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B-LIFT CURVES AND ITS RULED SURFACES

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Abstract. In this paper, we have described the B-Lift curve in Euclidean space as a curve obtained by combining the endpoints of the binormal vector of a unit speed curve. Subsequently, we have explored the Frenet frames of the B-Lift curves. Moreover, we have introduced the tangent, normal and binormal surfaces of the B-Lift curve and examined the geometric invariants of these surfaces. Finally, we have investigated the singularities of these surface and visualized the surfaces with MATLAB program.

1. Introduction

Ruled surfaces have important applications in kinematics, computer science, physics, etc. A ruled surface is defined by a straight line that is moving along a curve [1]. Many mathematicians have studied the ruled surfaces [2–8]. E. Ergün and M. Çalışkan [2] created ruled surfaces by accepting the natural lift of a curve as the base curve and they characterized these surfaces. The natural lift curve is described in an example in Thorpe's book. Generally, the natural lift curve is defined as the curve formed by combining the end points of the tangent vectors of the curve [9].

One of the main purposes of classical differential geometry is to investigate some classes of surfaces such as developable surfaces and minimal surfaces. Ruled surfaces are developable surfaces with zero Gaussian curvature suach that these surfaces are called minimal surfaces [10]. S. Izumiya and N. Takeuchi presented new results for the Gaussian curvature and the main curvature of the ruled surface [3].

A point is called the singular point of the surface if the tangent vector at any point does not lie in a plane. At the singular point, the surface intersects itself. If

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all points of a curve on a surface are singular, this curve is called a singular curve [1]. Recently, many studies have been done on the singularity of curves [11–16].

In this study, we define a new curve which is called B-Lift curve and we calculate its Frenet vectors. Furthermore, we examine the integral invariants of the tangent, principal normal and binormal surfaces of the B-Lift curve. Also, we study the singular points of the ruled surfaces of the B-Lift curve. Finally, we give examples of these situations and drawn our surfaces.

2. Preliminaries

Let a vector $\vec{x} = (x_1, x_2, x_3)$ be given in $\mathbb{R}^3$. The norm of $\vec{x} = (x_1, x_2, x_3)$ is defined by

$$\|\vec{x}\|= \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

A vector which its norm is 1 is called a unit vector. For the vectors $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$, the inner product $<,>: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined as

$$< \vec{x}, \vec{y} >= x_1y_1 + x_2y_2 + x_3y_3$$

which is called Euclidean inner product. If $\gamma^{'}(s) \neq 0$, $\gamma :$ I $\to\mathbb{R}^3$ is called regular curve , for all s $\in$ I. Let $\gamma :$ I $\to\mathbb{R}^3$ be a curve, if $\|\gamma^{'}(s)\|= 1$ then the curve is called unit speed curve [1].

A curve α is called general helix in $\mathbb{R}^3$ if tangent vector of the curve makes a constant angle with a fixed straight line. M. A. Lancret discovered that the ratio of curvatures of the general helix is constant in 1802 [17].

Let γ be a regular curve in $\mathbb{R}^3$. The set $\{T(s), N(s), B(s)\}$ is called Frenet frame given by tangent, principal normal and binormal vectors, respectively.

$$\begin{aligned} T(s)&=\gamma^{'}(s), \\ N(s)&=\frac{\gamma^{''}(s)}{\|\gamma^{''}(s)\|}, \\ B(s)&=T(s) \times N(s), \end{aligned}$$

Here $T(s)$, $N(s)$ and $B(s)$ are the unit tangent, principal normal and binormal vectors of $\gamma(s)$, respectively. Frenet-Serret formulas are following as [10]:

$$\begin{aligned} T^{'}(s)&=\kappa(s)N(s), \\ N^{'}(s)&=-\kappa(s)T(s) + \tau(s)B(s), \\ B^{'}(s)&=-\tau(s)N(s). \end{aligned}$$

When a point moves along a curve with unit speed, the rotation is determined by an angular velocity vector W that is called Darboux vector. The Darboux vector W is presented as $W = \tau T + \kappa B$. Moreover, $\kappa = \|W\| \cos\varphi$ and $\tau = \|W\| \sin\varphi$ are written. Here φ is the angle between Darboux vector and binormal vector of $\gamma(s)$ [10].

