sum_{n=0}^{\infty}\left[\frac{1}{\lambda s}\frac{\left(\frac{1}{k}(sA+(s-1)I)\right)_n}{n!}\right]w^n$$

Equating the coefficients of $w^n$ gives the equality (2.5).

**Theorem 3.** Let $n \in \mathbb{N}$, $s$ and $k$ be the positive integers with $s \geq 2$, $A_i \in \mathbb{C}^{r\times r}$ satisfying the spectral conditions $\mathrm{Re}(\mu) > -1$ for $\forall \mu \in \sigma(A_i)$ and $\lambda_i$ be a complex parameter with $\mathrm{Re}(\lambda_i) > 0$ for $i = 1,2,\ldots,s$. The following equality is valid for the $Y_n^{(A,\lambda)}(x;k)$ matrix polynomials

$$\sum_{m=0}^{n}\frac{\left(\frac{1}{k}(\mathbf{A}-A_i+(s-1)I)\right)_m Y_{n-m}^{(A_i,\lambda_1+\ldots+\lambda_s)}(sx;k)}{m!}$$
$$= \sum_{n_1+\ldots+n_s=n}Y_{n_1}^{(A_1,\lambda_1)}(sx;k)\ldots Y_{n_s}^{(A_s,\lambda_s)}(sx;k) \tag{2.7}$$

where $\mathbf{A} = A_1 + \ldots + A_s$ and the matrices $A_1,\ldots,A_s$ are assumed to be commutative.

**Proof.** Taking $(s-1)$ times the partial derivative with respect to $x$ to the both sides of

$$g(sx,w,\mathbf{A}) = (1-w)^{-\frac{1}{k}(\mathbf{A}+I)}\exp\left[-(\lambda_1+\ldots+\lambda_s)sx\left\{(1-w)^{-\frac{1}{k}}-1\right\}\right]$$

then by using the matrix generating function (1.8) and the equality $Ix = x^I$, we get

$$\frac{\partial^{(s-1)} g(sx,w,\mathbf{A})}{\partial x^{(s-1)}}$$
$$= (-1)^{s-1}\left[(\lambda_1+\ldots+\lambda_s)s\left(1-(1-w)^{\frac{1}{k}}\right)\right]^{s-1}$$
$$\times(1-w)^{-\frac{1}{k}(\mathbf{A}+sI)}\exp\left[-(\lambda_1+\ldots+\lambda_s)sx\left\{(1-w)^{-\frac{1}{k}}-1\right\}\right]$$
$$= (-1)^{s-1}\left[(\lambda_1+\ldots+\lambda_s)s\left(1-(1-w)^{\frac{1}{k}}\right)\right]^{s-1}(1-w)^{-\frac{1}{k}(\mathbf{A}-A_i+(s-1)I)}$$
$$\times(1-w)^{-\frac{1}{k}(A_i+I)}\exp\left[-(\lambda_1+\ldots+\lambda_s)sx\left\{(1-w)^{-\frac{1}{k}}-1\right\}\right]$$
$$= (-1)^{s-1}\left[(\lambda_1+\ldots+\lambda_s)s\left(1-(1-w)^{\frac{1}{k}}\right)\right]^{s-1}(1-w)^{-\frac{1}{k}(\mathbf{A}-A_i+(s-1)I)}$$
$$\times\sum_{n=0}^{\infty}Y_n^{(A_i,\lambda_1+\ldots+\lambda_s)}(sx;k)w^n. \tag{2.8}$$

According to Lemma 2 and the Cauchy product, (2.8) leads to

$$\frac{\partial^{(s-1)} g(sx,w,\mathbf{A})}{\partial x^{(s-1)}}$$
$$= (-1)^{s-1}\left[(\lambda_1+\ldots+\lambda_s)s\left(1-(1-w)^{\frac{1}{k}}\right)\right]^{s-1}$$
$$\times\sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}\frac{\left(\frac{1}{k}(\mathbf{A}-A_i+(s-1)I)\right)_m}{m!}Y_{n-m}^{(A_i,\lambda_1+\ldots+\lambda_s)}(sx;k)\right]w^n. \tag{2.9}$$

