Geometry of ∗-k-Ricci-Yamabe soliton and gradient ∗-k-Ricci-Yamabe soliton on Kenmotsu manifolds

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Abstract

The goal of the current paper is to characterize the ∗-k-Ricci-Yamabe soliton within the framework on Kenmotsu manifolds. Here, we have shown the nature of the soliton and found the scalar curvature when the manifold admits the ∗-k-Ricci-Yamabe soliton on the Kenmotsu manifold. Next, we have evolved the characterization of the vector field when the manifold satisfies the ∗-k-Ricci-Yamabe solitons. Also we have embellished some applications of vector field as torse-forming in terms of ∗-k-Ricci-Yamabe soliton on Kenmotsu manifold. Then, we studied the gradient ∗-k-Ricci-Yamabe soliton to yield the nature of the Riemannian curvature tensor. We have developed an example of a ∗-k-Ricci-Yamabe soliton on a 5-dimensional Kenmotsu manifold to prove our findings.

1. Introduction and motivations

Contact geometry has evolved from the mathematical formalism of classical mechanics. The contact geometry has an eminence to resolve many issues of basic sciences including geometric quantization, control theory, thermodynamics and technology and, therefore, it became familiar among the researchers. In very recent years, an interest in the study of Riemannian manifolds or pseudo-Riemannian manifolds has been furnished with a metric which gratifies some structural equations presuming Ricci curvature and some locally as well as globally defined vector fields. For instance, R. S. Hamilton [22] introduced the

concept of Ricci flow, which is an evolution equation for metrics on a Riemannian manifold and proved its existence in 1982. The Ricci flow equation is given by:

$$\frac{\partial g}{\partial t} = -2S \tag{1.1}$$

on a compact Riemannian manifold M with Riemannian metric g.

A self-similar solution to the Ricci flow ([22], [47]) is called a Ricci soliton [23] if it moves only by a one parameter family of diffeomorphism and scaling. The Ricci soliton equation is given by [32]:

$$\pounds_V g + 2S + 2\Lambda g = 0, \tag{1.2}$$

where $\pounds_V$ is the Lie derivative in the direction of V, S is Ricci tensor, g is Riemannian metric, V is a vector field and Λ is a scalar. The Ricci soliton is said to be shrinking, steady and expanding accordingly as Λ is negative, zero and positive respectively.

Recently, Wang-Gomes-Xia [49] extended the notion of almost Ricci soliton to k-almost Ricci soliton. A complete Riemannian manifold (M^n, g) is said to be a k-almost Ricci soliton, denoted by (M^n, g, X, k, Λ), if there exists smooth vector field X on M^n, a soliton function $\Lambda \in C^\infty(M^n)$ and a non-zero real valued function k on M^n such that

$$k\pounds_V g + 2S + 2\Lambda g = 0. \tag{1.3}$$

The concept of Yamabe flow was first introduced by Hamilton [23] to construct Yamabe metrics on compact Riemannian manifolds. On a Riemannian or pseudo-Riemannian manifold M, a time-dependent metric $g(\cdot, t)$ is said to evolve by the Yamabe flow if the metric g satisfies the given equation,

$$\frac{\partial}{\partial t} g(t) = -rg(t), \quad g(0) = g_0, \tag{1.4}$$

where r is the scalar curvature of the manifold M.

In 2-dimension the Yamabe flow is equivalent to the Ricci flow [22] (defined by $\frac{\partial}{\partial t} g(t) = -2S(g(t))$, where S denotes the Ricci tensor). But in dimension > 2 the Yamabe and Ricci flows do not agree, since the Yamabe flow preserves the conformal class of the metric but the Ricci flow does not in general.

A Yamabe soliton [2] correspond to self-similar solution of the Yamabe flow, is defined on a Riemannian or pseudo-Riemannian manifold (M, g) as:

$$\frac{1}{2} \pounds_V g = (r - \Lambda)g, \tag{1.5}$$

where $\pounds_V g$ denotes the Lie derivative of the metric g along the vector field V, r is the scalar curvature and Λ is a constant. Moreover a Yamabe soliton is said to be expanding, steady, shrinking depending on Λ being positive, zero, negative respectively. If Λ is a smooth function then (1.5) is called almost Yamabe soliton [2].

Very recently, Chen [9] introduced a new concept, named k-almost Yamabe soliton. According to Chen, a Riemannian metric is said to be a k-almost Yamabe soliton if there exists a smooth vector field V, a C^∞ function Λ and a nonzero function k such that

$$\frac{k}{2} \pounds_V g = (r - \Lambda)g \tag{1.6}$$

holds. If for any smooth function f, $V = Df$ then the previous equation is called gradient k-almost Yamabe soliton. If Λ is constant, then (1.6) is called k-Yamabe soliton.

Recently, many authors ([3, 5, 10, 19, 24–26, 33, 34, 36–38, 45]) have been studied some types of Ricci solitons and Yamabe solitons on contact and complex manifolds.

In 2019, S. Güler and M. Crasmareanu [20] introduced a new geometric flow which is a scalar combination of Ricci and Yamabe flow under the name Ricci-Yamabe map. This flow is also known as Ricci-Yamabe flow of the type (α, β).

Let (M^n, g) be a Riemannian manifold and $T_2^s(M)$ be the linear space of its symmetric tensor fields of $(0, 2)$-type and $Riem(M) \subsetneq T_2^s(M)$ be the infinite space of its Riemannian metrics. In [20], the authors have stated the following definition:

Definition 1.1 ([20]). A Riemannian flow on M is a smooth map:

$$g : I \subseteq \mathbb{R} \to Riem(M),$$

where I is a given open interval. We can call it also as time-dependent (or non-stationary) Riemannian metric.

Definition 1.2 ([20]). The map $RY^{(\alpha,\beta,g)} : I \to T_2^s(M)$ given by:

$$RY^{(\alpha,\beta,g)} := \frac{\partial}{\partial t}g(t) + 2\alpha S(t) + \beta r(t)g(t)$$

is called the (α, β)-Ricci-Yamabe map of the Riemannian flow (M^n, g), where α, β are some scalars. If $RY^{(\alpha,\beta,g)} \equiv 0$, then $g(\cdot)$ will be called an (α, β)-Ricci-Yamabe flow.