Let γ be a regular curve and ω be unit direction of a straight line in $\mathbb{R}^3$, then the ruled surface ϕ is the surface formed by the continuous moving of ω based on the curve γ. The parametric representation of the ruled surface ϕ is given as follows [10]:

$$\phi(s,v)=\gamma(s)+v\omega(s).$$

For the ruled surface ϕ(s,v), we can write

$$\phi_s\times\phi_v=\gamma^{'}(s)\times\omega(s)+v\omega^{'}(s)\times\omega(s).$$

Hence (s_0,v_0) is a singular point of ϕ(s,v) if and only if $\gamma^{'}(s_0)\times\omega(s_0)+v_0\omega^{'}(s_0)\times\omega(s_0)=0$. If $\omega^{'}(s)\times\omega(s)=0$, the ruled surface ϕ(s,v) is called a cylindrical surface. Therefore, if $\omega^{'}(s)\times\omega(s)\neq 0$ the ruled surface ϕ(s,v) is called non-cylindrical surface [10].

The foot of the common normal between two consecutive generators is called the striction point on a ruled surface. The striction curve formed by the set of striction points is as follows [10]:

$$b(s)=\gamma(s)-\frac{<\gamma^{'}(s),\omega^{'}(s)>}{<\omega^{'}(s),\omega^{'}(s)>}\omega(s).$$

The distribution parameter for a ruled surface is described as follows [10]:

$$P_w=\frac{det(\gamma^{'},\omega,\omega^{'})}{||\omega^{'}||^2}.$$

A ruled surface ϕ is developable if and only if $P_w=0$ [10].

Let ϕ(s,v) be a ruled surface. Then the Gaussian curvature of ϕ(s,v) is given by

$$K(s,v)=-\frac{(det(\gamma^{'}(s),\omega(s),\omega^{'}(s)))^2}{(EG-F^2)^2}$$

and mean curvature of ϕ(s,v) given by

$$H(s,v)=\frac{-2<\gamma^{'}(s),\omega(s)>det(\gamma^{'}(s),\omega(s),\omega^{'}(s))+det(\gamma^{''}(s)+v\omega^{''}(s),\gamma^{'}(s)+v\omega^{'}(s),\omega(s))}{2(EG-F^2)^{3/2}}$$

where $E=E(s,v)=||\gamma^{'}(s)+v\omega^{'}(s)||^2, F=F(s,v)=<\gamma^{'}(s),\omega(s)>$,
$G=G(s,v)=1$ [3].

Let γ be a regular curve in $\mathbb{R}^3$ and the set $\{T(s),N(s),B(s)\}$ be the Frenet vectors of the curve γ. Then the tangent, principal normal and binormal surfaces of the curve γ are given in the following equalities [3]:

$$\begin{aligned}\phi_T(s,v)&=\gamma(s)+vT(s)\\ \phi_N(s,v)&=\gamma(s)+vN(s)\\ \phi_B(s,v)&=\gamma(s)+vB(s).\end{aligned}$$

3. B-Lift Curves and its Ruled Surfaces

Definition 1. *Let $\gamma : I \to M \subset \mathbb{R}^3$ be a unit speed curve, then $\gamma_B : I \to TM$ is called the B-Lift curve and ensures the following equation:*

$$\gamma_B(s) = (\gamma(s), B(s)) = B(s)|_{\gamma(s)}. \quad (1)$$

Proposition 1. *Assume that γ_B is the B-Lift curve of a unit speed curve γ. Thus, the following equations are provided:*

$$\begin{aligned} T_B(s) &= -N(s), \\ N_B(s) &= \frac{\kappa(s)}{||W(s)||}T(s) - \frac{\tau(s)}{||W(s)||}B(s), \\ B_B(s) &= \frac{\tau(s)}{||W(s)||}T(s) + \frac{\kappa(s)}{||W(s)||}B(s) \end{aligned}$$

where $\{T(s), N(s), B(s)\}$ and $\{T_B(s), N_B(s), B_B(s)\}$ are the Frenet vectors of the curve γ and γ_B, respectively. (In particular, the torsion will be considered greater than zero.)

