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Conformal bi-slant Riemannian maps

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Abstract

In this study, conformal bi-slant Riemannian maps from an almost Hermitian manifold to a Riemannian manifold are defined. Integrability conditions of certain distributions on total manifolds are examined. Also, we studied that under which conditions, the distributions can define a totally geodesic foliation.

Keywords: Riemannian map, Conformal Riemannian map, Conformal bi-slant Riemannian map.

1. *Introduction*

At first, the notion of submersion was introduced by O'Neill (O'Neill 1966) and Gray (Gray 1967). Submersions between almost Hermitian manifolds were studied by Watson (Watson 1976). Then, this notion was studied in various types and generalized to Riemannian maps by Fischer (Fischer 1992). Riemannian maps between Riemannian manifolds are generalization of isometric immersions and Riemannian submersions. Let $\Phi:(\mathrm{M}_1,\mathrm{g}_1)\rightarrow(\mathrm{M}_2,\mathrm{g}_2)$ be a smooth map between Riemannian manifolds such that $0<rank\Phi<min\,\{dim(M_1),dim(M_2)\}$. Then, the tangent bundle of TM_1 of M_1 has the following decomposition:

$$TM_1=ker\Phi_*\oplus(ker\Phi_*)^{\perp}.$$

Since $rank\Phi<min\,\{dim(M_1),dim(M_2)\}$, we have $(range\Phi_*)^{\perp}$. Therefore, tangent bundle of TM_2 of M_2 has the following decomposition:

$$TM_2=range\Phi_*\oplus(range\Phi_*)^{\perp}.$$

A smooth map $\Phi:(\mathrm{M}_1^m,\mathrm{g}_1)\rightarrow(\mathrm{M}_2^m,\mathrm{g}_2)$ is called Riemannian map at $p_1\in M_1$ if the horizontal restriction $\Phi^h_{*p_1}:\left(ker\Phi_{*p_1}\right)^{\perp}\longrightarrow(range\Phi_*)$ is a linear isometry. Hence the Riemannian map satisfies the equation

$$g_1(X,Y)=g_2\big(\Phi_*(X),\Phi_*(Y)\big)$$

for $X,Y\in\Gamma((ker\Phi_*)^{\perp})$. So that isometric immersions and Riemannian submersions are particular Riemannian maps, respectively, with $\mathrm{ker}\Phi_*=\{0\}$ and $(\mathrm{range}\Phi_*)^{\perp}=\{0\}$ (Fischer 1992). Moreover, Şahin and Yanan searched conformal Riemannian maps (Şahin and Yanan 2018), (Şahin and Yanan 2019), (Yanan and Şahin 2022), see also (Yanan 2021). We say that $\Phi:(\mathrm{M}^{\mathrm{m}},\mathrm{g}_{\mathrm{M}})\longrightarrow(\mathrm{N}^{\mathrm{n}},\mathrm{g}_{\mathrm{N}})$ is a conformal Riemannian map at $p\in M$ if $0<rank\Phi_{*p}\leq min\{m,n\}$ and Φ_* maps the horizontal space $\left(\mathrm{ker}\left(\Phi_{*\mathrm{p}}\right)^{\perp}\right)$ conformally onto $range\left(\Phi_{*p}\right)$, i.e., there exist a number $\lambda^2(\mathrm{p})\neq0$ such that

$$g_N\left(\Phi_{*p}(X),\Phi_{*p}(Y)\right)=\lambda^2(p)g_M(X,Y)$$

for $X,Y\in\Gamma((ker\Phi_*)^{\perp})$. Also, Φ is called conformal Riemannian if Φ is conformal Riemannian at each $p\in M$. Here, λ is the dilation of Φ at a point $p\in M$ and it is a continuous function as $\lambda:M\rightarrow[0,\infty)$ (Şahin 2010), (Şahin 2017). If anyone wants to have more knowledge on submersion theory and bi-slant structure, the studies written by Aykurt Sepet could be seen (Akyol and Şahin 2019, Aykurt Sepet 2020), (Aykurt Sepet 2021).

An even-dimensional Riemannian manifold (M,g_M,J) is called an almost Hermitian manifold if there exists a tensor field J of type (1,1) on M such that $J^2=-I$ where I denotes the identity transformation of TM and

$$g_M(X,Y)=g_M(JX,JY),\forall X,Y\in\Gamma(TM).$$

Let (M,g_M,J) be an almost Hermitian manifold and its Levi-Civita connection ∇ with respect to g_M. If J is parallel with respect to ∇, i.e.

$$(\nabla_X J)Y=0,$$

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we say M is a Kaehler manifold (Yano and Kon 1984).

Therefore, we define conformal bi-slant Riemannian maps from an almost Hermitian manifold to a Riemannian manifold as a generalization of conformal anti-invariant Riemannian maps (Şahin and Yanan 2018), conformal semi-invariant Riemannian maps (Şahin and Yanan 2019), conformal semi-slant Riemannian maps (Yanan 2022b) and conformal hemi-slant Riemannian maps (Yanan 2022a). Also, an explicit example is given. Some geometric properties of this type maps are examined.