Also in [20], the authors characterized that the (α, β)-Ricci-Yamabe flow is said to be:
• Ricci flow [22] if $\alpha = 1$, $\beta = 0$.
• Yamabe flow [23] if $\alpha = 0$, $\beta = 1$.
• Einstein flow ([4], [35]) if $\alpha = 1$, $\beta = -1$.
A soliton to the Ricci-Yamabe flow is called Ricci-Yamabe solitons if it moves only by one parameter group of diffeomorphism and scaling. The metric of the Riemannain manifold (M^n, g), $n > 2$ is said to admit (α, β)-Ricci-Yamabe soliton or simply Ricci-Yamabe soliton (RYS) $(g, V, \Lambda, \alpha, \beta)$ if it satisfies the equation:

$$\pounds_V g + 2\alpha S + [2\Lambda - \beta r]g = 0, \tag{1.7}$$

where $\pounds_V g$ denotes the Lie derivative of the metric g along the vector field V, S is the Ricci tensor, r is the scalar curvature and Λ, α, β are real scalars.

In the above equation if the vector field V is the gradient of a smooth function f (denoted by Df, D denotes the gradient operator) then the equation (1.7) is called gradient Ricci-Yamabe soliton (GRYS) and it is defined as:

$$Hessf + \alpha S + \left[\Lambda - \frac{1}{2}\beta r\right]g = 0, \tag{1.8}$$

where $Hessf$ is the Hessian of the smooth function f.
Moreover, the RYS (or GRYS) is said to be expanding, steady, or shrinking according as Λ is positive, zero, negative respectively. Also if Λ become smooth function then (1.7) and (1.8) are called almost RYS and almost GRYS respectively.

Now using the previous identities (1.3), (1.6) and (1.7), we define new notation k-Ricci-Yamabe soliton(k-RYS). A complete Riemannian manifold (M^n, g) is said to be a k-almost Ricci-Yamabe soliton, denoted by (M^n, g, X, k, Λ), if there exists smooth vector field X on M^n, a soliton function $\Lambda \in C^\infty(M^n)$ and a non-zero real valued function k on M^n such that

$$k\pounds_V g + 2\alpha S + [2\Lambda - \beta r]g = 0. \tag{1.9}$$

In the previous equation if the vector field V is the gradient of a smooth function f (denoted by Df, D denotes the gradient operator) then the equation (1.9) is called gradient almost k-Ricci-Yamabe soliton (k-GRYS)and it is defined as:

$$kHessf + \alpha S + \left[\Lambda - \frac{1}{2}\beta r\right]g = 0. \tag{1.10}$$

Now if Λ is constant, then (1.9) and (1.10) are called k-Ricci-Yamabe soliton and gradient k-Ricci-Yamabe soliton. Also, k-almost Ricci-Yamabe soliton(k-RYS) is called
• k-almost Ricci soliton (or gradient k-almost Ricci soliton) if $\alpha = 1$ and $\beta = 0$.

- *k*-almost Yamabe soliton (or gradient *k*-almost Yamabe soliton) if $\alpha = 0$ and $\beta = 2$.
- *k*-almost Einstein soliton (or gradient *k*-almost Einstein soliton) if $\alpha = 1$ and $\beta = 1$.

The concept of ∗-Ricci tensor on almost Hermitian manifolds and ∗-Ricci tensor of real hypersurfaces in non-flat complex space were introduced by Tachibana [46] and Hamada [21] respectively where the ∗-Ricci tensor is defined by:

$$S^*(X, Y) = \frac{1}{2}(\text{Tr}\{\varphi \circ R(X, \varphi Y)\}) \tag{1.11}$$

for all vector fields X, Y on M^n, φ is a (1, 1)-tensor field and Tr denotes Trace.

If $S^*(X, Y) = \lambda g(X, Y) + \nu \eta(X) \eta(Y)$ for all vector fields X, Y where λ, ν are smooth functions and η is a 1-form, then the manifold is called ∗-η-Einstein manifold.

Further if $\nu = 0$ i.e $S^*(X, Y) = \lambda g(X, Y)$ for all vector fields X, Y then the manifold becomes ∗-Einstein.

In 2014, Kaimakamis and Panagiotidou [27] introduced the notion of ∗-Ricci soliton which can be defined as:

$$\pounds_V g + 2S^* + 2\Lambda g = 0 \tag{1.12}$$

for all vector fields X, Y on M^n and Λ being a constatnt.

In [48], authors have considered ∗-Ricci solitons and gradient almost ∗-Ricci solitons on Kenmotsu manifolds and obtained some beautiful results. Very recently, Dey et al. [13–17,29,30,34,38,41] have studied ∗-Ricci solitons and their generalizations in the framework of almost contact geometry.

Recently D. Dey [11] introduced the notion of ∗-Ricci-Yamabe soliton (∗-RYS) as follows :

Definition 1.3. A Riemannian or pseudo-Riemannian manifold (M, g) of dimension n is said to admit ∗-Ricci-Yamabe soliton (∗-RYS) if

$$\pounds_V g + 2\alpha S^* + [2\Lambda - \beta r^*]g = 0, \tag{1.13}$$

where $r^* = \text{Tr}(S^*)$ is the ∗- scalar curvature and Λ, α, β are real scalars.

The ∗-Ricci-Yamabe soliton (∗-RYS) is said to be expanding, steady, shrinking depending on Λ being positive, zero, negative respectively. If the vector field V is of gradient type i.e $V = grad(f)$, for f is a smooth function on M, then the equation (1.13) is called gradient ∗-Ricci-Yamabe soliton (∗-GRYS).

Using (1.3) and (1.13), we introduce ∗-*k*-Ricci-Yamabe soliton (∗-*k*-RYS) as:

Definition 1.4. A Riemannian or pseudo-Riemannian manifold (M, g) of dimension n is said to admit ∗-*k*-Ricci-Yamabe soliton (∗-*k*-RYS) if

$$k\pounds_V g + 2\alpha S^* + [2\Lambda - \beta r^*]g = 0, \tag{1.14}$$

where $\pounds_V g$ denotes the Lie derivative of the metric g along the vector field V, S^* is the ∗-Ricci tensor, $r^* = \text{Tr}(S^*)$ is the ∗-scalar curvature and Λ, α, β are real scalars.