Furthermore, taking into account the matrix generating function (1.8), we find

$$\begin{aligned}\frac{\partial^{(s-1)} g(sx,w,\mathbf{A})}{\partial x^{(s-1)}} &= (-1)^{s-1}\left[(\lambda_1+\ldots+\lambda_s)s\left(1-(1-w)^{\frac{1}{k}}\right)\right]^{s-1}\\ &\times(1-w)^{-\frac{1}{k}(A_1+\ldots A_s+sI)}\exp\left[-(\lambda_1+\ldots+\lambda_s)sx\left\{(1-w)^{-\frac{1}{k}}-1\right\}\right]\\ &= (-1)^{s-1}\left[(\lambda_1+\ldots+\lambda_s)s\left(1-(1-w)^{\frac{1}{k}}\right)\right]^{s-1}\\ &\times\left(\sum_{n_1=0}^{\infty}Y_{n_1}^{(A_1,\lambda_1)}(sx;k)w^{n_1}\right)\ldots\left(\sum_{n_s=0}^{\infty}Y_{n_s}^{(A_s,\lambda_s)}(sx;k)w^{n_s}\right)\\ &= (-1)^{s-1}\left[(\lambda_1+\ldots+\lambda_s)s\left(1-(1-w)^{\frac{1}{k}}\right)\right]^{s-1}\\ &\times\sum_{n=0}^{\infty}\left[\sum_{n_1+\ldots+n_s=n}Y_{n_1}^{(A_1,\lambda_1)}(sx;k)\ldots Y_{n_s}^{(A_s,\lambda_s)}(sx;k)\right]w^n.\end{aligned} \tag{2.10}$$

Comparing (2.9) with (2.10), we obtain the equality (2.7).

**Theorem 4.** Let $n\in\mathbb{N}$, $s$ and $k$ be the positive integers with $s\geq 2$, $A_i\in\mathbb{C}^{r\times r}$ satisfying the spectral conditions $\mathrm{Re}(\mu)>-1$ for $\forall\mu\in\sigma(A_i)$ and $\lambda_i$ be a complex parameter with $\mathrm{Re}(\lambda_i)>0$ for $i=1,2,\ldots,s$. $Y_n^{(A,\lambda)}(x;k)$ matrix polynomials fulfil the following equality

$$\sum_{n_1+\ldots+n_s=n}\int_0^{\infty}Y_{n_1}^{(A_1,\lambda_1)}(sx;k)\ldots Y_{n_s}^{(A_s,\lambda_s)}(sx;k)\exp[-(\lambda_1+\ldots+\lambda_s)sx]dx$$
$$=\frac{1}{s(\lambda_1+\ldots+\lambda_s)}\frac{\left(\frac{1}{k}(\mathbf{A}+(s-1)I)\right)_n}{n!} \tag{2.11}$$

where $\mathbf{A}=A_1+\ldots+A_s$ and the matrices $A_1,\ldots,A_s$ are assumed to be commutative.

**Proof.** Let us consider the following equality

$$\begin{aligned}&(1-w)^{-\frac{1}{k}(A_1+\ldots A_s+sI)}\exp\left[-(\lambda_1+\ldots+\lambda_s)sx\left\{(1-w)^{-\frac{1}{k}}-1\right\}\right]\\ &=\left(\sum_{n_1=0}^{\infty}Y_{n_1}^{(A_1,\lambda_1)}(sx;k)w^{n_1}\right)\ldots\left(\sum_{n_s=0}^{\infty}Y_{n_s}^{(A_s,\lambda_s)}(sx;k)w^{n_s}\right)\\ &=\sum_{n=0}^{\infty}\left[\sum_{n_1+\ldots+n_s=n}Y_{n_1}^{(A_1,\lambda_1)}(sx;k)\ldots Y_{n_s}^{(A_s,\lambda_s)}(sx;k)\right]w^n.\end{aligned} \tag{2.12}$$