2. Materials and Methods

In this section, we give several definitions and results to be used along the study for conformal bi-slant Riemannian maps. Let $\Phi:(M,g_M)\longrightarrow(N,g_N)$ be a smooth map between Riemannian manifolds. The second fundamental form of Φ is defined by

$$(\nabla\Phi_*)(X,Y)=\nabla_X^{\Phi}\Phi_*(Y)-\Phi_*(\nabla_X Y)$$

for $X,Y\in\Gamma(TM)$. The second fundamental form $(\nabla\Phi_*)$ is symmetric. Note that Φ is said to be totally geodesic map if $(\nabla F_*)(X,Y)=0$ for all $X,Y\in\Gamma(TM)$ (Nore 1986). Here, we define O'Neill's tensor fields $\mathcal{T}$ and $\mathcal{A}$ as

$$\mathcal{A}_X Y=h\nabla_{hX}vY+v\nabla_{hX}hY,$$
$$\mathcal{T}_X Y=h\nabla_{vX}vY+v\nabla_{vX}hY,$$

for $X,Y\in\Gamma(TM)$ with the Levi-Civita connection ∇ of g_M. Here, we denote by v and h the projections on the vertical distribution $ker\Phi_*$ and the horizontal distribution $(ker\Phi_*)^{\perp}$, respectively. For any $X\in\Gamma(TM)$, $\mathcal{T}_X$ and $\mathcal{A}_X$ are skew-symmetric operators on $(\Gamma(TM),g)$ reversing the horizontal and the vertical distributions. Also, $\mathcal{T}$ is vertical, $\mathcal{T}_X=\mathcal{T}_{vX}$ and $\mathcal{A}$ is horizontal, $\mathcal{A}_X=\mathcal{A}_{hX}$. Note that the tensor field $\mathcal{T}$ is symmetric on the vertical distribution (O'Neill 1966). In addition, by definitions of O'Neill's tensor fields, we have

$$\nabla_U V=\mathcal{T}_U V+v\nabla_U V,$$
$$\nabla_U X=h\nabla_U X+\mathcal{T}_U X,$$
$$\nabla_X V=\mathcal{A}_X V+v\nabla_X V,$$
$$\nabla_X Y=h\nabla_X Y+\mathcal{A}_X Y$$

for $X,Y\in\Gamma((ker\Phi_*)^{\perp})$ and $U,V\in\Gamma(ker\Phi_*)$ (Falcitelli et al. 2004).

If a vector field X on M is related to a vector field X' on N, we say X is a projectable vector field. If X is both a horizontal and a projectable vector field, we say X is a basic vector field on M (Baird and Wood 2003). Throughout this study, when we mention a horizontal vector field, we always consider a basic vector field.

On the other hand, let $\Phi:(M^m,g_M)\longrightarrow(N^n,g_N)$ be a conformal Riemannian map between Riemannian manifolds. Then, we have

$$\begin{aligned}(\nabla\Phi_*)(X,Y)\,|_{range\Phi_*} &= X(\ln\lambda)\Phi_*(Y)+Y(\ln\lambda)\Phi_*(X)\\ &\quad -g_M(X,Y)\Phi_*\big(grad(\ln\lambda)\big)\end{aligned}$$

where $X,Y\in\Gamma((ker\Phi_*)^{\perp})$ (Şahin 2010). Hence, we obtain $\nabla_X^{\Phi}\Phi_*(Y)$ as

$$\begin{aligned}\nabla_X^{\Phi}\Phi_*(Y) &= \Phi_*(h\nabla_X Y)+X(\ln\lambda)\Phi_*(Y)\\ &\quad +Y(\ln\lambda)\Phi_*(X)\\ &\quad -g_M(X,Y)\Phi_*\big(grad(\ln\lambda)\big)\\ &\quad +(\nabla\Phi_*)^{\perp}(X,Y)\end{aligned}$$

where $(\nabla\Phi_*)^{\perp}(X,Y)$ is the component of $(\nabla\Phi_*)(X,Y)$ on $(range\Phi_*)^{\perp}$ for $X,Y\in\Gamma((ker\Phi_*)^{\perp})$ (Şahin and Yanan 2018).

3. Results

In this section, we define conformal bi-slant Riemannian maps and give an example. In addition, we present conditions to be integrable and to define totally geodesic foliation for distributions.

Definition 3.1. Let (M,g_M,J) be an almost Hermitian manifold and (N,g_N) be a Riemannian manifold. Then, a conformal Riemannian map $\Phi:(M,g_M,J)\longrightarrow(N,g_N)$ is called a conformal bi-slant Riemannian map if and only if D_1 and D_2 are slant distributions with their slant angles θ_1 and θ_2, respectively, such that

$$ker\Phi_*=D_1\oplus D_2.$$

Here, if the slant angles satisfy that $\theta_1,\theta_2\neq 0,\frac{\pi}{2}$, Φ is called a proper conformal bi-slant Riemannian map.

Therefore, suppose that the dimensions of D_1 and D_2 are m_1 and m_2, respectively. Then, we have the next notions.

i. If $m_1 = 0$ and $\theta_2 = \frac{\pi}{2}$, then Φ is a conformal anti-invariant Riemannian map (Şahin and Yanan 2018),

ii. If $m_1, m_2 \neq 0$, $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$, then Φ is a conformal semi-invariant Riemannian map (Şahin and Yanan 2019),

iii. If $m_1, m_2 \neq 0$, $\theta_1 = 0$ and $0 < \theta_2 < \frac{\pi}{2}$, then then Φ is a conformal semi-slant Riemannian map (Yanan 2022b),

iv. If $m_1, m_2 \neq 0$, $\theta_1 = \frac{\pi}{2}$ and $0 < \theta_2 < \frac{\pi}{2}$, then then Φ is a conformal hemi-slant Riemannian map (Yanan 2022a).