If the vector field V is of gradient type i.e., $V = grad(f)$, for f is a smooth function on M, then the soliton equation changes to

$$kHessf + \alpha S^* + (\Lambda - \frac{\beta r^*}{2})g = 0, \tag{1.15}$$

then the equation (1.15) is called gradient ∗-*k*-Ricci-Yamabe soliton (∗-*k*-GRYS). The ∗-*k*-Ricci-Yamabe soliton or gradient ∗-*k*-Ricci-Yamabe soliton is said to be expanding, steady, shrinking depending on Λ being positive, zero, negative respectively. A ∗-*k*-RYS (or ∗-*k*-GRYS) is called an almost ∗-*k*-RYS (or ∗-*k*-GRYS) if Λ is a smooth function on M.

The above notation generalizes a large class of solitons. We can also define some solitons in the following way. A ∗-*k*-RYS (or ∗-*k*-GRYS) is called

- ∗-k-Ricci soliton (or gradient ∗-k-Ricci soliton) if $\alpha = 1$ and $\beta = 0$.
- ∗-k-Yamabe soliton (or gradient ∗-k-Yamabe soliton) if $\alpha = 0$ and $\beta = 2$.
- ∗-k-Einstein soliton (or gradient ∗-k-Einstein soliton) if $\alpha = 1$ and $\beta = 1$.

On the other hand, a nowhere vanishing vector field τ on a Riemannian or pseudo-Riemannian manifold (M, g) is called torse-forming [52] if

$$\nabla_X \tau = \psi X + \omega(X)\tau, \tag{1.16}$$

where ∇ is the Levi-Civita connection of g, ψ is a smooth function and ω is a 1-form. Moreover, The vector field τ is called concircular ([6], [51]) if the 1-form ω vanishes identically in the equation (1.16). The vector field τ is called concurrent ([42], [53]) if in (1.16) the 1-form ω vanishes identically and the function $\psi = 1$. The vector field τ is called recurrent if in (1.16) the function $\psi = 0$. Finally if in (1.16) $\psi = \omega = 0$, then the vector field τ is called a parallel vector field.

In 2017, Chen [7] introduced a new vector field called torqued vector field. If the vector field τ staisfies (1.16) with $\omega(\tau) = 0$, then τ is called torqued vector field. Also in this case, ψ is known as the torqued function and the 1-form ω is the torqued form of τ.

In [28], a new class of almost contact metric manifolds was introduced and studied, satisfying certain conditions which is known as Kenmotsu manifold. This type of manifold is very closely related to the warped product spaces. Kenmotsu proved that a locally Kenmotsu manifold is a warped product $I \times_f N$ of an interval I and a Kähler manifold N with warping function $f(t) = ce^t$, where c is a non-zero constant. Recent years, many authors have been studied Ricci soliton, Ricci-Yamabe soliton, ∗-Ricci-Yamabe soliton and its characterizations on contact geometry. First, Sharma [43] initiated the study of Ricci solitons in contact geometry. D. Dey [11] studied new type of soliton namely ∗-Ricci-Yamabe soliton on contact geometry. Recently, P. Zhang et al. [56] have strained ∗-conformal η-Ricci-Yamabe soliton on α-cosymplectic manifolds. Siddiqi and Akyol [44] have discussed the notion of η-Ricci-Yamabe soliton to set up the geometrical structure on Riemannian submersions admitting η-Ricci-Yamabe soliton with the potential field. Very recently, Yoldaş [54] measured Kenmotsu metric in terms of η-Ricci-Yamabe soliton. Next, Chen [8] considered a real hypersurface of a non-flat complex space form which admits a ∗-Ricci soliton whose potential vector field belongs to the principal curvature space and the holomorphic distribution. Recently, Wang [50] proved that if the metric of a Kenmotsu 3-manifold represents a ∗-Ricci soliton, then the manifold is locally isometric to the hyperbolic space $\mathbb{H}^3(-1)$. Moreover, we can get more latest studies equipped with soliton geometry in [12, 18, 31, 39, 40, 55].

In the ongoing paper, we will discuss about ∗-k-Ricci-Yamabe soliton and gradient ∗-k-Ricci-Yamabe soliton on Kenmotsu manifold.

The outline of the article goes as follows:

In section 2, after a brief introduction, we have discussed some preliminaries of kenmotsu manifold. In section 3, we have studied ∗-k-Ricci-Yamabe soliton admitting Kenmotsu manifold and obtained the nature of soliton, Poisson's equation of the smooth function. We have also proved that the manifold is η-Einstein when the manifold satisfies ∗-k-Ricci-Yamabe soliton and the vector field is conformal Killing. Next section, we have demonstrated some applications of torse-forming potential vector field admitting ∗-k-Ricci-Yamabe soliton on Kenmotsu manifold. Section 5 deals with gradient ∗-k-Ricci-Yamabe soliton and obtain the curvature tensor. In section 6, we have constructed an example to illustrate the existence of ∗-k-Ricci-Yamabe soliton on 5-dimensional Kenmotsu manifold.

2. Notes on contact metric manifolds

Let M be a $(2n+1)$-dimensional connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) where ϕ is a $(1, 1)$ tensor field, ξ is a vector field,

η is a 1-form and g is the compatible Riemannian metric such that

$$\phi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \eta \circ \phi = 0, \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X) \tag{2.4}$$

for all vector fields $X, Y \in \chi(M)$.

An almost contact metric manifold is said to be a Kenmotsu manifold [28] if

$$(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X, \tag{2.5}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.6}$$

where ∇ denotes the Riemannian connection of g.

In a Kenmotsu manifold the following relations hold ([1], [36]):

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{2.7}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.8}$$

$$R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.9}$$

where R is the Riemannian curvature tensor.

$$S(X, \xi) = -2n\eta(X), \tag{2.10}$$

$$S(\phi X, \phi Y) = S(X, Y) + 2n\eta(X)\eta(Y), \tag{2.11}$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y) \tag{2.12}$$

for all vector fields $X, Y, Z \in \chi(M)$.

Now, we know

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) \tag{2.13}$$

for all vector fields $X, Y, \in \chi(M)$.