(i) Let γ_B be B-Lift curve of the regular curve γ. Then the tangent surface of B-Lift curve is given as follows:

$$\phi_{T_B}(s, v) = \gamma_B(s) + vT_B(s). \quad (2)$$

From (1) and Proposition 1, we have

$$\phi_{T_B}(s, v) = B(s) + v(-N(s)). \quad (3)$$

Now, we investigate the singular point of the ruled surface ϕ_{T_B}

$$\begin{aligned} (\phi_{T_B})_s \times (\phi_{T_B})_v &= (B^{'}(s) \times (-N(s)) + v(\kappa(s)T(s) - \tau(s)N(s)) \times -N(s) \\ &= -v\kappa(s)B(s). \end{aligned}$$

Since for every $(s_0, v_0) \in I \times (\mathbb{R} - \{0\})$, $(\phi_{T_B})_{s_0} \times (\phi_{T_B})_{v_0} = -v_0\kappa(s_0)B(s_0) \neq 0$, the ruled surface ϕ_{T_B} has no singular point. Since for every $(s_0, v_0) \in I \times (\mathbb{R} - \{0\})$, $\omega^{'}(s_0) \times \omega(s_0) = \kappa(s_0)B(s_0) \neq 0$, the ruled surface ϕ_{T_B} is non-cylindrical surface. The distribution parameter of the tangent surface ϕ_{T_B} is

$$P_{T_B} = \frac{det(B^{'}, -N, -N^{'})}{|| - N^{'}||^2} = 0.$$

The striction curve of the ruled surface ϕ_{T_B} is

$$\begin{aligned} b_{T_B}(s) &= \gamma_B(s) - \frac{< \gamma_B^{'}(s), T_B^{'}(s) >}{< T_B^{'}(s), T_B^{'}(s) >}T_B(s) \\ &= B(s) - \frac{< -\tau N, \kappa T - \tau B >}{< \kappa T - \tau B, \kappa T - \tau B >}(\kappa T - \tau B) \\ &= B(s). \end{aligned}$$

The Gaussian curvature of the ruled surface ϕ_{T_B} is

$$K_{T_B}(s,v) = -\frac{(det(-\tau N, -N, \kappa T - \tau B))^2}{(EG-F^2)^2} = 0.$$

The mean curvature of the ruled surface ϕ_{T_B} is

$$\begin{aligned} H_{T_B}(s,v) &= \frac{det(\kappa\tau T - \tau^{'} N - \tau^2 B + v(\kappa^{'} T + (\kappa^2+\tau^2)N - \tau^{'} B, -\tau N + v(\kappa T - \tau B), -N)}{2(EG-F^2)^{3/2}} \\ &= \frac{v^2(\frac{\tau}{\kappa})^{'}\kappa^2}{2(EG-F^2)^{3/2}}. \end{aligned}$$

Corollary 1. *The ruled surface ϕ_{T_B} is developable.*

Corollary 2. *Let the curve $\gamma : I \to \mathbb{R}^3$ be a general helix curve. Then the ruled surface ϕ_{T_B} is a minimal surface.*

(ii) Let γ_B be B-Lift curve of the regular curve γ. Then the principal normal surface of B-Lift curve is given as

$$\phi_{N_B}(s,v) = \gamma_B(s) + vN_B(s). \tag{4}$$

From (1) and Proposition 1, we get

$$\phi_{N_B}(s,v) = B(s) + v(\frac{\kappa(s)}{||W(s)||}T(s) - \frac{\tau(s)}{||W(s)||}B(s)). \tag{5}$$

$$(\phi_{N_B})_s \times (\phi_{N_B})_v = (-\tau + \frac{\tau^2}{||W||}, v(\frac{\kappa^{'}\tau - \kappa\tau^{'}}{||W||^2}), -\kappa + \frac{\kappa\tau}{||W||}). \tag{6}$$

The distribution parameter of the principal normal surface of the curve γ_B is

$$\begin{aligned} P_{N_B} &= \frac{det(B^{'}, N_B, N_B^{'})}{||N_B^{'}||^2} \\ &= \frac{\tau(-\frac{\kappa\tau^{'}}{||W||^2} + \frac{\kappa^{'}\tau}{||W||^2})}{(\frac{\kappa^{'}}{||W||})^2 + (\frac{\kappa^2+\tau^2}{||W||})^2 + (\frac{\tau^{'}}{||W||})^2}. \end{aligned}$$