After these cases, we give an explicit example for proper conformal bi-slant Riemannian map.

Example 3.2. Let Φ be a map defined as

$$\Phi: R^8 \longrightarrow R^5: e^2\left(\frac{x_1 - x_7}{\sqrt{2}}, x_4, \frac{x_5 - x_6}{\sqrt{2}}, x_2, \gamma\right)$$

where γ is the real number. The almost complex structure J_β on R^8 is

$$J_\beta = (\cos\beta)J_1 + (\sin\beta)J_2, 0 < \beta \leq \frac{\pi}{2}$$

where

$$J_1 = (-a_2, a_1, -a_4, a_3, -a_6, a_5, -a_8, a_7)$$

and

$$J_2 = (-a_8, -a_7, -a_6, -a_5, a_4, a_3, a_2, a_1).$$

Then, we have the horizontal distribution as

$$(ker\Phi_*)^\perp = \{X_1 = \frac{e^2}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_7}\right), X_2 = e^2\frac{\partial}{\partial x_4},$$

$$X_3 = \frac{e^2}{\sqrt{2}}\left(\frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_6}\right), X_4 = e^2\frac{\partial}{\partial x_2}\}$$

and the vertical distribution as

$$ker\Phi_* = \{U_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_7}\right), U_2 = \frac{\partial}{\partial x_3},$$

$$U_3 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6}\right), X_4 = \frac{\partial}{\partial x_8}\}.$$

Here, we say Φ is a conformal Riemannian map with $\lambda = e^2$ and $rank\Phi_* = 4$. Then, by some calculations we obtain the slant distributions as $D_1 = \{U_1, U_2\}$ and $D_2 = \{U_3, U_4\}$. Hence, Φ is a proper conformal bi-slant Riemannian map with respect to the slant distributions

$$D_1 = \{U_1, U_2\}, D_2 = \{U_3, U_4\}$$

and the slant angles

$$\cos\theta_1 = \frac{1}{\sqrt{2}}(\cos\beta + \sin\beta), \ \cos\theta_2 = \frac{1}{\sqrt{2}}\sin\beta.$$

Now, we explain decomposition of distributions for a conformal bi-slant Riemannian map.

Assume that Φ be a conformal bi-slant Riemannian map from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (N, g_N). For any $U \in \Gamma(ker\Phi_*)$, we have

$$U = PU + QU,$$

where $PU \in \Gamma(D_1)$ and $QU \in \Gamma(D_2)$. On the other hand, we have

$$JU = \psi U + \phi U,$$

for $U \in \Gamma(ker\Phi_*)$ where $\phi U \in \Gamma((ker\Phi_*)^\perp)$ and $\psi U \in \Gamma(ker\Phi_*)$. Also, for any $X \in \Gamma((ker\Phi_*)^\perp)$, we write

$$JX = BX + CX,$$

where $BX \in \Gamma(ker\Phi_*)$ and $CX \in \Gamma((ker\Phi_*)^\perp)$. Therefore, the horizontal distribution $(ker\Phi_*)^\perp$ can be decomposed as

$$(ker\Phi_*)^\perp = \phi D_1 \oplus \phi D_2 \oplus \mu,$$

where μ is the orthogonal complementary distribution of $\phi D_1 \oplus \phi D_2$ in $(ker\Phi_*)^\perp$.

We have the following theorem same for conformal bi-slant Riemannian maps (Aykurt Sepet 2021).

Theorem 3.3. Let Φ be a conformal bi-slant Riemannian map from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (N, g_N) with slant angles θ_1 and θ_2. Then, we have

$$\psi^2 U_i = -(\cos^2\theta_i)U_i$$

for $U_i \in \Gamma(D_i)$, $i = 1,2$.

After then, we examine integrability conditions for certain distributions.

Theorem 3.4. Let Φ be a proper conformal bi-slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) with slant angles θ_1 and θ_2. Then, the distribution D_1 is integrable if and only if

$$g_N\left((\nabla\Phi_*)(V_1, U_2), \Phi_*\left(\phi(JU_1)\right)\right)$$

$$= \lambda^2 \cos^2\theta_1\, g_M\left(v\nabla_{V_1}U_1, U_2\right)$$

$$-g_M\left(v\nabla_{U_1}V_1 + v\nabla_{V_1}\psi\phi U_1, U_2\right)$$

for $U_1, V_1 \in \Gamma(D_1)$ and $U_2 \in \Gamma(D_2)$.

Proof. Since the vertical distribution $ker\Phi_*$ is always integrable, we only examine $0 = g_M([U_1, V_1], U_2)$ for $U_1, V_1 \in \Gamma(D_1)$ and $U_2 \in \Gamma(D_2)$. To get this equality, we write

$$\begin{aligned}[U_1, V_1] &= \nabla_{U_1} V_1 - \nabla_{V_1} U_1 \\ &= v\nabla_{U_1} V_1 - \nabla_{V_1} \psi^2 U_1 + \nabla_{V_1} \phi\psi U_1 \\ &\quad + \nabla_{V_1} \psi\phi U_1 \\ &= v\nabla_{U_1} V_1 - \cos^2 \theta_1 \nabla_{V_1} U_1 + h\nabla_{V_1} \phi\psi U_1 \\ &\quad + \mathcal{T}_{V_1} \phi\psi U_1 + h\nabla_{V_1} \phi^2 U_1 + \mathcal{T}_{V_1} \phi^2 U_1 \\ &\quad + \mathcal{T}_{V_1} \psi\phi U_1 + v\nabla_{V_1} \psi\phi U_1.\end{aligned}$$