Then we use the identities (2.6) and (2.13) to find

$$(\mathcal{L}_\xi g)(X, Y) = 2[g(X, Y) - \eta(X)\eta(Y)]. \tag{2.14}$$

Proposition 2.1 ([48]). *On a $(2n+1)$-dimensional Kenmotsu manifold, the $*$-Ricci tensor is given by*

$$S^*(X, Y) = S(X, Y) + (2n - 1)g(X, Y) + \eta(X)\eta(Y). \tag{2.15}$$

Also, we plug $X = e_i$, $Y = e_i$ in the above equation, where e_i's are a local orthonormal frame and summing over $i = 1, 2, ..., (2n + 1)$ to yield

$$r^* = r + 4n^2, \tag{2.16}$$

where r^* is the $*$- scalar curvature of M.

3. Kenmotsu metric as ∗-k-Ricci-Yamabe soliton

Let M be a $(2n+1)$-dimensional Kenmotsu manifold. Now we consider $V = \xi$ into identity (1.14) on M to achieve

$$k(\pounds_\xi g)(X,Y) + 2\alpha S^*(X,Y) + [2\Lambda - \beta r^*]g(X,Y) = 0 \tag{3.1}$$

for all vector fields $X, Y, \in \chi(M)$.
So, we have the following major theorems as follows:

Theorem 3.1. *If the metric g of a (2n+1)-dimensional Kenmotsu manifold satisfies the ∗-k-Ricci-Yamabe soliton $(g, \xi, \Lambda, \alpha, \beta)$, where ξ is the reeb vector field, then the soliton is expanding, steady, shrinking according as $\beta(r + 4n^2) \gneqq 0$.*

Proof. Making the use of the equations (2.14) and (2.15), the equation (3.1) provides

$$\alpha S(X,Y) + [\Lambda + \alpha(2n - k) + 1 - \frac{\beta r^*}{2}]g(X,Y) + [\alpha - k]\eta(X)\eta(Y) = 0. \tag{3.2}$$

Now, we plug $Y = \xi$ in the above equation and from the identities (2.1), (2.10) to yield

$$[\Lambda - \frac{\beta r^*}{2}]\eta(X) = 0. \tag{3.3}$$

Since $\eta(X) \neq 0$, the above equation takes the form

$$\Lambda = \frac{\beta r^*}{2}. \tag{3.4}$$

Using (2.16), we acquire

$$\Lambda = \frac{\beta(r + 4n^2)}{2}. \tag{3.5}$$

Hence, we end the proof.. □

From the above theorem, we can state

Corollary 3.2. *If the metric g of a (2n+1)-dimensional Kenmotsu manifold, which is flat, satisfies the ∗-k-Ricci-Yamabe soliton $(g, \xi, \Lambda, \alpha, \beta)$, where ξ is the reeb vector field, then the soliton is expanding, steady, shrinking according as $\beta \gneqq 0$.*

Proof. If the manifold becomes flat i.e., $r = 0$ then (3.5) becomes $\Lambda = 2\beta n^2$ and it reaches the result. □

Next, we have

Theorem 3.3. *If the metric g of a (2n+1)-dimensional Kenmotsu manifold admits the ∗-k-Ricci-Yamabe soliton $(g, V, \Lambda, \alpha, \beta)$, where V is the gradient of a smooth function f, then the Poisson's equation satisfied by f is*

$$\Delta(f) = -\frac{(r + 4n^2)}{k}\Big[\alpha - \frac{\beta(2n + 1)}{2}\Big] - \frac{\Lambda}{k}(2n + 1).$$

Proof. We consider a ∗-k-Ricci-Yamabe soliton $(g, V, \Lambda, \alpha, \beta)$ on M as:

$$k(\pounds_V g)(X,Y) + 2\alpha S^*(X,Y) + [2\Lambda - \beta r^*]g(X,Y) = 0 \tag{3.6}$$

for all vector fields $X, Y, \in \chi(M)$.
We take $X = e_i$, $Y = e_i$, in the above equation, where e_i's are a local orthonormal frame and summing over $i = 1, 2, ..., (2n + 1)$ and using (2.16) to obtain

$$divV + \frac{(r + 4n^2)}{k}\Big[\alpha - \frac{\beta(2n + 1)}{2}\Big] + \frac{\Lambda}{k}(2n + 1) = 0. \tag{3.7}$$

If we take the vector field V is of gradient type i.e., $V = grad(f)$, for f is a smooth function on M, then the identity (3.7) becomes

$$\Delta(f) = -\frac{(r + 4n^2)}{k}\Big[\alpha - \frac{\beta(2n + 1)}{2}\Big] - \frac{\Lambda}{k}(2n + 1), \tag{3.8}$$

where $\Delta(f)$ is the Poisson's equation satisfied by f. This end our proof. $\square$

Now if we choose $\alpha = 1, \beta = 0$, (1.14) reduces to $*$-k-Ricci soliton and (3.8) becomes $\Delta(f) = -\frac{(r+4n^2)}{k} - \frac{\Lambda}{k}(2n + 1)$.
If $\alpha = 0, \beta = 2$, (1.14) reduces to $*$-k-Yamabe soliton and (3.8) takes the form, $\Delta(f) = \frac{1}{k}[r + 4n^2 - \Lambda](2n + 1)$.
Moreover if $\alpha = \beta = 1$, (1.14) reduces to $*$-k-Einstein soliton and (3.8) provides $\Delta(f) = -\frac{(r+4n^2)}{k}\Big[1 - \frac{(2n+1)}{2}\Big] - \frac{\Lambda}{k}(2n + 1)$.

Definition 3.4. A vector field V is said to be a conformal Killing vector field iff the following relation holds:

$$(\pounds_V g)(X, Y) = 2\Omega g(X, Y), \tag{3.9}$$

where Ω is some function of the coordinates(conformal scalar).
Moreover if Ω is not constant the conformal Killing vector field V is said to be proper. Also when Ω is constant, V is called homothetic vector field and when the constant Ω becomes non zero, V is said to be proper homothetic vector field. If $\Omega = 0$ in the above equation, then V is called Killing vector field.