Multiplying the both sides of the equality (2.12) by $\exp(-(\lambda_1+\ldots+\lambda_s)sx)$, then integrating with respect to $x$ over the interval $(0,\infty)$, by virtue of Lemma 2, (2.12) becomes

$$\begin{aligned}&\sum_{n=0}^{\infty}\left[\sum_{n_1+\ldots+n_s=n}\int_0^{\infty}Y_{n_1}^{(A_1,\lambda_1)}(sx;k)\ldots Y_{n_s}^{(A_s,\lambda_s)}(sx;k)\exp[-(\lambda_1+\ldots+\lambda_s)sx]dx\right]w^n\\ &=(1-w)^{-\frac{1}{k}(A_1+\ldots A_s+sI)}\\ &\times\int_0^{\infty}\exp\left[-(\lambda_1+\ldots+\lambda_s)sx\left\{(1-w)^{-\frac{1}{k}}-1\right\}\right]\exp[-(\lambda_1+\ldots+\lambda_s)sx]dx\\ &=\frac{1}{s(\lambda_1+\ldots+\lambda_s)}(1-w)^{-\frac{1}{k}(\mathbf{A}+(s-1)I)}=\sum_{n=0}^{\infty}\left[\frac{1}{s(\lambda_1+\ldots+\lambda_s)}\frac{\left(\frac{1}{k}(\mathbf{A}+(s-1)I)\right)_n}{n!}\right]w^n\end{aligned}$$

Equating the coefficients of $w^n$ gives the equality (2.11).

**Theorem 5.** Let $n\in\mathbb{N}$, $s$ and $k$ be the positive integers with $s\geq 2$, $A_i\in\mathbb{C}^{r\times r}$ satisfying the spectral conditions $\mathrm{Re}(\mu)>-1$ for $\forall\mu\in\sigma(A_i)$ and $\lambda_i$ be a complex parameter with $\mathrm{Re}(\lambda_i)>0$ for $i=1,2,\ldots,s$. Then, we have

$$\sum_{m=0}^{n}\frac{\left(\frac{1}{k}(\mathbf{A}-A_i+(s-1)I)\right)_m Y_{n-m}^{(A_i,\lambda_i)}\left(\frac{\lambda_1x_1+\ldots+\lambda_sx_s}{\lambda_i};k\right)}{m!}$$
$$=\sum_{n_1+\ldots+n_s=n}Y_{n_1}^{(A_1,\lambda_1)}(x_1;k)\ldots Y_{n_s}^{(A_s,\lambda_s)}(x_s;k) \tag{2.13}$$

where $\mathbf{A}=A_1+\ldots+A_s$ and the matrices $A_1,\ldots,A_s$ are assumed to be commutative.

**Proof.** Applying $(s-1)$ times the partial derivative operator with respect to $x_i$ for $i=1,2,\ldots,s$ to the both sides of

$$g(x_1,\ldots,x_s,w,\mathbf{A})=(1-w)^{-\frac{1}{k}(\mathbf{A}+I)}\exp\left[-(\lambda_1x_1+\ldots+\lambda_sx_s)\left\{(1-w)^{-\frac{1}{k}}-1\right\}\right]$$

then by using the matrix generating function (1.8) and the equality $Ix=x^I$, we get

$$\frac{\partial^{(s-1)} g(x_1,\ldots,x_s,w,\mathbf{A})}{\partial x_i^{(s-1)}}$$
$$= (-1)^{s-1}\left[\lambda_i\left(1-(1-w)^{\frac{1}{k}}\right)\right]^{s-1}$$
$$\times (1-w)^{-\frac{1}{k}(\mathbf{A}+sI)}\exp\left[-(\lambda_1 x_1+\ldots+\lambda_s x_s)\left\{(1-w)^{-\frac{1}{k}}-1\right\}\right]$$
$$= (-1)^{s-1}\left[\lambda_i\left(1-(1-w)^{\frac{1}{k}}\right)\right]^{s-1}(1-w)^{-\frac{1}{k}(\mathbf{A}-A_i+(s-1)I)}$$
$$\times (1-w)^{-\frac{1}{k}(A_i+I)}\exp\left[-\lambda_i\left(\frac{\lambda_1 x_1+\ldots+\lambda_s x_s}{\lambda_i}\right)\left\{(1-w)^{-\frac{1}{k}}-1\right\}\right]$$
$$= (-1)^{s-1}\left[\lambda_i\left(1-(1-w)^{\frac{1}{k}}\right)\right]^{s-1}(1-w)^{-\frac{1}{k}(\mathbf{A}-A_i+(s-1)I)}$$
$$\times \sum_{n=0}^{\infty} Y_n^{(A_i,\lambda_i)}\left(\frac{\lambda_1 x_1+\ldots+\lambda_s x_s}{\lambda_i};k\right)w^n. \tag{2.14}$$