Now, for $U_2 \in \Gamma(D_2)$, we get

$$\begin{aligned} g_M([U_1, V_1], U_2) &= g_M\big(v\nabla_{U_1} V_1 - \cos^2 \theta_1 \nabla_{V_1} U_1, U_2\big) \\ &\quad + g_M\big(\mathcal{T}_{V_1} \phi(JU_1) + v\nabla_{V_1} \psi\phi U_1, U_2\big).\end{aligned}$$

Since $\mathcal{T}$ is an anti-symmetric tensor field with respect to g_M, we have

$$g_M\big(\mathcal{T}_{V_1} \phi(JU_1), U_2\big) = -g_M\left(\mathcal{T}_{V_1} U_2, \phi(JU_1)\right).$$

Then, since the map Φ is conformal by using definition of second fundamental form of the map, we get

$$\begin{aligned} -g_M\left(\mathcal{T}_{V_1} U_2, \phi(JU_1)\right) &= \frac{1}{\lambda^2} g_N\left((\nabla\Phi_*)(V_1, U_2), \Phi_*\big(\phi(JU_1)\big)\right).\end{aligned}$$

At last, we obtain

$$\begin{aligned} g_M([U_1, V_1], U_2) &= -\cos^2 \theta_1 g_M\big(v\nabla_{V_1} U_1, U_2\big) \\ &\quad + g_M\big(v\nabla_{U_1} V_1 + v\nabla_{V_1} \psi\phi U_1, U_2\big) \\ &\quad + \frac{1}{\lambda^2} g_N\left((\nabla\Phi_*)(V_1, U_2), \Phi_*\big(\phi(JU_1)\big)\right).\end{aligned}$$

The proof is complete.

Theorem 3.5. Let Φ be a proper conformal bi-slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) with slant angles θ_1 and θ_2. Then, the distribution D_2 is integrable if and only if

$$\begin{aligned} g_N\left((\nabla\Phi_*)(V_2, U_1), \Phi_*\big(\phi(JU_2)\big)\right) &= \lambda^2 \cos^2 \theta_2\, g_M\big(v\nabla_{V_2} U_2, U_1\big) \\ &\quad - g_M\big(v\nabla_{U_2} V_2 + v\nabla_{V_2} \psi\phi U_2, U_1\big)\end{aligned}$$

for $U_2, V_2 \in \Gamma(D_2)$ and $U_1 \in \Gamma(D_1)$.

Proof. The proof of the Theorem 3.5. can be done in a similar way as Theorem 3.4.

Theorem 3.6. Let Φ be a proper conformal bi-slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) with slant angles θ_1 and θ_2. Then, the distribution D_1 defines a totally geodesic foliation on M if and only if

i. $$\begin{aligned} & g_N\big((\nabla\Phi_*)(U_1, \psi V_1), \Phi_*(\phi U_2)\big) - g_N\big((\nabla\Phi_*)(U_1, V_1), \Phi_*(\phi\psi U_2)\big) \\ &= -\lambda^2 \cos^2 \theta_2\, g_M\big(v\nabla_{U_1} U_2, V_1\big) - \lambda^2 g_M\big(h\nabla_{U_1} \phi U_2, \phi V_1\big)\end{aligned}$$

ii. $$\begin{aligned} g_N\big((\nabla\Phi_*)(U_1, BX), \Phi_*(\phi V_1)\big) &= \lambda^2 g_M\big(h\nabla_{U_1} \phi\psi V_1, X\big) \\ &\quad + \lambda^2 g_M\big(h\nabla_{U_1} \phi V_1, CX\big)\end{aligned}$$

for $U_1, V_1 \in \Gamma(D_1)$, $U_2 \in \Gamma(D_2)$ and $X \in \Gamma((ker\Phi_*)^\perp)$.

Proof. If the distribution D_1 defines a totally geodesic foliation on M, we have $0 = g_M\big(\nabla_{U_1} V_1, U_2\big)$ and $0 = g_M\big(\nabla_{U_1} V_1, X\big)$ for $U_1, V_1 \in \Gamma(D_1)$, $U_2 \in \Gamma(D_2)$ and $X \in \Gamma((ker\Phi_*)^\perp)$. At first, we get

$$\begin{aligned} g_M\big(\nabla_{U_1} V_1, U_2\big) &= g_M\big(\nabla_{U_1} \psi^2 U_2, V_1\big) \\ &\quad + g_M\big(\nabla_{U_1} \phi\psi U_2, V_1\big) \\ &\quad + g_M\big(\mathcal{T}_{U_1} \psi V_1, \phi U_2\big) \\ &\quad - g_M\big(h\nabla_{U_1} \phi U_2, \phi V_1\big)\end{aligned}$$

for $U_1, V_1 \in \Gamma(D_1)$ and $U_2 \in \Gamma(D_2)$. On the other hand, we have from definition of the second fundamental form of the map Φ and $\mathcal{T}$ is an anti-symmetric tensor field with respect to g_M

$$\begin{aligned} g_M\big(\mathcal{T}_{U_1} \phi\psi U_2, V_1\big) &= -g_M\big(\mathcal{T}_{U_1} \mathcal{V}_1, \phi\psi \mathcal{U}_2\big) \\ &= \frac{1}{\lambda^2} g_N\big((\nabla\Phi_*)(U_1, V_1), \Phi_*(\phi\psi \mathcal{U}_2)\big)\end{aligned}$$

and

$$g_M\big(\mathcal{T}_{U_1} \psi V_1, \phi U_2\big) = -\frac{1}{\lambda^2} g_N\big((\nabla\Phi_*)(U_1, \psi V_1), \Phi_*(\phi \mathcal{U}_2)\big).$$