Theorem 3.5. *If the metric g of a ($2n$+1)-dimensional Kenmotsu manifold endows the $*$-k-Ricci-Yamabe soliton $(g, V, \Lambda, \alpha, \beta)$, where V is a conformal Killing vector field, then the manifold becomes η-Einstein, provided $\alpha \neq 0$.*

Proof. As $(g, V, \Lambda, \alpha, \beta)$ is a $*$-Ricci-Yamabe soliton on a ($2n$+1)-dimensional Kenmotsu manifold M, where V is a conformal Killing vector field. Then from (1.14), (2.15) and (3.9), we obtain

$$\alpha S(X, Y) = -\Big[\alpha(2n - 1) + \Lambda + k\Omega - \frac{\beta r^*}{2}\Big]g(X, Y) - \alpha\eta(X)\eta(Y). \tag{3.10}$$

Now, we know, a Kenmotsu manifold (M^{2n+1}, g) is said to η-Einstein if its Ricci tensor S of type (0, 2) is of the form

$$S = ag + b\eta \otimes \eta, \tag{3.11}$$

where a and b are smooth function on (M^{2n+1}, g).
In view of (3.11), (3.10) leads to the fact that the given manifold is η-Einstein, provided $\alpha \neq 0$ and it completes the proof. $\square$

Theorem 3.6. *Let the metric g of a ($2n$+1)-dimensional Kenmotsu manifold endows the $*$-k-Ricci-Yamabe soliton $(g, V, \Lambda, \alpha, \beta)$, where V is a conformal Killing vector field. Then V is*

(i) *proper vector field if $\frac{1}{k}\Big[\frac{\beta(r+4n^2)}{2} - \Lambda\Big]$ is not constant.*

(ii) *homothetic vector field if $\frac{1}{k}\Big[\frac{\beta(r+4n^2)}{2} - \Lambda\Big]$ is constant.*

(iii) *proper homothetic vector field if $\frac{1}{k}\Big[\frac{\beta(r+4n^2)}{2} - \Lambda\Big]$ is non-zero constant.*

(iv) *Killing vector field if $\Lambda = \frac{\beta(r+4n^2)}{2}$.*

Proof. We plug $Y = \xi$ in (3.10) and using (2.1), (2.10) to achieve

$$\Big[2\alpha n - \alpha(2n - 1) - \Lambda - k\Omega + \frac{\beta r^*}{2} - \alpha\Big]\eta(X) = 0. \tag{3.12}$$

Since $\eta(X) \neq 0$, we get

$$\Omega = \frac{1}{k}\Big[\frac{\beta r^*}{2} - \Lambda\Big]. \tag{3.13}$$

Then with the help of the equation (2.16), (3.13) reads

$$\Omega = \frac{1}{k}\Big[\frac{\beta(r + 4n^2)}{2} - \Lambda\Big]. \tag{3.14}$$

Thus we can conclude the result. □

Theorem 3.7. *If $(M, \phi, g, \xi, \Lambda, a, b)$ is ∗-k-Ricci-Yamabe soliton on an η-Einstein Kenmotsu manifold, then $\Lambda = -a\alpha - 2n\alpha + \frac{\beta}{2}\big(r + 4n^2\big) - b\alpha$.*

Proof. We employ the identities (3.1), (2.15) and (2.16) to acquire

$$\begin{aligned}k[g(\nabla_X V, Y) &+ g(X, \nabla_V Y)] + 2\alpha S(X, Y) + [2\Lambda - \beta(r + 4n^2) \\ &+ 2\alpha(2n - 1)]g(X, Y) + 2\alpha\eta(X)\eta(Y) = 0.\end{aligned} \tag{3.15}$$

We insert the identities (3.11) and (3.10) into the previous equation to yield

$$\begin{aligned}k[g(\nabla_X V, Y) &+ g(X, \nabla_V Y)] + (2a\alpha + 2\alpha(2n - 1) + 2\Lambda \\ &- \beta(r + 4n^2)]g(X, Y) + 2\alpha(b + 1)\eta(X)\eta(Y) = 0.\end{aligned} \tag{3.16}$$

Now, we plug $X = Y = \xi$ into (3.16) to achieve

$$2kg(\nabla_\xi V, \xi) = (2a\alpha + 2\alpha(2n - 1) + 2\Lambda - \beta(r + 4n^2) + 2\alpha(b + 1). \tag{3.17}$$

We fetch $V = \xi$ into identity (3.17) to obtain

$$kg(\nabla_\xi \xi, \xi) = \Big[a\alpha + 2n\alpha + \Lambda - \frac{\beta}{2}(r + 4n^2) + \alpha b\Big]. \tag{3.18}$$

As it is well known that

$$g(\nabla_\xi \xi, \xi) = 0$$

for all vector fields on M. It follows that

$$\Lambda = -a\alpha - 2n\alpha + \frac{\beta}{2}\big(r + 4n^2\big) - b\alpha.$$

Hence the proof. □

4. Application of torse forming vector field on Kenmotsu manifold admitting ∗-k-Ricci-Yamabe soliton

Here, we will discuss some applications of torse forming potential vector field admitting ∗-k-Ricci-Yamabe soliton on Kenmotsu manifolds. Now, we have the following theorem.

Theorem 4.1. *If the metric g of a $(2n+1)$-dimensional Kenmotsu manifold admits the ∗-k-Ricci-Yamabe soliton $(g, \tau, \Lambda, \alpha, \beta)$, where τ is a torse-forming vector field, then $\Lambda = \frac{\beta(r+4n^2)}{2} - k\psi - \alpha(2n - 1) - \frac{\alpha r + \alpha + k\omega(\tau)}{(2n+1)}$ and the soliton is expanding, steady, shrinking according as $\frac{\beta(r+4n^2)}{2} - k\psi - \alpha(2n - 1) - \frac{\alpha r + \alpha + k\omega(\tau)}{(2n+1)} \gtreqless 0$.*