The striction curve of the ruled surface ϕ_{N_B} is

$$\begin{aligned} b_{N_B}(s) &= \gamma_B(s) - \frac{< \gamma_B^{'}(s), N_B^{'}(s) >}{< N_B^{'}(s), N_B^{'}(s) >} N_B(s) \\ &= B(s) - \frac{< -\tau N, \frac{\kappa^{'}}{||W||}T + \frac{\kappa^2+\tau^2}{||W||}N - \frac{\tau^{'}}{||W||}B >}{< \frac{\kappa^{'}}{||W||}T + \frac{\kappa^2+\tau^2}{||W||}N - \frac{\tau^{'}}{||W||}B, \frac{\kappa^{'}}{||W||}T + \frac{\kappa^2+\tau^2}{||W||}N - \frac{\tau^{'}}{||W||}B >}(\frac{\kappa}{||W||}T - \frac{\tau}{||W||}B). \end{aligned}$$

The Gaussian curvature of the ruled surface ϕ_{N_B} is

$$K_{N_B}(s,v) = -\frac{(det(\gamma_B^{'}, N_B, N_B^{'}))^2}{(EG-F^2)^2}$$

$$= \frac{\tau(-\frac{\kappa\tau'}{||W||^2}+\frac{\kappa'\tau}{||W||^2})}{(EG-F^2)^2}.$$

The mean curvature of the ruled surface ϕ_{N_B} is

$$\begin{aligned} H_{N_B}(s,v) &= \frac{det(\gamma_B''+vN_B'',\gamma_B'+vN_B',N_B)}{2(EG-F^2)^{3/2}} \\ &= \frac{v^2(\frac{3\kappa\kappa'+3\tau\tau'}{||W||^3})(\kappa'\tau-\kappa\tau')+v^2(\frac{\kappa^2+\tau^2}{||W||^3})(\kappa\tau''-\kappa''\tau)+v\tau'(\frac{\kappa\tau'-\tau\kappa'}{||W||^2})+v\tau(\frac{\kappa''\tau-\tau''\kappa}{||W||^2})}{2(EG-F^2)^{3/2}}. \end{aligned}$$

Corollary 3. *Assume that $\gamma : I \to \mathbb{R}^3$ is a general helix curve. Hence the ruled surface ϕ_{N_B} is a developable surface.*

Corollary 4. *Let $\gamma : I \to \mathbb{R}^3$ be a general helix curve. Then the ruled surface ϕ_{N_B} is a minimal surface.*

(iii) Let γ_B be B-Lift curve of the regular curve γ. Then the binormal surface of B-Lift curve is given by

$$\phi_{B_B}(s,v) = \gamma_B(s) + vB_B(s). \tag{7}$$

From (1) and Proposition 1, we know

$$\phi_{B_B}(s,v) = B(s) + v(\frac{\tau(s)}{||W(s)||}T(s) + \frac{\kappa(s)}{||W(s)||}B(s)). \tag{8}$$

$$(\phi_{B_B})_s \times (\phi_{B_B})_v = (-\frac{\kappa\tau}{||W||}, v(\frac{\kappa'\tau-\kappa\tau'}{||W||^2}), \frac{\tau^2}{||W||}). \tag{9}$$

From (9), the ruled surface ϕ_{B_B} has no singular point and since $B_B \times B_B' \neq 0$, ϕ_{B_B} is non-cylindrical surface.
The distribution parameter of the ruled surface ϕ_{B_B} is

$$\begin{aligned} P_{B_B} &= \frac{det(B', B_B, B_B')}{||B_B'||^2} \\ &= \frac{\tau(-\frac{\kappa\tau'}{||W||^2}+\frac{\kappa'\tau}{||W||^2})}{(\tau')^2+(\kappa')^2}. \end{aligned}$$

The striction curve of the ruled surface ϕ_{B_B} is

$$\begin{aligned} b_{B_B}(s) &= \gamma_B(s) - \frac{<\gamma_B'(s), B_B'(s)>}{<B_B'(s), B_B'(s)>}B_B(s) \\ &= B(s). \end{aligned}$$

The Gaussian curvature of the ruled surface ϕ_{B_B} is

$$K_{B_B}(s,v) = -\frac{(det(\gamma_B', B_B, B_B'))^2}{(EG-F^2)^2}$$

$$= \frac{\tau(-\frac{\kappa\tau'}{||W||^2} + \frac{\kappa'\tau}{||W||^2})}{(EG - F^2)^2}.$$