By using these equalities and from Theorem 3.3., we obtain

$$\begin{aligned} g_M\big(\nabla_{U_1} V_1, U_2\big) &= -\cos^2 \theta_2\, g_M\big(v\nabla_{U_1} U_2, V_1\big) \\ &\quad - g_M\big(h\nabla_{U_1} \phi U_2, \phi V_1\big) \\ &\quad + \frac{1}{\lambda^2} g_N\big((\nabla\Phi_*)(U_1, V_1), \Phi_*(\phi\psi \mathcal{U}_2)\big) \\ &\quad - \frac{1}{\lambda^2} g_N\big((\nabla\Phi_*)(U_1, \psi V_1), \Phi_*(\phi \mathcal{U}_2)\big).\end{aligned}$$

From last equation, we have the proof of i. Now, we examine $0 = g_M\big(\nabla_{U_1} V_1, X\big)$ for $U_1, V_1 \in \Gamma(D_1)$ and $X \in \Gamma((ker\Phi_*)^\perp)$. By some similar computations, we have

$$\begin{aligned} g_M\big(\nabla_{U_1} V_1, X\big) &= \cos^2 \theta_1\, g_M\big(\nabla_{U_1} V_1, X\big) \\ &\quad - g_M\big(h\nabla_{U_1} \phi\psi V_1, X\big) \\ &\quad + g_M\big(\mathcal{T}_{U_1} \phi V_1, BX\big) \\ &\quad - g_M\big(h\nabla_{U_1} \phi V_1, CX\big)\end{aligned}$$

$$sin^2\theta_1\, g_M(\nabla_{U_1}V_1, X) = -g_M(h\nabla_{U_1}\phi\psi V_1, X) - g_M(h\nabla_{U_1}\phi V_1, CX) - g_M(\mathcal{T}_{U_1}BX, \phi V_1)$$

$$sin^2\theta_1\, g_M(\nabla_{U_1}V_1, X) = -g_M(h\nabla_{U_1}\phi\psi V_1, X) - g_M(h\nabla_{U_1}\phi V_1, CX) + \frac{1}{\lambda^2}g_N((\nabla\Phi_*)(U_1, BX), \Phi_*(\phi V_1)).$$

From the last equation, we obtain ii. Hence, the proof of Theorem 3.6. is complete.

Theorem 3.7. Let Φ be a proper conformal bi-slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) with slant angles θ_1 and θ_2. Then, the distribution D_2 defines a totally geodesic foliation on M if and only if

i. $$g_N((\nabla\Phi_*)(U_2, \psi U_1), \Phi_*(\phi V_2)) - g_N((\nabla\Phi_*)(U_2, U_1), \Phi_*(\phi\psi V_2)) = \lambda^2 g_M(h\nabla_{U_2}\phi V_2, \phi U_1)$$

ii. $$g_N((\nabla\Phi_*)(U_2, BX), \Phi_*(\phi V_2)) = \lambda^2 g_M(h\nabla_{U_2}\phi V_2, CX) - \lambda^2 g_M(h\nabla_{U_2}\phi\psi V_2, X)$$

for $U_1 \in \Gamma(D_1)$, $U_2, V_2 \in \Gamma(D_2)$ and $X \in \Gamma((ker\Phi_*)^\perp)$.

Proof. Here, we search $0 = g_M(\nabla_{U_2}V_2, U_1)$ and $0 = g_M(\nabla_{U_2}V_2, X)$ to get conditions i. and ii. for $U_1 \in \Gamma(D_1)$, $U_2, V_2 \in \Gamma(D_2)$ and $X \in \Gamma((ker\Phi_*)^\perp)$. Firstly, we have

$$g_M(\nabla_{U_2}V_2, U_1) = cos^2\theta_2\, g_M(\nabla_{U_2}V_2, U_1) - g_M(\nabla_{U_2}\phi\psi V_2, U_1) + g_M(\mathcal{T}_{U_2}\phi V_2, \psi U_1) + g_M(h\nabla_{U_2}\phi V_2, \phi U_1)$$

for $U_1 \in \Gamma(D_1)$ and $U_2, V_2 \in \Gamma(D_2)$. By using anti-symmetry property of $\mathcal{T}$ and from Theorem 3.3., we get

$$sin^2\theta_2\, g_M(\nabla_{U_2}V_2, U_1) = g_M(\mathcal{T}_{U_2}U_1, \phi\psi U_1) - g_M(\mathcal{T}_{U_2}\psi U_1, \phi V_2) + g_M(h\nabla_{U_2}\phi V_2, \phi U_1).$$

Since the map Φ is conformal, we obtain

$$sin^2\theta_2\, g_M(\nabla_{U_2}V_2, U_1) = g_M(h\nabla_{U_2}\phi V_2, \phi U_1) - \frac{1}{\lambda^2}g_N((\nabla\Phi_*)(U_2, \psi U_1), \Phi_*(\phi V_2)) + \frac{1}{\lambda^2}g_N((\nabla\Phi_*)(U_2, U_1), \Phi_*(\phi\psi V_2)).$$