Proof. Let $(g, \tau, \Lambda, \alpha, \beta)$ be a ∗-k-Ricci-Yamabe soliton on a $(2n+1)$-dimensional Kenmotsu manifold M, where τ is a torse-forming vector field. Then from (1.14), (2.15) and (2.16), we have

$$\begin{aligned}k(\pounds_\tau g)(X, Y) + 2\alpha[S(X, Y) &+ (2n - 1)g(X, Y) + \eta(X)\eta(Y)] \\ &+ [2\Lambda - \beta(r + 4n^2)]g(X, Y) = 0,\end{aligned} \tag{4.1}$$

where $\mathcal{L}_\tau g$ denotes the Lie derivative of the metric g along the vector field τ.
Now, we utilize the equation (1.16) to obtain

$$
\begin{aligned}
(\mathcal{L}_\tau g)(X,Y) &= g(\nabla_X \tau, Y) + g(X, \nabla_Y \tau) \\
&= 2\psi g(X,Y) + \omega(X)g(\tau, Y) + \omega(Y)g(\tau, X)
\end{aligned}
\tag{4.2}
$$

for all $X, Y \in M$.
Then we combine two identities (4.2) and (4.1) to yield

$$
\left[\frac{\beta(r + 4n^2)}{2} - \Lambda - k\psi - \alpha(2n-1) \right] g(X,Y) - \alpha S(X,Y) - \alpha \eta(X)\eta(Y)
$$
$$
= \frac{k}{2} \Big[\omega(X)g(\tau, Y) + \omega(Y)g(\tau, X) \Big]. \quad (4.3)
$$

Now, we take contraction of (4.3) over X and Y to acquire

$$
\left[\frac{\beta(r + 4n^2)}{2} - \Lambda - k\psi - \alpha(2n-1) \right](2n+1) - \alpha r - \alpha = k\omega(\tau), \tag{4.4}
$$

which leads to

$$
\Lambda = \frac{\beta(r + 4n^2)}{2} - k\psi - \alpha(2n-1) - \frac{\alpha r + \alpha + k\omega(\tau)}{(2n+1)}. \tag{4.5}
$$

Now, using the nature of the soliton, we can prove the remaining part of the theorem.
Thus, we end our proof. $\qquad\square$

Now in (4.5), if the 1-form ω vanishes identically, then $\Lambda = \frac{\beta(r+4n^2)}{2} - k\psi - \alpha(2n-1) - \frac{\alpha r + \alpha}{(2n+1)}$.

If the 1-form ω vanishes identically and the function $\psi = 1$ in (4.5), then $\Lambda = \frac{\beta(r+4n^2)}{2} - k - \alpha(2n-1) - \frac{\alpha r + \alpha}{(2n+1)}$.

In (4.5), if the function $\psi = 0$, then $\Lambda = \frac{\beta(r+4n^2)}{2} - \alpha(2n-1) - \frac{\alpha r + \alpha + k\omega(\tau)}{(2n+1)}$.

If $\psi = \omega = 0$ in (4.5), then $\Lambda = \frac{\beta(r+4n^2)}{2} - \alpha(2n-1) - \frac{\alpha r + \alpha}{(2n+1)}$.

Finally in (4.5), if $\omega(\tau) = 0$, then $\Lambda = \frac{\beta(r+4n^2)}{2} - k\psi - \alpha(2n-1) - \frac{\alpha r + \alpha}{(2n+1)}$.
Then we have

Corollary 4.2. *Let the metric g of a $(2n+1)$-dimensional Kenmotsu manifold satisfy the $*$-k-Ricci-Yamabe soliton $(g, \tau, \Lambda, \alpha, \beta)$, where τ is a torse-forming vector field, then if τ is*

(i) *concircular, then $\Lambda = \frac{\beta(r+4n^2)}{2} - k\psi - \alpha(2n-1) - \frac{\alpha r + \alpha}{(2n+1)}$ and the soliton is expanding, steady, shrinking according as $\frac{\beta(r+4n^2)}{2} - k\psi - \alpha(2n-1) - \frac{\alpha r + \alpha}{(2n+1)} \gtreqless 0$,*

(ii) *concurrent, then $\Lambda = \frac{\beta(r+4n^2)}{2} - 1 - \alpha(2n-1) - \frac{\alpha r + \alpha}{(2n+1)}$ and the soliton is expanding, steady, shrinking according as $\Lambda = \frac{\beta(r+4n^2)}{2} - 1 - \alpha(2n-1) - \frac{\alpha r + \alpha}{(2n+1)} \gtreqless 0$,*

(iii) *recurrent, then $\Lambda = \frac{\beta(r+4n^2)}{2} - \alpha(2n-1) - \frac{\alpha r + \alpha + k\omega(\tau)}{(2n+1)}$ and the soliton is expanding, steady, shrinking according as $\Lambda = \frac{\beta(r+4n^2)}{2} - \alpha(2n-1) - \frac{\alpha r + \alpha + k\omega(\tau)}{(2n+1)} \gtreqless 0$,*

(iv) *parallel, then $\Lambda = \frac{\beta(r+4n^2)}{2} - \alpha(2n-1) - \frac{\alpha r + \alpha}{(2n+1)}$ and the soliton is expanding, steady, shrinking according as $\Lambda = \frac{\beta(r+4n^2)}{2} - \alpha(2n-1) - \frac{\alpha r + \alpha}{(2n+1)} \gtreqless 0$,*

(v) *torqued, then $\Lambda = \frac{\beta(r+4n^2)}{2} - k\psi - \alpha(2n-1) - \frac{\alpha r + \alpha}{(2n+1)}$ and the soliton is expanding, steady, shrinking according as $\Lambda = \frac{\beta(r+4n^2)}{2} - k\psi - \alpha(2n-1) - \frac{\alpha r + \alpha}{(2n+1)} \gtreqless 0$.*

5. Gradient ∗-k-Ricci-Yamabe soliton on Kenmotsu manifolds

This section is devoted to the study of Kenmotsu manifolds admitting gradient ∗-k-Ricci-Yamabe soliton and we try to characterize the potential vector field of the soliton. Here, we prove some lemma on the Kenmotsu manifold.