According to Lemma 2 and the Cauchy product, (2.14) becomes

$$\frac{\partial^{(s-1)} g(x_1,\ldots,x_s,w,\mathbf{A})}{\partial x_i^{(s-1)}}$$
$$= (-1)^{s-1}\left[\lambda_i\left(1-(1-w)^{\frac{1}{k}}\right)\right]^{s-1}$$
$$\times \sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}\frac{\left(\frac{1}{k}(\mathbf{A}-A_i+(s-1)I)\right)_m Y_{n-m}^{(A_i,\lambda_i)}\left(\frac{\lambda_1 x_1+\ldots+\lambda_s x_s}{\lambda_i};k\right)}{m!}\right]w^n. \tag{2.15}$$

On the other hand, in view of the matrix generating function (1.8), we have

$$\frac{\partial^{(s-1)} g(x_1,\ldots,x_s,w,\mathbf{A})}{\partial x_i^{(s-1)}}$$
$$= (-1)^{s-1}\left[\lambda_i\left(1-(1-w)^{\frac{1}{k}}\right)\right]^{s-1}$$
$$\times (1-w)^{-\frac{1}{k}(A_1+\ldots+A_s+sI)}\exp\left[-(\lambda_1 x_1+\ldots+\lambda_s x_s)\left\{(1-w)^{-\frac{1}{k}}-1\right\}\right]$$
$$= (-1)^{s-1}\left[\lambda_i\left(1-(1-w)^{\frac{1}{k}}\right)\right]^{s-1}$$
$$\times\left(\sum_{n_1=0}^{\infty} Y_{n_1}^{(A_1,\lambda_1)}(x_1;k)w^{n_1}\right)\ldots\left(\sum_{n_s=0}^{\infty} Y_{n_s}^{(A_s,\lambda_s)}(x_s;k)w^{n_s}\right)$$
$$= (-1)^{s-1}\left[\lambda_i\left(1-(1-w)^{\frac{1}{k}}\right)\right]^{s-1}$$
$$\times\sum_{n=0}^{\infty}\left[\sum_{n_1+\ldots+n_s=n} Y_{n_1}^{(A_1,\lambda_1)}(x_1;k)\ldots Y_{n_s}^{(A_s,\lambda_s)}(x_s;k)\right]w^n. \tag{2.16}$$

Comparing (2.15) with (2.16), we obtain the equality (2.13).

**Theorem 6.** Let $n\in\mathbb{N}$, $s$ and $k$ be the positive integers with $s\geq 2$ , $A_i\in\mathbb{C}^{r\times r}$ satisfying the spectral conditions $\mathrm{Re}(\mu)>-1$ for $\forall\mu\in\sigma(A_i)$ and $\lambda_i$ be a complex parameter with $\mathrm{Re}(\lambda_i)>0$ for $i=1,2,\ldots,s$ . Then, we get

$$\sum_{n_1+\ldots+n_s=n}\left[\int_{x_1=0}^{\infty}\ldots\int_{x_s=0}^{\infty} Y_{n_1}^{(A_1,\lambda_1)}(x_1;k)\ldots Y_{n_s}^{(A_s,\lambda_s)}(x_s;k)\right.$$
$$\left.\times\exp[-(\lambda_1 x_1+\ldots+\lambda_s x_s)]dx_s\ldots dx_1\right]$$
$$= \frac{\left(\frac{1}{k}\mathbf{A}\right)_n}{\lambda_1\ldots\lambda_s n!} \tag{2.17}$$

where $\mathbf{A}=A_1+\ldots+A_s$ and the matrices $A_1,\ldots,A_s$ are assumed to be commutative.