From the last equation, we have the proof of i. Now, we examine $0 = g_M(\nabla_{U_2}V_2, X)$ for $U_2, V_2 \in \Gamma(D_2)$ and $X \in \Gamma((ker\Phi_*)^\perp)$. By similar calculations, we have

$$g_M(\nabla_{U_2}V_2, X) = cos^2\theta_2\, g_M(\nabla_{U_2}V_2, X) - g_M(h\nabla_{U_2}\phi\psi V_2, X) - g_M(\mathcal{T}_{U_2}BX, \phi V_2) + g_M(h\nabla_{U_2}\phi V_2, CX)$$

for $U_2, V_2 \in \Gamma(D_2)$ and $X \in \Gamma((ker\Phi_*)^\perp)$. At last, from conformality of the map, we obtain

$$sin^2\theta_2\, g_M(\nabla_{U_2}V_2, X) = g_M(h\nabla_{U_2}\phi V_2, CX) - g_M(h\nabla_{U_2}\phi\psi V_2, X) - \frac{1}{\lambda^2}g_N((\nabla\Phi_*)(U_2, BX), \Phi_*(\phi V_2)).$$

Hence, we have the proof of ii. clearly.

Theorem 3.8. Let Φ be a proper conformal bi-slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) with slant angles θ_1 and θ_2. Then the vertical distribution $ker\Phi_*$ is a locally product as $M_{D_1} \times M_{D_2}$ if and only if the equations in Theorem 3.6. and Theorem 3.7. are hold where M_{D_1} and M_{D_2} are integral manifolds of the distributions D_1 and D_2, respectively.

Theorem 3.9. Let Φ be a proper conformal bi-slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) with slant angles θ_1 and θ_2. Then, the distribution $(ker\Phi_*)^\perp$ defines a totally geodesic foliation on M if and only if

$$\lambda^2\{X(ln\,\lambda)g_M(CY, \phi U_i) + CY(ln\,\lambda)g_M(X, \phi U_i) - \phi U_i(ln\,\lambda)g_M(X, CY) - X(ln\,\lambda)g_M(Y, \phi\psi U_i) - Y(ln\,\lambda)g_M(X, \phi\psi U_i) + \phi\psi U_i(ln\,\lambda)g_M(X, Y)\}$$
$$= g_N\left((\nabla\Phi_*)(X, BY) + \nabla_X^\Phi\Phi_*(CY), \Phi_*(\phi U_i)\right) - g_N\left(\nabla_X^\Phi\Phi_*(Y), \Phi_*(\phi\psi U_i)\right), \qquad i = 1,2$$

for $X, Y \in \Gamma((ker\Phi_*)^\perp)$, $U_1 \in \Gamma(D_1)$ and $U_2 \in \Gamma(D_2)$.

Proof. Since Φ is a proper conformal bi-slant Riemannian map, we have two orthogonal complement distribution that D_1 and D_2 in $ker\Phi_*$, respectively. So, we examine $0 = g_M(\nabla_X Y, U_1)$ and $0 = g_M(\nabla_X Y, U_2)$ for $X, Y \in \Gamma((ker\Phi_*)^\perp)$, $U_1 \in \Gamma(D_1)$ and $U_2 \in \Gamma(D_2)$. Since we will use the same calculations for these two cases, we examine just one for U_1. Then, it will be same for U_2. Firstly, since M is a Kaehler manifold, we get

$$g_M(\nabla_X Y, U_1) = g_M(\mathcal{A}_X BY, \phi U_1) + g_M(h\nabla_X CY, \phi U_1)$$

$$+\cos^2\theta_1\, g_M(\nabla_X Y, U_1)$$
$$-g_M(h\nabla_X Y, \phi\psi U_1).$$

Since the map Φ is conformal Riemannian map, we obtain

$$\sin^2\theta_1\, g_M(\nabla_X Y, U_1)$$
$$= \frac{1}{\lambda^2} g_N\big((\nabla\Phi_*)(X, BY), \Phi_*(\phi U_1)\big)$$
$$+\frac{1}{\lambda^2}\{g_N\left(\nabla_X^{\Phi}\Phi_*(CY), \Phi_*(\phi U_1)\right)$$
$$-X(\ln\lambda) g_N\left(\Phi_*(CY), \Phi_*(\phi U_1)\right)$$
$$-CY(\ln\lambda) g_N\left(\Phi_*(X), \Phi_*(\phi U_1)\right)$$
$$+g_M(X, CY) g_N\left(\Phi_*\big(grad(\ln\lambda)\big), \Phi_*(\phi U_1)\right)\}$$
$$-\frac{1}{\lambda^2}\{g_N\left(\nabla_X^{\Phi}\Phi_*(Y), \Phi_*(\phi\psi U_1)\right)$$
$$-X(\ln\lambda) g_N\left(\Phi_*(Y), \Phi_*(\phi\psi U_1)\right)$$
$$-Y(\ln\lambda) g_N\left(\Phi_*(X), \Phi_*(\phi\psi U_1)\right)$$

$$+g_M(X, Y) g_N\left(\Phi_*\big(grad(\ln\lambda)\big), \Phi_*(\phi\psi U_1)\right)\}$$

$$\sin^2\theta_1\, g_M(\nabla_X Y, U_1)$$
$$= \frac{1}{\lambda^2} g_N\big((\nabla\Phi_*)(X, BY), \Phi_*(\phi U_1)\big)$$
$$+\frac{1}{\lambda^2}\{g_N\left(\nabla_X^{\Phi}\Phi_*(CY), \Phi_*(\phi U_1)\right)$$
$$-g_N\left(\nabla_X^{\Phi}\Phi_*(Y), \Phi_*(\phi\psi U_1)\right)$$
$$-X(\ln\lambda) g_M(CY, \phi U_1)$$
$$-CY(\ln\lambda) g_M(X, \phi U_1)$$
$$+g_M(X, CY)\, \phi U_1(\ln\lambda)$$
$$+X(\ln\lambda) g_M(Y, \phi\psi U_1)$$
$$+Y(\ln\lambda) g_M(X, \phi\psi U_1)$$
$$-g_M(X, Y)\, \phi\psi U_1(\ln\lambda).$$