Lemma 5.1. *For a (2n+1)-dimensional Kenmotsu manifold, the following relation holds*

$$(\nabla_Y Q^*)\xi - (\nabla_\xi Q^*)Y = \nabla_Y \xi.$$

Proof. Now from (2.15), we can write

$$Q^* X = -X + \eta(X)\xi. \tag{5.1}$$

Differentiating the above equation covariantly with respect to Y, we get

$$\begin{aligned}
\nabla_Y Q^* X &= -\nabla_Y X + [g(X,Y) - \eta(X)\eta(Y) + \eta(\nabla_Y X) - \eta(X)\eta(Y)]\xi \\
&+ \eta(X)Y.
\end{aligned} \tag{5.2}$$

Now, we use the above two identities (5.1) and (5.2) to yield

$$\begin{aligned}
(\nabla_Y Q^*)X &= \nabla_Y Q^* X - Q^*(\nabla_Y X) \\
&= g(X,Y)\xi - 2\eta(X)\eta(Y)\xi + \eta(X)Y.
\end{aligned} \tag{5.3}$$

We plug $X = \xi$ into identity (5.3) to achieve

$$(\nabla_Y Q^*)\xi = Y - \eta(Y)\xi = \nabla_Y \xi. \tag{5.4}$$

Now, we insert $Y = \xi$ into (5.3) to find

$$(\nabla_\xi Q^*)Y = 0. \tag{5.5}$$

From the identities (5.4) and (5.5), we obtain

$$(\nabla_Y Q^*)\xi - (\nabla_\xi Q^*)Y = \nabla_Y \xi. \tag{5.6}$$

So, this ends our proof. □

Lemma 5.2. *If $(g, V, \alpha, \beta, \Lambda)$ is a gradient ∗-k-Ricci-Yamabe soliton on a (2n+1)-dimensional Kenmotsu manifold (M, g, ϕ, ξ, η), then the Riemannian curvature tensor R satisfies (5.12).*

Proof. Since the metric is gradient ∗-k-Ricci-Yamabe soliton, so using (1.15), (2.15) and (2.16), we can write

$$k\,Hess f(X,Y) = -\alpha S(X,Y) - \Big[\Lambda - \frac{\beta(r+4n^2)}{2} + (2n-1)\Big]g(X,Y) - \alpha\eta(X)\eta(Y) \tag{5.7}$$

for all $X, Y \in M$.
Now, the foregoing equation can be rewritten as

$$k\nabla_X Df = -\alpha QX - \Big[\Lambda - \frac{\beta(r+4n^2)}{2} + (2n-1)\Big]X - \alpha\eta(X)\xi. \tag{5.8}$$

We differentiate covariantly the previous equation along an arbitrary vector field Y and using (2.6) to achieve

$$\begin{aligned}
k\nabla_Y \nabla_X Df + (Yk)\nabla_X Df &= -\alpha\nabla_Y QX - \Big[\Lambda - \frac{\beta(r+4n^2)}{2} + (2n-1)\Big]\nabla_Y X \\
&+ \frac{X\beta}{2}(Yr) - \alpha[\nabla_Y \eta(X)\xi \\
&+ (Y - \eta(Y)\xi)\eta(X)].
\end{aligned} \tag{5.9}$$

Now, we replace X and Y into the identity (5.9) to yield

$$
\begin{aligned}
k\nabla_X\nabla_Y Df + (Xk)\nabla_Y Df &= -\alpha\nabla_X QY - \left[\Lambda - \frac{\beta(r+4n^2)}{2} + (2n-1)\right]\nabla_X Y \\
&+ \frac{Y\beta}{2}Xr - \alpha[\nabla_X\eta(Y)\xi \\
&+ (X-\eta(X)\xi)\eta(Y)].
\end{aligned}
\tag{5.10}
$$

Also in view of (5.8), we acquire

$$
\begin{aligned}
k\nabla_{[X,Y]}Df &= -\alpha Q(\nabla_X Y - \nabla_Y X) - \left[\Lambda - \frac{\beta(r+4n^2)}{2} + (2n-1)\right](\nabla_X Y \\
&- \nabla_Y X) - \alpha\eta(\nabla_X Y - \nabla_Y X)\xi.
\end{aligned}
\tag{5.11}
$$

Now, we plug the values of (5.9), (5.10) and (5.11) into the very well known Riemannian curvature formula

$$
R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z
$$

to achieve

$$
\begin{aligned}
kR(X,Y)Df &= \alpha[(\nabla_Y Q)X - (\nabla_X Q)Y] + \frac{\beta}{2}[X(r)Y - Y(r)X] \\
&+ \frac{\alpha}{k}[(Xk)QY - (Yk)QX] + \frac{1}{k}\left[\Lambda - \frac{\beta(r+4n^2)}{2}\right. \\
&+ (2n-1)\Big][(Xk)Y - (Yk)X] + \frac{\alpha}{k}[(Xk)\eta(Y) \\
&- (Yk)\eta(X)]\xi + \alpha[Y\eta(X) - X\eta(Y)].
\end{aligned}
\tag{5.12}
$$

This completes the proof. $\qquad\square$

6. Example of a 5-dimensional Kenmotsu manifold admitting $*$-k-Ricci-Yamabe soliton

Example 6.1. Let us consider the set $M = \{(x,y,z,u,v) \in \mathbb{R}^5\}$ as our manifold where (x,y,z,u,v) are the standard coordinates in $\mathbb{R}^5$. The vector fields defined below:

$$
e_1 = e^{-v}\frac{\partial}{\partial x}, \qquad e_2 = e^{-v}\frac{\partial}{\partial y}, \qquad e_3 = e^{-v}\frac{\partial}{\partial z}, \qquad e_4 = e^{-v}\frac{\partial}{\partial u}, \qquad e_5 = \frac{\partial}{\partial v}
$$

are linearly independent at each point of M. We define the metric g as

$$
g(e_i, e_j) = \begin{cases} 1, & \text{if } i = j \text{ and } i,j \in \{1,2,3,4,5\} \\ 0, & \text{otherwise.} \end{cases}
$$

Let η be a 1-form defined by $\eta(X) = g(X, e_5)$, for arbitrary $X \in \chi(M)$. Let us define (1,1)-tensor field ϕ as:

$$
\phi(e_1) = e_3, \qquad \phi(e_2) = e_4, \qquad \phi(e_3) = -e_1, \qquad \phi(e_4) = -e_2, \qquad \phi(e_5) = 0.
$$

Then it satisfy the relations $\eta(\xi) = 1$, $\phi^2(X) = -X + \eta(X)\xi$ and $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$, where $\xi = e_5$ and X, Y is arbitrary vector field on M. So, (M, ϕ, ξ, η, g) defines an almost contact structure on M.