**Proof.** Let us consider the following equality

$$(1-w)^{-\frac{1}{k}(\mathbf{A}+sI)}\exp\left[-(\lambda_1 x_1+\ldots+\lambda_s x_s)\left\{(1-w)^{-\frac{1}{k}}-1\right\}\right]$$
$$=\sum_{n=0}^{\infty}\left[\sum_{n_1+\ldots+n_s=n} Y_{n_1}^{(A_1,\lambda_1)}(x_1;k)\ldots Y_{n_s}^{(A_s,\lambda_s)}(x_s;k)\right]w^n. \tag{2.18}$$

Multiplying the both sides of the equality (2.18) by $\exp[-(\lambda_1 x_1+\ldots+\lambda_s x_s)]$ , then integrating over the domain

$$\Lambda=\left\{(x_1,\ldots,x_s):0<x_i<\infty\ ,\ \ i=1,\ldots,s\right\},$$

taking into account of Lemma 2, we obtain

$$\sum_{n=0}^{\infty}\left\{\sum_{n_1+\ldots+n_s=n}\int_{x_1=0}^{\infty}\ldots\int_{x_s=0}^{\infty} Y_{n_1}^{(A_1,\lambda_1)}(x_1;k)\ldots Y_{n_s}^{(A_s,\lambda_s)}(x_s;k)\right.$$
$$\left.\times\exp[-(\lambda_1 x_1+\ldots+\lambda_s x_s)]dx_s\ldots dx_1\right\}w^n$$
$$= (1-w)^{-\frac{1}{k}(\mathbf{A}+sI)}\int_{x_1=0}^{\infty}\ldots\int_{x_s=0}^{\infty}\exp\left[-(\lambda_1 x_1+\ldots+\lambda_s x_s)\left\{(1-w)^{-\frac{1}{k}}-1\right\}\right]$$
$$\times\exp[-(\lambda_1 x_1+\ldots+\lambda_s x_s)]dx_s\ldots dx_1$$
$$= \frac{1}{\lambda_1\ldots\lambda_s}(1-w)^{-\frac{1}{k}\mathbf{A}} = \sum_{n=0}^{\infty}\frac{\left(\frac{1}{k}\mathbf{A}\right)_n}{\lambda_1\ldots\lambda_s n!}w^n.$$

Equating the coefficients of $w^n$ gives the equality (2.17) .

**Remark 3.** It is worthy to note that for $k=1$ , all results obtained in Theorems $1-6$ reduce to Laguerre matrix polynomials case given in [15].

## 3. RAISING OPERATOR and RODRIGUES FORMULA FOR $Z_n^{(A,\lambda)}(x;k)$ MATRIX POLYNOMIALS

In this section, we obtain a raising operator and a Rodrigues formula for $Z_n^{(A,\lambda)}(x;k)$ matrix polynomials. We need following lemma which is used in the sequel.

**Lemma 3.** (Jodar et al. [7]) Let $A\in\mathbb{C}^{r\times r}$ be a matrix satisfying the spectral condition $\mathrm{Re}(\mu)>-1$ for $\forall\mu\in\sigma(A)$ and $\lambda$ be a complex parameter with $\mathrm{Re}(\lambda) > 0$ . Then any fixed matrix polynomial $P(t)$ satisfies the following limit relations

$$\lim_{t\to 0^+} e^{-\lambda t} t^{A+I} P(t) = \mathbf{0} \text{ and } \lim_{t\to\infty} e^{-\lambda t} t^{A+I} P(t) = \mathbf{0}$$

where $t^{A+I} = \exp[(A+I)\ln t]$ for $t > 0$ .