It is clear that the distribution $(ker\Phi_*)^\perp$ defines a totally geodesic foliation on M for $X, Y \in \Gamma((ker\Phi_*)^\perp)$ and $U_1 \in \Gamma(D_1)$.

Theorem 3.10. Let Φ be a proper conformal bi-slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) with slant angles θ_1 and θ_2. Then, the distribution $ker\Phi_*$ defines a totally geodesic foliation on M if and only if

$$g_N\big((\nabla\Phi_*)(\phi\psi V, U), \Phi_*(X)\big)$$
$$= \lambda^2\cos^2\theta_1\, g_M(\nabla_U PV, X)$$
$$+ \lambda^2\cos^2\theta_2\, g_M(\nabla_U QV, X)$$
$$+ \lambda^2 g_M(\nabla_U \phi V, JX)$$

for $X \in \Gamma((ker\Phi_*)^\perp)$ and $U, V \in \Gamma(ker\Phi_*)$.

Proof. Now, we examine $0 = g_M(\nabla_U V, X)$ to show that the distribution $ker\Phi_*$ defines a totally geodesic foliation on M. Hence, we get

$$g_M(\nabla_U V, X) = -g_M(\nabla_U J\psi PV, X)$$
$$+g_M(h\nabla_U\phi PV + \mathcal{T}_U\phi PV, BX + CX)$$
$$-g_M(\nabla_U J\psi QV, X)$$
$$+g_M(h\nabla_U\phi QV + \mathcal{T}_U\phi QV, BX + CX)$$
$$g_M(\nabla_U V, X) = -g_M(\nabla_U\psi^2 PV + \nabla_U\phi\psi PV, X)$$
$$- g_M(\nabla_U\psi^2 QV + \nabla_U\phi\psi QV, X)$$
$$+ g_M(h\nabla_U\phi PV + h\nabla_U\phi QV, CX)$$
$$+ g_M(\mathcal{T}_U\phi PV + \mathcal{T}_U\phi QV, BX)$$
$$g_M(\nabla_U V, X) = \cos^2\theta_1\, g_M(\nabla_U PV, X)$$
$$- g_M(h\nabla_U\phi\psi PV, X)$$
$$+ \cos^2\theta_2\, g_M(\nabla_U QV, X)$$
$$- g_M(h\nabla_U\phi\psi QV, X)$$
$$+ g_M(h\nabla_U\phi V, CX)$$
$$+ g_M(\mathcal{T}_U\phi V, BX)$$
$$g_M(\nabla_U V, X) = \cos^2\theta_1\, g_M(\nabla_U PV, X)$$
$$+ \cos^2\theta_2\, g_M(\nabla_U QV, X)$$
$$-g_M(h\nabla_U\phi\psi V, X)$$
$$+g_M(\nabla_U\phi V, JX)$$

for $X \in \Gamma((ker\Phi_*)^\perp)$ and $U, V \in \Gamma(ker\Phi_*)$. By using symmetry properties of second fundamental form of the map and conformality of the map, we obtain

$$g_M(\nabla_U V, X) = \cos^2\theta_1\, g_M(\nabla_U PV, X)$$
$$+ \cos^2\theta_2\, g_M(\nabla_U QV, X)$$
$$-\frac{1}{\lambda^2} g_N\left(\Phi_*(\mathcal{A}_{\phi\psi V} U), \Phi_*(X)\right)$$
$$+g_M(\nabla_U\phi V, JX)$$
$$g_M(\nabla_U V, X) = \cos^2\theta_1\, g_M(\nabla_U PV, X)$$
$$+ \cos^2\theta_2\, g_M(\nabla_U QV, X)$$
$$-\lambda^{-2} g_N\big((\nabla\Phi_*)(\phi\psi V, U), \Phi_*(X)\big)$$
$$+g_M(\nabla_U\phi V, JX).$$

From the last equation, we obtain the proof.

Theorem 3.11. Let Φ be a proper conformal bi-slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) with slant angles θ_1 and θ_2. Then the total space M is a locally product manifold as $M_{D_1} \times M_{D_2} \times M_{(ker\Phi_*)^\perp}$ if and only if the equations in Theorem 3.6., Theorem 3.7. and Theorem 3.9. are hold where M_{D_1}, M_{D_2} and $M_{(ker\Phi_*)^\perp}$ are integral manifolds of the distributions D_1, D_2 and $(ker\Phi_*)^\perp$, respectively.

Theorem 3.12. Let Φ be a proper conformal bi-slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) with slant angles θ_1 and θ_2. Then the total space M is a locally product manifold as $M_{(ker\Phi_*)^\perp} \times M_{ker\Phi_*}$ if and only if the equations in Theorem 3.9. and Theorem 3.10. are hold where $M_{(ker\Phi_*)^\perp}$ and $M_{ker\Phi_*}$ are integral manifolds of the distributions $(ker\Phi_*)^\perp$ and $ker\Phi_*$, respectively.