The mean curvature of the ruled surface ϕ_{B_B} is

$$\begin{aligned} H_{B_B}(s,v) &= \frac{det(\gamma_B'' + vB_B'', \gamma_B' + vB_B', B_B)}{2(EG - F^2)^{3/2}} \\ &= \frac{\frac{\tau v}{||W||^2}(-\kappa'\tau + \tau'\kappa + \kappa''\tau - \kappa\tau'') - \frac{\tau^2}{||W||}(\kappa^2 + \tau^2) - \frac{v^2}{||W||^3}(\tau'\kappa - \kappa'\tau)^2}{2(EG - F^2)^{3/2}}. \end{aligned}$$

Corollary 5. *Let $\gamma : I \to \mathbb{R}^3$ be a general helix curve. Then the ruled surface ϕ_{B_B} is a developable surface.*

Example 1. *Let us consider the unit speed general helix curve that is given as following equality:*

$$\gamma(s) = (\frac{\sqrt{3}}{3}s^{3/2}, \frac{\sqrt{3}}{3}(1-s)^{3/2}, \frac{s}{2}).$$

Then the curve γ_B is given as follows:

$$\gamma_B(s) = (-\frac{1}{2}s^{1/2}, \frac{1}{2}(1-s)^{1/2}, \frac{\sqrt{3}}{2}).$$

The Frenet vectors of the curve γ_B can be calculated by

$$\begin{aligned} T_B(s) &= (-(1-s)^{1/2}, -s^{1/2}, 0), \\ N_B(s) &= (s^{1/2}, -(1-s)^{1/2}, 0), \\ B_B(s) &= (0, 0, 1). \end{aligned}$$

From (3), (5) and (8), the tangent, normal and binormal surfaces are calculated as follows:

$$\begin{aligned} \phi_{T_B}(s,v) &= \gamma_B(s) + vT_B(s) \\ &= (-\frac{1}{2}s^{1/2}, \frac{1}{2}(1-s)^{1/2}, \frac{\sqrt{3}}{2}) + v(-(1-s)^{1/2}, -s^{1/2}, 0) \\ \phi_{N_B}(s,v) &= \gamma_B(s) + vN_B(s) \\ &= (-\frac{1}{2}s^{1/2}, \frac{1}{2}(1-s)^{1/2}, \frac{\sqrt{3}}{2}) + v(s^{1/2}, -(1-s)^{1/2}, 0) \\ \phi_{B_B}(s,v) &= \gamma_B(s) + vB_B(s) \\ &= (-\frac{1}{2}s^{1/2}, \frac{1}{2}(1-s)^{1/2}, \frac{\sqrt{3}}{2}) + v(0, 0, 1). \end{aligned}$$

The distrubition parameters of the ruled surfaces ϕ_{T_B}, ϕ_{N_B} and ϕ_{B_B} are

$$P_{T_B} = \frac{det(B', T_B, T_B')}{||T_B'||^2} = 0,$$

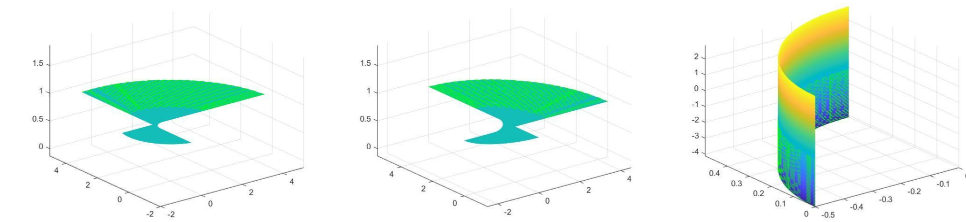

FIGURE 1. Illustration of the ruled surfaces ϕ_{T_B}, ϕ_{N_B} and ϕ_{B_B}, respectively.

$$
\begin{aligned}
P_{N_B} &= \frac{det(B', N_B, N_B')}{||N_B'||^2} = 0, \\
P_{B_B} &= \frac{det(B', B_B, B_B')}{||B_B'||^2} = 0.
\end{aligned}
$$