**Theorem 7.**Let $A \in \mathbb{C}^{r\times r}$ be a positive stable matrix and $\lambda$ be a complex parameter with $\mathrm{Re}(\lambda) > 0$ . The raising operator of $Z_n^{(A,\lambda)}(x;k)$ matrix polynomials is given with the following equality

$$\frac{d}{dx}\left[x^A e^{-\lambda x} Z_n^{(A,\lambda)}(x;k)\right] = \lambda^{1-k}(n+1)x^{A-kI}e^{-\lambda x} Z_{n+1}^{(A-kI,\lambda)}(x;k).$$

(3.1)

**Proof.** Let us consider the left-hand side of the equality (3.1). Then we have

$$\begin{aligned}&\frac{d}{dx}\left[x^A e^{-\lambda x} Z_n^{(A,\lambda)}(x;k)\right]\\ &= x^{A-kI}e^{-\lambda x}\left\{x^k \frac{d}{dx}\left[Z_n^{(A,\lambda)}(x;k)\right] + Z_n^{(A,\lambda)}(x;k)[Ax^{(k-1)I} - \lambda x^{kI}]\right\}\\ &= x^{A-kI}e^{-\lambda x}\Theta_{n+1}(x)\end{aligned}$$

(3.2)

where

$$\Theta_{n+1}(x) := x^k \frac{d}{dx}\left[Z_n^{(A,\lambda)}(x;k)\right] + Z_n^{(A,\lambda)}(x;k)\left[Ax^{(k-1)I} - \lambda x^{kI}\right]$$

Note that $\Theta_{n+1}(x)$ is a polynomial of degree $n+1$ with respect to $x^k$ . Multiplying the both sides of the equality (3.2) by $(\lambda x)^i$ for $i = 0,1,\ldots,n$, then integrating over the interval $(0,\infty)$ , we obtain according to Lemma 3 and the relation (1.7)

$$\begin{aligned}\int_0^\infty x^{A-kI}e^{-\lambda x}\Theta_{n+1}(x)(\lambda x)^i dx &= \int_0^\infty \frac{d}{dx}\left[x^A e^{-\lambda x} Z_n^{(A,\lambda)}(x;k)\right](\lambda x)^i dx\\ &= \lambda^i x^{A+iI} e^{-\lambda x} Z_n^{(A,\lambda)}(x;k)\Big|_0^\infty\\ &\quad - \lambda i \int_0^\infty x^A e^{-\lambda x} Z_n^{(A,\lambda)}(x;k)(\lambda x)^{i-1} dx\\ &= \mathbf{0}.\end{aligned}$$

(3.3)

This result means that $\Theta_{n+1}(x)$ is a biorthogonal matrix polynomial with respect to matrix weight function $x^{A-kI}e^{-\lambda x}$ over the interval $(0,\infty)$ . In view of the relation (1.7), it is understood from (3.3) that

$$\Theta_{n+1}(x) = CZ_{n+1}^{(A-kI,\lambda)}(x;k).$$

By comparing the coefficients of $(x^k)^{n+1}$ , we find $C = \lambda^{1-k}(n+1)$ . We obtain the desired result by substituting $\Theta_{n+1}(x)$ in the equality (3.2).

**Theorem 8.**Let $A \in \mathbb{C}^{r\times r}$ be a positive stable matrix and $\lambda$ be a complex parameter with $\mathrm{Re}(\lambda) > 0$ . The Rodrigues formula for $Z_n^{(A,\lambda)}(x;k)$ matrix polynomials is given with the following equality

$$Z_n^{(A,\lambda)}(x;k) = \frac{1}{n!}\lambda^{n(k-1)} x^{-A} e^{\lambda x} \frac{d^n}{dx^n}[x^{A+knI}e^{-\lambda x}].$$

(3.4)

**Proof.** Applying the raising operator given with the equality (3.1) to the matrix polynomials

$$Z_0^{(A+nkI,\lambda)}(x;k) = I$$

successively. Then we obtain that

$$\frac{d^n}{dx^n}[x^{A+knI}e^{-\lambda x}] = \lambda^{n(1-k)} n! x^A e^{-\lambda x} Z_n^{(A,\lambda)}(x;k),$$

which gives the equality (3.4) .

**Remark 4.**For the special case $k = 1$ , the raising operator (3.1) reduces to the raising operator for Laguerre matrix polynomials obtained in [15] and the Rodrigues Formula (3.4) returns to the Rodrigues formula for Laguerre matrix polynomials given in [7].

## CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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