Theorem 3.13. Let Φ be a proper conformal bi-slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) with slant angles θ_1 and θ_2. Then, the map Φ is a totally geodesic map if and only if

$$\begin{aligned} g_N\left(\nabla^\Phi_{hE}\Phi_*(hG) - \nabla^\Phi_E\Phi_*(hG), \Phi_*(F)\right) &= \lambda^2 \cos^2\theta_1\, g_M(\nabla_{vE}PvG, F) \\ &+ \lambda^2 \cos^2\theta_2\, g_M(\nabla_{vE}QvG, F) \\ &+ \lambda^2 g_M(h\nabla_{vE}\phi\psi vG + \phi\mathcal{T}_{v\mathcal{E}}\phi vG \\ &+ Ch\nabla_{vE}\phi vG - \mathcal{A}_{\hbar\mathcal{E}}vG \\ &- h\nabla_{vE}hG, F) \\ &+ \lambda^2\{hE(\ln\lambda)g_M(hG, F) \\ &+ hG(\ln\lambda)g_M(hE, F) \\ &- F(\ln\lambda)g_M(hE, hG)\} \end{aligned}$$

for $E, F, G \in \Gamma(TM)$.

Proof. Now, recall that Φ is said to be totally geodesic map if $(\nabla F_*)(E, G) = 0$ for all $E, G \in \Gamma(TM)$. By using this notion, we have

$$\begin{aligned} (\nabla\Phi_*)(E, G) &= \nabla^\Phi_E\Phi_*(hG) \\ &- \Phi_*(\nabla_{vE}vG + \mathcal{A}_{\hbar\mathcal{E}}vG + h\nabla_{vE}hG) \\ &+ (\nabla\Phi_*)(hE, hG) - \nabla^\Phi_{hE}\Phi_*(hG) \\ &= \nabla^\Phi_E\Phi_*(hG) \\ &-\Phi_*(\nabla_{vE}vG + \mathcal{A}_{\hbar\mathcal{E}}vG + h\nabla_{vE}hG) \\ &+(\nabla\Phi_*)^\perp(hE, hG) - \nabla^\Phi_{hE}\Phi_*(hG) \\ &+hE(\ln\lambda)\Phi_*(hG) \\ &+hG(\ln\lambda)\Phi_*(hE) \\ &-g_M(hE, hG)\Phi_*\big(grad(\ln\lambda)\big). \end{aligned}$$

On the other hand, we get

$$\begin{aligned} -\Phi_*(\nabla_{vE}vG) &= \cos^2\theta_1\,\Phi_*(\nabla_{vE}PvG) \\ &+ \cos^2\theta_2\,\Phi_*(\nabla_{vE}QvG) \\ &+ \Phi_*(h\nabla_{vE}\phi\psi vG + \phi\mathcal{T}_{v\mathcal{E}}\phi vG \\ &+ Ch\nabla_{vE}\phi vG). \end{aligned}$$

Hence, by putting this equation into $(\nabla\Phi_*)(E, G)$, we obtain,

$$\begin{aligned} (\nabla\Phi_*)(E, G) &= \nabla^\Phi_E\Phi_*(hG) - \nabla^\Phi_{hE}\Phi_*(hG) \\ &+ \cos^2\theta_1\,\Phi_*(\nabla_{vE}PvG) \\ &+ \cos^2\theta_2\,\Phi_*(\nabla_{vE}QvG) \\ &+ \Phi_*(h\nabla_{vE}\phi\psi vG + \phi\mathcal{T}_{v\mathcal{E}}\phi vG \\ &+ Ch\nabla_{vE}\phi vG) - \Phi_*(\mathcal{A}_{\hbar\mathcal{E}}vG \\ &+ h\nabla_{vE}hG) + hE(\ln\lambda)\Phi_*(hG) \\ &+hG(\ln\lambda)\Phi_*(hE) \\ &-g_M(hE, hG)\Phi_*\big(grad(\ln\lambda)\big). \end{aligned}$$

For $F \in \Gamma(TM)$, by applying $\Phi_*(F)$ to last equation and since the map is conformal Riemannian, we obtain

$$\begin{aligned} &g_N\big((\nabla\Phi_*)(E, G), \Phi_*(F)\big) = \\ &g_N\left(\nabla^\Phi_E\Phi_*(hG) - \nabla^\Phi_{hE}\Phi_*(hG), \Phi_*(F)\right) \\ &+\lambda^2\cos^2\theta_1\, g_M(\nabla_{vE}PvG, F) \\ &+\lambda^2\cos^2\theta_2\, g_M(\nabla_{vE}QvG, F) \\ &+\lambda^2 g_M(h\nabla_{vE}\phi\psi vG + \phi\mathcal{T}_{v\mathcal{E}}\phi vG + Ch\nabla_{vE}\phi vG \\ &\qquad - \mathcal{A}_{\hbar\mathcal{E}}vG - h\nabla_{vE}hG, F) \\ &+\lambda^2\{hE(\ln\lambda)g_M(hG, F) + hG(\ln\lambda)g_M(hE, F) \\ &\qquad - F(\ln\lambda)g_M(hE, hG)\}. \end{aligned}$$

Therefore, the proof is clear.

4. Discussion

Since we have the definition of conformal Riemannian map and bi-slant structure properties, these notions are combined as conformal bi-slant Riemannian map. In this study, we examine its some geometric properties.

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