We can now deduce that,

$$
\begin{array}{llll}
[e_1, e_2] = 0 & [e_1, e_3] = 0 & [e_1, e_4] = 0 & [e_1, e_5] = e_1 \\
[e_2, e_1] = 0 & [e_2, e_3] = 0 & [e_2, e_4] = 0 & [e_2, e_5] = e_2 \\
[e_3, e_1] = 0 & [e_3, e_2] = 0 & [e_3, e_4] = 0 & [e_3, e_5] = e_3 \\
[e_4, e_1] = 0 & [e_4, e_2] = 0 & [e_4, e_3] = 0 & [e_4, e_5] = e_4 \\
[e_5, e_1] = -e_1 & [e_5, e_2] = -e_2 & [e_5, e_3] = -e_3 & [e_5, e_4] = -e_4.
\end{array}
$$

Let ∇ be the Levi-Civita connection of g. Then from $Koszul's formula$ for arbitrary $X, Y, Z \in \chi(M)$ given by:

$$
\begin{aligned}
2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\
&- g(Y, [X, Z]) + g(Z, [X, Y]),
\end{aligned}
$$

we can have:

$$
\begin{array}{lllll}
\nabla_{e_1} e_1 = -e_5 & \nabla_{e_1} e_2 = 0 & \nabla_{e_1} e_3 = 0 & \nabla_{e_1} e_4 = 0 & \nabla_{e_1} e_5 = e_1 \\
\nabla_{e_2} e_1 = 0 & \nabla_{e_2} e_2 = -e_5 & \nabla_{e_2} e_3 = 0 & \nabla_{e_2} e_4 = 0 & \nabla_{e_2} e_5 = e_2 \\
\nabla_{e_3} e_1 = 0 & \nabla_{e_3} e_2 = 0 & \nabla_{e_3} e_3 = -e_5 & \nabla_{e_3} e_4 = 0 & \nabla_{e_3} e_5 = e_3 \\
\nabla_{e_4} e_1 = 0 & \nabla_{e_4} e_2 = 0 & \nabla_{e_4} e_3 = 0 & \nabla_{e_4} e_4 = -e_5 & \nabla_{e_4} e_5 = e_4 \\
\nabla_{e_5} e_1 = 0 & \nabla_{e_5} e_2 = 0 & \nabla_{e_5} e_3 = 0 & \nabla_{e_5} e_4 = 0 & \nabla_{e_5} e_5 = 0.
\end{array}
$$

Therefore $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$ is satisfied for arbitrary $X, Y \in \chi(M)$. So (M, ϕ, ξ, η, g) becomes a Kenmotsu manifold.

The non-vanishing components of curvature tensor are:

$$
\begin{array}{lll}
R(e_1, e_2)e_2 = -e_1 & R(e_1, e_3)e_3 = -e_1 & R(e_1, e_4)e_4 = -e_1 \\
R(e_1, e_5)e_5 = -e_1 & R(e_1, e_2)e_1 = e_2 & R(e_1, e_3)e_1 = e_3 \\
R(e_1, e_4)e_1 = e_4 & R(e_1, e_5)e_1 = e_5 & R(e_2, e_3)e_2 = e_3 \\
R(e_2, e_4)e_2 = e_4 & R(e_2, e_5)e_2 = e_5 & R(e_2, e_3)e_3 = -e_2 \\
R(e_2, e_4)e_4 = -e_2 & R(e_2, e_5)e_5 = -e_2 & R(e_3, e_4)e_3 = e_4 \\
R(e_3, e_5)e_3 = e_5 & R(e_3, e_4)e_4 = -e_3 & R(e_4, e_5)e_4 = e_5 \\
R(e_5, e_3)e_5 = e_3 & R(e_5, e_4)e_5 = e_4.
\end{array}
$$

Now from the above results we have, $S(e_i, e_i) = -4$ for $i = 1, 2, 3, 4, 5$ and

$$
S(X, Y) = -4g(X, Y) \quad \forall X, Y \in \chi(M). \tag{6.1}
$$

Contracting this we have $r = \sum_{i=1}^{5} S(e_i, e_i) = -20 = -2n(2n+1)$ where dimension of the manifold $2n + 1 = 5$. Also, we have

$$
S^*(e_i, e_i) = \begin{cases} -1, & \text{if } i = 1, 2, 3, 4 \\ 0, & \text{if } i = 5. \end{cases}
$$

So,

$$
S^*(X, Y) = -g(X, Y) + \eta(X)\eta(Y) \quad \forall X, Y \in \chi(M). \tag{6.2}
$$

Hence,

$$
r^* = \text{Tr}(S^*) = -4. \tag{6.3}
$$

Now, we consider a vector field V as

$$
V = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u} + \frac{\partial}{\partial v}. \tag{6.4}
$$

Then from the above results we can justify that

$$
(\pounds_V g)(X, Y) = 4\{g(X, Y) - \eta(X)\eta(Y)\}, \tag{6.5}
$$

which holds for all $X, Y \in \chi(M)$. Hence in light of (6.5), we infer

$$
\sum_{i=1}^{5} (\pounds_V g)(e_i, e_i) = 16. \tag{6.6}
$$

Now, we put $X = Y = e_i$ in the (1.14), summing over $i = 1, 2, 3, 4, 5$ and using (6.3) and (6.6) to obtain

$$\Lambda = \frac{4\alpha - 10\beta - 8k}{5}. \tag{6.7}$$

As this Λ, defined as above satisfies (3.7), so g defines a $*$-k-Ricci-Yamabe soliton on the 5-dimensional Kenmotsu manifold M.

Also, we can state

Remark 6.2. Case-I: When $\alpha = 1, \beta = 0$, (6.7) provides $\Lambda = \frac{4-8k}{5}$ and hence (g, V, Λ) is a $*$-k-Ricci soliton which is shrinking when $k > \frac{1}{2}$, expanding when $k < \frac{1}{2}$ and steady if $k = \frac{1}{2}$.

Case-II: When $\alpha = 0, \beta = 2$, (6.7) provides $\Lambda = \frac{-20-8k}{5}$ and hence (g, V, Λ) is a $*$-k-Yamabe soliton which is shrinking if $k > -\frac{5}{2}$, expanding if $k < -\frac{5}{2}$ and shrinking if $k = -\frac{5}{2}$.

Case-III: When $\alpha = 1, \beta = 1$, (6.7) provides $\Lambda = \frac{-6-8k}{5}$ and hence (g, V, Λ) is a $*$-k-Einstein soliton which is shrinking if $k > -\frac{3}{4}$, expanding if $k < -\frac{3}{4}$ and steady if $k = -\frac{3}{4}$.

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