Since $P_{T_B} = P_{N_B} = P_{B_B} = 0$, the ruled surfaces ϕ_{T_B}, ϕ_{N_B} and ϕ_{B_B} are developable. The striction lines of the ruled surfaces ϕ_{T_B}, ϕ_{N_B} and ϕ_{B_B} are given by

$$
\begin{aligned}
b_{T_B}(s) &= \gamma_B(s) - \frac{<\gamma_B'(s), T_B'(s)>}{<T_B'(s), T_B'(s)>} T_B(s) \\
&= B(s) \\
&= (-\frac{1}{2}s^{1/2}, \frac{1}{2}(1-s)^{1/2}, \frac{\sqrt{3}}{2}). \\
b_{N_B}(s) &= \gamma_B(s) - \frac{<\gamma_B'(s), N_B'(s)>}{<N_B'(s), N_B'(s)>} N_B(s) \\
&= (-\frac{1}{2}s^{1/2}, \frac{1}{2}(1-s)^{1/2}, \frac{\sqrt{3}}{2}) + \frac{1}{2}(s^{1/2}, -(1-s)^{1/2}, 0) \\
&= (0, 0, \frac{\sqrt{3}}{2}). \\
b_{B_B}(s) &= \gamma_B(s) - \frac{<\gamma_B'(s), B_B'(s)>}{<B_B'(s), B_B'(s)>} B_B(s) \\
&= B(s) \\
&= (-\frac{1}{2}s^{1/2}, \frac{1}{2}(1-s)^{1/2}, \frac{\sqrt{3}}{2}).
\end{aligned}
$$

Gaussian curvatures of the ruled surfaces ϕ_{T_B}, ϕ_{N_B} and ϕ_{B_B} are given as follows:

$$
K_{T_B}(s,v) = -\frac{(det(\gamma_B', T_B, T_B'))^2}{(EG - F^2)^2}
$$

$$
\begin{aligned}
&= 0 \\
K_{N_B}(s,v) &= -\frac{(det(\gamma_B^{'}, N_B, N_B^{'}))^2}{(EG-F^2)^2} \\
&= 0 \\
K_{B_B}(s,v) &= -\frac{(det(\gamma_B^{'}, B_B, B_B^{'}))^2}{(EG-F^2)^2} \\
&= 0.
\end{aligned}
$$

Mean curvatures of the ruled surfaces ϕ_{T_B}, ϕ_{N_B} and ϕ_{B_B} are calculated as

$$
\begin{aligned}
H_{T_B}(s,v) &= \frac{-2 < \gamma^{'}(s), T_B(s) > det(\gamma^{'}(s), T_B(s), T_B^{'}(s))}{2(EG-F^2)^{3/2}} \\
&+ \frac{det(\gamma^{''}(s) + vT_B^{''}(s), \gamma^{'}(s) + vT_B^{'}(s), T_B(s))}{2(EG-F^2)^{3/2}} \\
&= 0 \\
H_{N_B}(s,v) &= \frac{-2 < \gamma^{'}(s), N_B(s) > det(\gamma^{'}(s), N_B(s), N_B^{'}(s))}{2(EG-F^2)^{3/2}} \\
&+ \frac{det(\gamma^{''}(s) + vN_B^{''}(s), \gamma^{'}(s) + vN_B^{'}(s), N_B(s))}{2(EG-F^2)^{3/2}} \\
&= 0 \\
H_{B_B}(s,v) &= \frac{-2 < \gamma^{'}(s), B_B(s) > det(\gamma^{'}(s), B_B(s), B_B^{'}(s))}{2(EG-F^2)^{3/2}} \\
&+ \frac{det(\gamma^{''}(s) + vB_B^{''}(s), \gamma^{'}(s) + vB_B^{'}(s), B_B(s))}{2(EG-F^2)^{3/2}} \\
&= 0.
\end{aligned}
$$

Since $H_{T_B}(s,v) = H_{N_B}(s,v) = H_{B_B}(s,v) = 0$, the ruled surfaces ϕ_{T_B}, ϕ_{N_B} and ϕ_{B_B} are minimal surfaces.

4. Conclusion

In this study, based on Thorpe's definition [9], we have introduced the B-lift curve and calculated the Frenet vectors of the B-Lift curves. Furthermore, we have given the tangent, normal, and binormal surfaces of the B-Lift curves and calculated the integral invariants of these surfaces.

Author Contribution Statements The authors jointly worked on the results and they read and approved the final manuscript.

Declaration of Competing Interests The authors declare that they have no competing interest.

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